Section 9.7: Steady-State Temperature and Laplace's Equation Pt 1

So far, we have solved

1D Heat Eqn $u_t = k u_{xx}$
1D Wave Eqn $u_{tt} = a^2 u_{xx}$

We can extend these to 2D.

2D Heat Eqn $u_t = k(u_{xx} + u_{yy})$
2D Wave Eqn $u_{tt} = a^2(u_{xx} + u_{yy})$

So looking at the 2D PDEs

$u_t = k(u_{xx} + u_{yy})$ and $u_{tt} = a^2(u_{xx} + u_{yy})$

We see they both have $u_{xx} + u_{yy}$ this is called the Laplacian.

<u>Notation:</u> $u_{xx} + u_{yy} = \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$

First thing we want to do in this section is find the steady state solution (which happens when $\underset{\text{Heat}}{\underset{\downarrow}{u_t = 0}}$ or $\underset{\text{Wave}}{\underset{\downarrow}{u_{tt} = 0}}$)

So 2D Heat Eqn Steady State is

$u_t = k(u_{xx} + u_{yy})$

and $u_t = 0$ when $u_{xx} + u_{yy} = 0$

Similarly 2D Wave Eqn Steady State is

$u_{tt} = a^2(u_{xx} + u_{yy})$

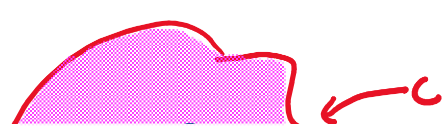
and $u_{tt} = 0$ when $u_{xx} + u_{yy} = 0$

Hence both of these satisfy the Laplace Eqn

$$u_{xx} + u_{yy} = 0$$

To understand the steady state solution, it is enough to understand the dirichlet problem, and solve it.

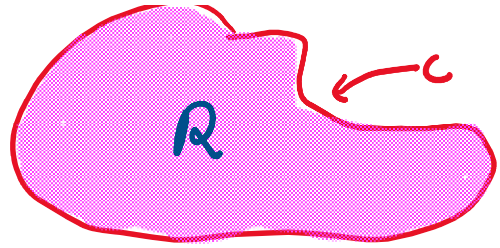
<u>Dirichlet Problem</u> helps finds a solution to Laplace's equation in a region R with given boundary values on a curve C.



values on a curve C.

region R (2D)

Curve C (1D)

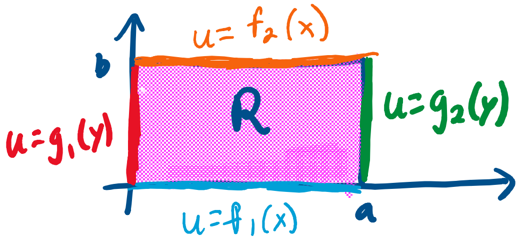


Dirichlet Problem is the following

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x,y) \text{ in region } R \\ u(x,y) = f(x,y) & \text{for } (x,y) \text{ in curve } C \end{cases}$$

Rectangular Domain R Case

Let region R be a rectangular.



The curve C is a set of 4 Straight lines

Dirichlet Problem on the Rectangle

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = g_1(y) \\ u(a,y) = g_2(y) \end{cases}$$

To solve this, we use Separation of Variables.

Ex 1: $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(a,y) = u(x,b) = 0 \\ u(x,0) = f(x) \end{cases}$ $\quad 0 \le x \le a,\ 0 \le y \le b$

Note: $f_1(x) = f(x)$ and $f_2 = g_1 = g_2 = 0$

So again Separation of Variables, we assume the solution is

$$u(x,y) = X(x)\,Y(y)$$

$$u(x,y) = X(x)\,Y(y)$$

Plug this into PDE: $u_{xx} + u_{yy} = 0$

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

Separate X terms and Y terms

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$ separation constant

So we get 2 ODEs.

$$\frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

Characteristic Eqn

$$r^2 + \lambda = 0$$

$$r = \pm i\sqrt{\lambda}$$

$$X(x) = c_1 \sin(\sqrt{\lambda}\,x) + c_2 \cos(\sqrt{\lambda}\,x)$$

$$\frac{-Y''}{Y} = -\lambda$$

$$Y'' - \lambda Y = 0$$

Characteristic Eqn:

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$Y(y) = c_3 e^{\sqrt{\lambda}\,y} + c_4 e^{\sqrt{\lambda}\,y}$$

Now let's look at the boundary conditions

$$u(0,y) = 0 = X(0)\,Y(y) \rightarrow X(0) = 0$$

$$u(a,y) = 0 = X(a)\,Y(y) \rightarrow X(a) = 0$$

$$u(x,0) = f(x) = X(x)\,Y(y)$$ pin this for later

$$u(x,b) = 0 = X(x)\,Y(b) \rightarrow Y(b) = 0$$

Remember $Y(y)$ or $X(x) \neq 0$ b/c then we get the trivial solution to the PDE.

So our ODEs become

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 = X(a) \end{cases}$$

Which we found to be

$$X(x) = c_1 \sin(\sqrt{\lambda}\,x) + c_2 \cos(\sqrt{\lambda}\,x)$$

$$\begin{cases} Y'' - \lambda Y = 0 \\ Y(b) = 0 \end{cases}$$

Which we found to be

$$Y(y) = c_3 e^{\sqrt{\lambda}\,y} + c_4 e^{-\sqrt{\lambda}\,y}$$

$$X(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$$

So when $X(0) = 0$

$$0 = X(0) = c_1 \sin(0) + c_2 \cos(0)$$

$$0 = c_2$$

So $X(x) = c_1 \sin(\sqrt{\lambda}\, x)$

When $X(a) = 0$

$$0 = X(a) = c_1 \sin(\sqrt{\lambda}\, a)$$

$$\sin(\sqrt{\lambda}\, a) = 0$$ b/c if $c_1 = 0$ then trivial solution

$\sqrt{\lambda}\, a = n\pi$ for $n = 1, 2, \dots$

$$\lambda = \left(\frac{n\pi}{a}\right)^2 \text{ for } n = 1, 2, \dots$$

with $X_n = \sin\left(\frac{n\pi}{a} x\right)$

$$Y(y) = c_3 e^{\sqrt{\lambda}\cdot y} + c_4 e^{-\sqrt{\lambda}\cdot y}$$

with $\lambda = \left(\frac{\pi n}{a}\right)^2$

$$Y(y) = c_3 e^{\pi n y/a} + c_4 e^{-\pi n y/a}$$

Note: for rectangular regions we want to use sinh and cosh instead of e^{ry}, e^{-ry}.

Recall: $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$

$$\sinh(t) = \frac{e^t - e^{-t}}{2}$$

$$Y = c_3 e^{ry} + c_4 e^{-ry}$$

$$= B_1\left(\frac{e^{ry} + e^{-ry}}{2}\right) + B_2\left(\frac{e^{ry} - e^{-ry}}{2}\right)$$

$$= B_1 \cosh(ry) + B_2 \sinh(ry)$$

where $B_1 + B_2 = C_1$

$B_1 - B_2 = C_2$

So $Y = c_3 \cosh\left(\frac{n\pi y}{a}\right) + c_4 \sinh\left(\frac{n\pi y}{a}\right)$

Now let's use $y(b) = 0$

$$0 = c_3 \cosh\left(\frac{n\pi b}{a}\right) + c_4 \sinh\left(\frac{n\pi b}{a}\right)$$

Solve for c_4.

$$c_4 = -\frac{c_3 \cosh(n\pi b/a)}{\sinh(n\pi b/a)}$$

So $Y = c_3 \cosh\left(\frac{n\pi y}{a}\right) - \dfrac{c_3 \cosh(n\pi b/a)}{\sinh(n\pi b/a)} \cdot \sinh\left(\frac{n\pi y}{a}\right)$

Let's pull $\dfrac{c_3}{\sinh\left(\frac{n\pi b}{a}\right)}$

$$Y = \frac{c_1}{\sinh\left(\frac{n\pi b}{a}\right)}\left[\sinh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right]$$

$$Y = \frac{c_1}{\sinh\left(\frac{n\pi b}{a}\right)}\left[\sinh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right]$$

Hyperbolic Trig Identity

$$\sinh(\alpha - \beta) = \sinh(\alpha)\cosh(\beta) - \cosh(\alpha)\sinh(\beta)$$

Let $\alpha = \frac{n\pi b}{a}$ $\quad \beta = \frac{n\pi y}{a}$ $\quad \Rightarrow \alpha - \beta = \frac{n\pi}{a}(b - y)$

So $Y_n = \boxed{\frac{c_1}{\sin\left(\frac{n\pi b}{a}\right)}} \sinh\left(\frac{n\pi}{a}(b-y)\right)$

Just a constant
so call it c_n

$$Y_n = c_n \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

Since our family of solutions is $u_n = X_n(x)Y_n(y)$ and the Principle of Superposition:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}(b-y)\right)\sin\left(\frac{n\pi x}{a}\right)$$

BUT we aren't done. We need to use that last condition.

$$u(x,0) = f(x)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}\cdot b\right)\sin\left(\frac{n\pi x}{a}\right) = f(x)$$

Not quite a Fourier series but think of $\underbrace{c_n \sinh\left(\frac{n\pi}{a}\cdot b\right)}_{b_n}$ as a constant that it is. Now we have a Fourier sine series

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

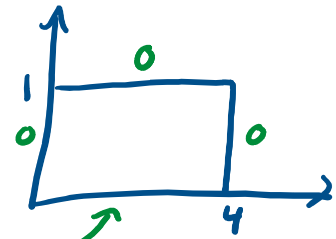
$$\Rightarrow c_n = \frac{2}{a\sinh\left(\frac{\pi n b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

namely when the other sides are nonzero

To do the other cases, namely when the other sides are nonzero you repeat the process.

Ex 2: Let $a=4,\ b=1$

$$u_{xx}+u_{yy}=0$$
$$u(0,y)=u(1,y)=u(4,y)=0$$
$$u(x,0)=\begin{cases} x, & 0\le x\le 2 \\ 4-x, & 2\le x\le 4 \end{cases}$$



So the general solution is

$$u(x,y)=\sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}(b-y)\right)\sin\left(\frac{n\pi x}{a}\right)$$
$$=\sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{4}(1-y)\right)\sin\left(\frac{n\pi x}{4}\right)$$

Now find $c_n=\dfrac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\displaystyle\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

$$=\frac{2}{4\sinh(n\pi/4)}\int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$=\frac{1}{2\sinh(n\pi/4)}\left[\int_0^2 x\sin\left(\frac{n\pi x}{4}\right)dx+\int_2^4 (4-x)\sin\left(\frac{n\pi x}{4}\right)dx\right]$$

$$\begin{array}{cl} u & dv \\ x\ + & \sin(n\pi x/4) \\ 1\ - & -\cos(n\pi x/4)\cdot\frac{4}{n\pi} \\ 0 & -\sin(n\pi x/4)\cdot\frac{16}{n^2\pi^2} \end{array} \qquad \begin{array}{cl} u & dv \\ 4-x\ + & \sin\left(\frac{n\pi x}{4}\right) \\ -1 & -\cos\left(\frac{n\pi x}{4}\right)\cdot\frac{4}{n\pi} \\ 0 & -\sin\left(\frac{n\pi x}{4}\right)\cdot\frac{16}{n^2\pi^2} \end{array}$$

$$=\frac{1}{2\sinh\left(\frac{n\pi}{4}\right)}\Bigg[\left(-x\cos\left(\frac{n\pi x}{4}\right)\cdot\frac{4}{n\pi}+\sin\left(\frac{n\pi x}{4}\right)\cdot\frac{16}{n^2\pi^2}\right)\Bigg]_0^2$$
$$+\left(-(4-x)\cos\left(\frac{n\pi x}{4}\right)\cdot\frac{4}{n\pi}-\sin\left(\frac{n\pi x}{4}\right)\cdot\frac{16}{n^2\pi^2}\right)\Bigg]_2^4\Bigg]$$

$$=\frac{1}{2\sinh\left(\frac{n\pi}{4}\right)}\left[\cancelto{0}{-2\cos\left(\frac{n\pi}{2}\right)\cdot\frac{4}{n\pi}}+\sin\left(\frac{n\pi}{2}\right)\cdot\frac{16}{n^2\pi^2}-\cancelto{0}{\left(0+\sin(0)\cdot\frac{16}{n^2\pi^2}\right)}\right.$$

$$= \frac{1}{2\sinh\left(\frac{n\pi}{4}\right)}\left[\cancel{-2\cos\left(\frac{n\pi}{2}\right)\cdot\frac{4}{n\pi}}^{\,0} + \sin\left(\frac{n\pi}{2}\right)\cdot\frac{16}{n^2\pi^2} - \cancel{\left(0+\sin(0)\cdot\frac{16}{n^2\pi^2}\right)}^{\,0}\right.$$

$$\left. + \cancel{\left(-0-\sin(n\pi)\cdot\frac{16}{n^2\pi^2}\right)}^{\,0} - \left(\cancel{-2\cos\left(\frac{n\pi}{2}\right)}^{\,0} - \sin\left(\frac{n\pi}{2}\right)\cdot\frac{16}{n^2\pi^2}\right)\right]$$

<u>Note:</u> $\cos\left(\frac{n\pi}{2}\right)=0$ b/c $n=1,2,3,\ldots$

$\sin\left(\frac{n\pi}{2}\right)=(-1)^{n+1}$ b/c $n=1,2,3,\ldots$

$$= \frac{1}{2\sinh(n\pi/4)}\left[\frac{16}{n^2\pi^2}(-1)^{n+1} + (-1)^{n+1}\frac{16}{n^2\pi^2}\right]$$

$$= \frac{16\cdot(-1)^{n+1}}{n^2\pi^2\sinh(n\pi/4)}$$