Lesson 35. Potential theory, Electrostatic fields

Potential theory is the theory of **harmonic functions**, that is, solutions to Laplace's equation $\nabla^2\Phi = 0$. In applications, electrostatic and gravitational potential, steady-state heat flow, and velocity potential of incompressible fluid flow, are harmonic.

Analytic functions are useful for two-dimensional (but not for three-dimensional) potential theory, since the real part $\operatorname{Re} f(z)$ and the imaginary part $\operatorname{Im} f(z)$ of an analytic function $f(z)$ are harmonic.

The level curves of a two-variable harmonic function Φ are the **equipotential lines**.

In two dimensions, $\nabla^2\Phi = \dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial^2\Phi}{\partial y^2} =$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\, r\, \frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}.$$

Example 1. Parallel plates at constant potential Φ_0, Φ_1. Assume the plates are $x = 0$ and $x = 1$. Then, Φ does not depend on y or z, so Laplace's equation becomes $\dfrac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = 0$. Then, $\Phi = Ax + B$. Imposing the boundary conditions, $\Phi = (\Phi_1 - \Phi_0)x + \Phi_0$.

Example 2. Infinite coaxial cable with constant potentials Φ_1 and Φ_2 at $r = r_1$ and $r = r_2$. Then, Φ does not depend on θ, so Laplace's equation becomes

$$r \frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + \frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0 \Rightarrow \frac{\Phi''}{\Phi'} = -\frac{1}{r} \Rightarrow \ln(\Phi')' = -\frac{1}{r} \Rightarrow$$

$$\ln(\Phi') = -\ln r + C = \ln \frac{a}{r} \Rightarrow \Phi' = \frac{a}{r} \Rightarrow \Phi = a \ln r + b.$$

Imposing the boundary conditions,

$$\Phi = \frac{\Phi_2 - \Phi_1}{\ln \frac{r_2}{r_1}} \ln \frac{r}{r_1} + \Phi_1.$$

Example 3. Sector $-\frac{\alpha}{2} < \theta < \frac{\alpha}{2}$ with the constant potentials Φ_- and Φ_+ on its sides. As $\theta = \operatorname{Arg} z$ is harmonic, we have $\Phi = a + b\theta$. Imposing the boundary conditions, $\Phi = \frac{1}{2}(\Phi_- + \Phi_+) + \frac{1}{\alpha}(\Phi_+ - \Phi_-)\theta$.

It is convenient to use the analytic **complex potential** $F(z) = \Phi(x, y) + i\Psi(x, y)$ where $z = x + iy$ and Ψ is a harmonic conjugate of Φ. For a complex potential, the level curves $\Phi = \operatorname{Re} F =$ const are equipotential lines, and the curves $\Psi = \operatorname{Im} F =$ const are the **lines of force** (or the **stream lines** in two-dimensional fluid flow).

Since $F(z)$ is **conformal**, the equipotential lines and the lines of force meet at right angle when $F'(z) \neq 0$.
In Example 1, $\Psi = ay$ so $F(z) = az + b$.
In Example 2, $\Psi = a\theta$ (multi-valued) so $F(z) = a \ln z + b$.
In Example 3, $F(z) = a + b\theta - ib \ln r = a - ib \operatorname{Ln} z$.

The sum of harmonic functions is harmonic, so we can use **superposition**.

Example. Opposite charges $\mp 2\pi K$ at $z = \pm c$ (real).
Let $\Phi_1 = K \,\text{Ln}\, |z - c|$ and $\Phi_2 = -K \,\text{Ln}\, |z + c|$.
Then, $F(z) = K\,(\text{Ln}\,(z-c) - \text{Ln}\,(z+c)) = K\,\text{Ln}\,\left(\dfrac{z-c}{z+c}\right)$.
Equipotential lines are given by $\text{Re}\, F(z) = \text{const}$, thus
$\text{Re}\,\text{Ln}\,\left(\dfrac{z-c}{z+c}\right) = \text{Ln}\,\left|\dfrac{z-c}{z+c}\right| = \text{const} \Rightarrow \left|\dfrac{z-c}{z+c}\right| = \text{const}.$

As $w = \dfrac{z-c}{z+c}$ is LFT, equipotential lines $|w| = \text{const}$ are circles (except $|w| = 1$ which is the imaginary axis).

If a function $\Phi(u,v) = \operatorname{Re} F(w)$ is harmonic and $w = f(z)$ is analytic, then $\Phi(f(z)) = \operatorname{Re} F(f(z))$ is harmonic.

Example. $e^u \cos v = \operatorname{Re} e^w$ is harmonic, $w = z^2 = (x^2 - y^2) + i(2xy)$ is analytic, so $e^{x^2-y^2} \cos(2xy)$ is harmonic.

Using this fact, we can transplant boundary value problems from one domain to another: If $w = f(z)$ maps a domain D onto a simpler domain D', we can solve a problem for $\Phi(w)$ in D', then transport it back to solve a problem for $\Phi(f(z))$ in D.

Example. Find the potential $\Phi(x,y)$ in the unit disk, $\Phi(x,y) = \Phi_0$ for $y = \sqrt{1-x^2}$ and $\Phi(x,y) = -\Phi_0$ for $y = -\sqrt{1-x^2}$.

Instead of solving this problem, we solve a problem in the right half plane $\text{Re}\, w > 0$, with $\Psi(w) = \pm\Phi_0$ on the imaginary axis. This is a special case of a sector (with $\alpha = \pi$) from Example 3. Then,

$$\Psi(w) = \frac{2\Phi_0}{\pi}\text{Arg}\, w.$$

Now we transport the solution back to the unit disk by the LFT $w = \frac{1+z}{1-z}$ mapping the unit disk onto the right half plane. Solution to the original problem is

$$\Phi(z) = \Psi(w(z)) = \frac{2\Phi_0}{\pi}\,\text{Arg}\,\frac{1+z}{1-z}.$$