Section 10.1 Sturm-Liouville Problems (Pt 1)

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So far, all of our Fourier and separation of variables methods relied on very special cases – constant coefficients, uniform materials, and simple sine/cosine bases. Strum-Liouville problems tells us:

> Whenever we have a $\underbrace{\text{self-adjoint}}_{\langle Ax,y\rangle = \langle x,Ay\rangle}$ differential operator, we'll still get real eigenvalues and orthogonal eigenfunctions.

That means we can build Fourier-like Series even for complicated systems.

In the previous lectures, we solved the PDEs using Separation of Variables. As a result, we often end up with an eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

And we got that

eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots$

eigenfunctions: $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

By the Principle of Superposition, we can represent the general Solution of the ODE as a linear combination

$$X(x) = \sum_{n=1}^{\infty} c_n X_n = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

Def: A Sturm-Liouville Problem is an endpoint value problem of the form:

$$\Bigg(\frac{d}{dx}\left[p(x)\,y'\right] - q(x)y + \lambda r(x) y = 0 \qquad a < x < b$$

both cannot be 0

$$\begin{cases} \dfrac{d}{dx}\left[p(x)\,y'\right] - q(x)y + \lambda r(x) y = 0 & a < x < b \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 & \text{where } \alpha_1, \alpha_2 \text{ constant, but both cannot be 0} \\ \beta_1 y(b) + \beta_2 y'(b) = 0 & \text{where } \beta_1, \beta_2 \text{ constant, but both cannot be 0} \end{cases}$$

where λ is the eigenvalue.

Note: Sturm-Liouville problems always have the trivial solution $y(x) \equiv 0$.

BUT what we want is all the eigenvalues that result in nontrivial solutions $(y_n(x) \not\equiv 0)$

Thm 1: (Sturm-Liouville Eigenvalues)

Suppose $p(x), p'(x), q(x), r(x)$ are all continuous on $[a,b]$ and that

$p(x) > 0$ and $r(x) > 0$ at each point in $[a,b]$

Then the eigenvalues form an increasing sequence of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$$

with $\lim_{n \to \infty} \lambda_n = +\infty$

Furthermore, if $q(x) \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ then all λ's are also nonnegative $(\lambda_n \geq 0)$

Ex 1: Again take our ODE from the last few lectures

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

which has $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ $\quad n = 1, 2, 3, 4, \ldots$

Suppose we didn't know λ_n can we use the Thm to tell us about the eigenvalues?

$\alpha_1 = 1 \quad \alpha_2 = 0$

Suppose we didn't know λ_n can we use the Thm to tell us about the eigenvalues?

we see $p(x)=1$ $\quad \alpha_1=1 \quad \alpha_2=0$

$r(x)=1$ $\quad \beta_1=1 \quad \beta_2=0$

$q(x)=0$

Since $q(x)=0$, then by the Sturm-Liouville Eigenvalue Thm, this ODE has only nonnegative eigenvalues which is true since they are

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

Ex 2: Find the eigenvalues and associated eigenfunctions of the Sturm-Liouville Problem.

$$\begin{cases} y''+\lambda y=0 \\ y'(0)=0 \\ y(1)+y'(1)=0 \end{cases}$$

We know $y''+\lambda y=0 \Rightarrow y=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$

$\Rightarrow y'=-\sqrt{\lambda}A\sin(\sqrt{\lambda}x)+B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

So with $y'(0)=0$ $\Rightarrow$ $y'(0)=0=-\sqrt{\lambda}A\sin(0)+B\sqrt{\lambda}\cos(0)$

So $B\sqrt{\lambda}=0$

So $B=0$ or $\lambda=0$

If $\lambda=0$, then $y''=0$ and we know $y'(0)=0$ then

$y(x)=c$ - a constant

So with $y(1)+y'(1)=0$

$c+0=0$

$c=0$ which is the trivial solution

So let's look at the case when $B=0$

Then $y=A\cos(\sqrt{\lambda}x)$

$y'=-\sqrt{\lambda}A\sin(\sqrt{\lambda}x)$

So $y(1)+y'(1)=0$

$A\cos(\sqrt{\lambda})-\sqrt{\lambda}A\sin(\sqrt{\lambda})=0$

Note $A\neq 0$ b/c then we

$$A\cos(\sqrt{\lambda}) - \sqrt{\lambda}\, A\sin(\sqrt{\lambda}) = 0$$

$$A\underbrace{\left[\cos(\sqrt{\lambda}) - \sqrt{\lambda}\,\sin(\sqrt{\lambda})\right]}_{=0} = 0$$

Note $A \neq 0$ b/c then we have the trivial solution

$$\cos(\sqrt{\lambda}) - \sqrt{\lambda}\,\sin(\sqrt{\lambda}) = 0$$

$$\cos(\sqrt{\lambda}) = \sqrt{\lambda}\,\sin(\sqrt{\lambda})$$

$$\cot(\sqrt{\lambda}) = \frac{\cos(\sqrt{\lambda})}{\sin(\sqrt{\lambda})} = \sqrt{\lambda}$$

This happens to be a transcendental eqn for λ, but there are an infinite number of roots.

$$\lambda \approx 0.74017\ldots$$
$$\lambda \approx 11.73861\ldots$$
$$\vdots$$

with eigenfunction being

$$y_n = \cos(\sqrt{\lambda}\, x)$$

Next class, we will talk about properties of eigenfunctions and eigenfunction expansions