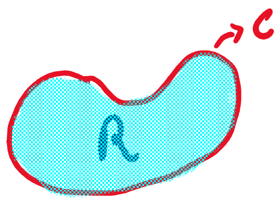
Section 9.7: Steady-State Temperature and Laplace's Equation (Pt 2)

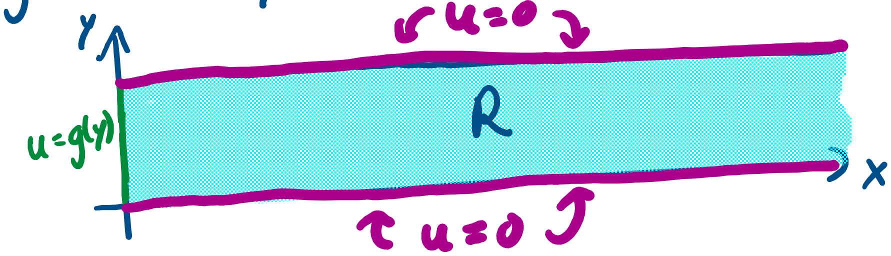
Thursday, October 30, 2025 7:34 PM

Recall from last class, we want to work with

Dirichlet Problem: $\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x,y \text{ in Region } R \\ u(x,y) = f(x,y) & \text{for } x,y \text{ on curve } C. \end{cases}$



In particular, we work with the region being a rectangular region. Today we want to work with semi-infinite strip.



"semi-infinite" because R is indefinite in the right direction but has a hard boundary at x=0.

$$R = \{(x,y) \mid x>0, 0<y<b$$

Solve Laplace Equation on R

$$\begin{cases} u_{xx} + u_{yy} = 0 & x>0,\ 0<y<b \\ u(x,0) = u(x,b) = 0 \\ u(0,y) = g(y) \\ u(x,y) \text{ is bounded as } x \to \infty \end{cases}$$

this is a typical Boundary condition for unbounded region

<u>Note:</u> $u(x,y)$ is bounded as $x \to +\infty$ can also be written as

"There exists some number $0 \le M < \infty$ such that

$$\lim_{x \to +\infty} |u(x,y)| \le M"$$

(so $u(x,y)$ cannot grow to $\pm\infty$)

(so $u(x,y)$ cannot grow to $\pm\infty$)

Solve using separation of variables

$$u(x,y) = X(x)\,Y(y)$$

Plug into PDE

$$u_{xx} + u_{yy} = 0$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$ separation constant

Separate X-terms and Y-terms and we get

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0$$

Let's evaluate the boundary conditions

$u(x,0) = 0 = X(x)\,Y(0) \longrightarrow Y(0) = 0$

$u(x,b) = 0 = X(x)\,Y(b) \longrightarrow Y(b) = 0$

Again $X(x) \neq 0$ b/c if not we have the trivial Solution

$u(0,y) = g(y) = X(0)\,Y(y)$ ·pin this for later

$$\lim_{x\to\infty} |u(x,y)| \leq M \iff \lim_{x\to\infty} |X(x)\,Y(y)| \leq M \longrightarrow \lim_{x\to\infty} |X(x)| \leq M_2$$

So the new ODEs

$$\begin{cases} X'' + \lambda X = 0 \\ \lim_{x\to\infty} |X(x)| \leq M \end{cases} \qquad \begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

Note when we have 2 boundary conditions we want $+\lambda$

So our separation constant should be $+\lambda$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ \lim\limits_{x\to\infty} |X(x)| \le M \end{cases} \qquad \begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

Let's solve the Y- ODE first. (Note we have solve this many times).

$$Y_n(y) = \sin\left(\frac{n\pi y}{b}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{b}\right)^2 \quad \text{for } n = 1, 2, 3, \ldots$$

Solve the X equation.

$$X'' - \lambda X = 0$$

Assume $X_n = e^{rx}$

$$r^2 - \lambda = 0$$
$$r = \pm\sqrt{\lambda} = \pm \frac{n\pi}{b}$$

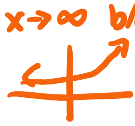
General solution: $X_n(x) = c_1 e^{n\pi x/b} + c_2 e^{-n\pi x/b}$

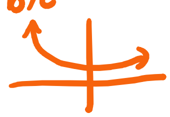
Note: For the semi-infinite strip, leave as exponentials

Now let's look at the boundary condition.

$$\lim_{x\to\infty} |X_n(x)| \le M$$

$$\lim_{x\to\infty} \Big| \underbrace{c_1 e^{n\pi x/b}}_{\to\infty \text{ as } x\to\infty \text{ b/c}} + \underbrace{c_2 e^{-n\pi x/b}}_{\to 0 \text{ as } x\to\infty \text{ b/c}} \Big|$$ and we want the limit to $\le M$.





So let $c_1 = 0$. Then

$$X_n(x) = e^{-n\pi x/b} \quad \text{for } n = 1, 2, 3, \ldots$$

So we have the family of solutions $u_n(x,y) = X_n(x) Y_n(y)$ and by the principle of superposition

$$u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right)$$

$$u(x,y)=\sum_{n=1}^{\infty} b_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right)$$

Now let's look at the condition we pinned: $u(0,y)=g(y)$

$$g(y)=u(0,y)=\underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{b}\right)}_{\text{Fourier sine series!}}$$

So $u(x,y)=\sum_{n=1}^{\infty} b_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right)$

with $b_n=\frac{2}{b}\int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right)dy$

<u>Ex 1:</u> Let $g(y)=36^\circ$ be constant. Let $b=6$

<u>Dirichlet Problem:</u>

$$\begin{cases} u_{xx}+u_{yy}=0 & 0<y<6,\ x>0 \\ u(x,0)=u(x,6)=0 \\ u(0,y)=36 \\ \lim\limits_{x\to\infty} |u(x,y)|\le M \end{cases}$$

So $u(x,y)=\sum_{n=1}^{\infty} b_n e^{-n\pi x/6} \sin\left(\frac{n\pi y}{6}\right)$

where $b_n=\frac{2}{6}\int_0^6 36\sin\left(\frac{n\pi y}{6}\right)dy$

$$=\frac{1}{3}\cdot 36\left(-\cos\left(\frac{n\pi y}{6}\right)\right)\cdot\frac{6}{n\pi}\Big]_0^6$$

$$=\frac{-72}{n\pi}\left[\cos(n\pi)-1\right]$$

$$=\frac{-72}{n\pi}\left[(-1)^n-1\right]$$

$$= \begin{cases} \dfrac{144}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$