TOWARDS CONNECTIVITY FOR CODIMENSION 2 CYCLES: INFINITESIMAL DEFORMATIONS

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Abstract. Let X be a smooth projective variety over an algebraically closed field $k \subset \mathbb{C}$ of characteristic zero, and $Y \subset X$ a smooth complete intersection. The Weak Lefschetz theorem states that the natural restriction map $\mathrm{H}^i(X(\mathbb{C}),\mathbb{Q}) \to \mathrm{H}^i(Y(\mathbb{C}),\mathbb{Q})$ on singular cohomology is an isomorphism for all $i < \dim(Y)$. The Bloch-Beilinson conjectures on the existence of certain filtrations on Chow groups combined with standard conjectures in the theory of motives imply that a similar result should be true for Chow groups, and, more generally, for motivic cohomology. In this note, we prove a consequence of the *Motivic Weak Lefschetz conjecture* (see Conjecture 1.2) for codimension 2 cycles.

1. Introduction

In this note we study a motivic analog of the Weak Lefschetz Theorem. Throughout, X will be a smooth projective variety over an algebraically closed field $k \subset \mathbb{C}$ of characteristic zero. Let $\mathrm{H}^*(X) := \mathrm{H}^i(X(\mathbb{C}),\mathbb{Q})$ denote the singular cohomology with rational coefficients. Let $Y \subset X$ be a smooth complete intersection of dimension n. Then one has the following result:

Theorem 1.1 (The Weak Lefschetz theorem). *The restriction map* $\mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ *is an isomorphism for* $i < n$ *and injective for* $i = n$.

It is known that étale cohomology, and all other good cohomology theories on smooth projective varieties over k also satisfy the above property. The following conjecture is a motivic analog of the Weak Lefschetz Theorem (see, for instance, [11], 1.5):

Conjecture 1.2. *Let X be a smooth projective variety over k and $Y \subset X$ a smooth complete intersection. The natural restriction map* $\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(Y)_{\mathbb{Q}}$ *is an isomorphism for all* $p < dim(Y)/2$.

The above conjecture can be deduced from more general conjectures of Bloch and Beilinson (see [8]) on the existence of certain filtrations on Chow groups. In fact, one also expects Weak Lefschetz theorems more generally for higher Chow groups and motivic cohomology (cf. §2). The case $p = 1$ of the above conjecture is a classical theorem of Lefschetz and Grothendieck (see [4]). In this case, the conjecture even holds integrally.

Theorem 1.3 (Grothendieck-Lefschetz). *Let X be a smooth projective variety and $Y \subset X$ a smooth complete intersection such that* $\dim(Y) \geq 3$. *Then the natural restriction map*

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

The Grothendieck-Lefschetz theorem is proved by factoring the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ as a composition

$$\mathrm{Pic}(X) \to \mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(Y),$$

where $\mathfrak{X}$ is the formal completion of X along Y, and showing that each of these arrows is an isomorphism. If we adopt a similar strategy to study the higher codimension case, then as a first step, we would like to write the restriction map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y)$ as a composition

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p(\mathfrak{X}) \to \mathrm{CH}^p(Y).$$

In particular, we would like to define the middle term.

Recall that the Bloch-Quillen formula, which relates the Chow groups of smooth projective varieties to K-cohomology groups, gives a natural isomorphism

$$\mathrm{CH}^p(X) \cong \mathrm{H}^p(X, \mathcal{K}_{p,X}).$$

Here $\mathcal{K}_{p,X}$ is the sheaf associated to the presheaf which sends an open subset $U \subset X$ to $\mathrm{K}_p(U)$, where $\mathrm{K}_p(U)$ is the p-th Quillen K-theory group of the exact category of locally free sheaves on U. Using this formula, we may define the Chow group of codimension p cycles $\mathrm{CH}^p(\mathfrak{X})$ to be $\mathrm{H}^p(\mathfrak{X}, \mathcal{K}_{p,\mathfrak{X}})$ where $\mathcal{K}_{p,\mathfrak{X}}$ is defined in an analogous manner. One can then show that the restriction map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y)$ factors as

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p(\mathfrak{X}) \to \mathrm{CH}^p(Y).$$

We refer to §3 for details.

There is however, yet another definition for the Chow groups of the formal scheme $\mathfrak{X}$. For any projective system of sheaves $(\mathcal{F}_n)$ on a scheme V, following Jannsen [7], we can consider the *continuous cohomology groups* denoted by $\mathrm{H}^p_{cont}(V, (\mathcal{F}_n))$. The formalism of continuous cohomology, together with the Bloch-Quillen formula then suggests another definition for the Chow group of codimension p cycles of $\mathfrak{X}$, namely $\mathrm{H}^p_{cont}(Y, (\mathcal{K}_{p,Y_n}))$. Here Y_n is the n-th infinitesimal thickening of Y in X such that $Y_0 = Y$. For the purposes of this article, we will work with this definition and set

$$\mathrm{CH}^p_{cont}(\mathfrak{X}) := \mathrm{H}^p_{cont}(Y, (\mathcal{K}_{p,Y_n})).$$

One can show that the restriction map $\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y)$ factors as a composition (cf. §3)

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p_{cont}(\mathfrak{X}) \to \mathrm{CH}^p(Y). \tag{1}$$

The motivic weak Lefschetz conjecture implies, in particular, that the last morphism in the sequence of maps (1) is a surjection (after tensoring with $\mathbb{Q}$) for all $p < dim(Y)/2$. The main result of this article is the following integral version of this consequence:

Theorem 1.4. *Let X and Y be as before with $\dim(Y) \geq 5$. Then the natural morphism*

$$\mathrm{CH}^2_{cont}(\mathfrak{X}) \to \mathrm{CH}^2(Y)$$

is an isomorphism.

The proof proceeds by first reducing to showing that $\forall\, n > 0$, one has isomorphisms

$$\mathrm{H}^2(Y, \mathcal{K}_{2,Y_n}) \to \mathrm{H}^2(Y, \mathcal{K}_{2,Y}).$$

On the other hand, a theorem of Bloch (Corollary 5.2), allows one to understand the kernel of $\mathcal{K}_{2,Y_n} \to \mathcal{K}_{2,Y}$, denoted by $\mathcal{K}_{2,(Y_n,Y)}$, in terms of the sheaf of relative 1-forms over $\mathbb{Z}$. Therefore, we are reduced to showing certain cohomology vanishing statements for this sheaf. A standard argument allows one to further reduce to showing analogous vanishing statements for the sheaf of relative 1-forms over k. Finally, an application of Kodaira-Nakano vanishing gives the desired vanishing statements.

An approach analogous to the one considered here has also been recently suggested by Bloch, Esnault, and Kerz (see [2]) in the context of the p-adic variational Hodge conjecture. In that setting, one has a smooth projective scheme X over the Witt vectors, and one would like to deform cycles from the special fiber to the ambient scheme X. A similar approach to deforming algebraic cycles was also proposed by Green-Griffiths (see [3]) in the context of the usual variational Hodge conjecture. In both of these situations, one is interested in deforming cycles from the special fiber to the ambient variety. The situation we consider here however is different. In the setting of this paper, the central issue we will be concerned with is to deform cycles from the closed complete intersection subvariety Y to each of its nilpotent thickenings Y_n in the ambient variety X.

We conclude this introduction with a brief description of the following sections. In §2, we recall how the Motivic Weak Lefschetz conjecture follows from standard conjectures in the theory of motives. In §3, we recall some standard facts about continuous cohomology groups and discuss the two definitions of Chow groups of formal schemes mentioned above. In §4, we apply Kodaira-Nakano to show the vanishing of the cohomology of certain sheaves of differentials over k. In §5, we prove the analogous vanishing statements for sheaves of relative differentials over $\mathbb{Z}$, and, use a theorem of Bloch to conclude that the latter vanishing gives the desired result.

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2. Weak Lefschetz Conjecture

In this section, we recall Weak Lefschetz type conjectures on motivic cohomology. Let k be a fixed algebraically closed field of characteristic zero. The motivic philosophy predicts the existence of a Tannakian category $\mathcal{M}(k)$ of pure motives. A construction of $\mathcal{M}(k)$ was proposed by Grothendieck. The resulting category should conjecturally satisfy the following properties ([8], Conjecture 4.8). In the following, we shall always work with the $\mathbb{Q}$-linear category of motives.

M1 $\mathcal{M}(k)$ comes equipped with a contravariant symmetric monoidal functor $H : \mathrm{Var(k)} \to \mathcal{M}(k)$. Here Var(k) denotes the category of smooth projective varieties over k.

M2 There is a natural decomposition $\mathrm{H}(X) = \oplus_{i=0}^{i=2n} h^i(X)$, where $n = \dim X$.
M3 For $k \hookrightarrow \mathbb{C}$, there is a fully faithful realization functor $R_H : \mathcal{M}(k) \to Hdg$ to the category of Hodge structures. Furthermore, $R_H(h^i(X)) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$
M4 There is a Tate object $\mathbb{Q}(1)$ whose realization is the Tate Hodge structure.

There is a deep connection between extension groups in the category $\mathcal{M}(k)$ and Higher Chow groups. In particular, Beilinson predicts the existence of filtrations on motivic cohomology whose graded quotients are determined by certain Ext groups in $\mathcal{M}(k)$. We recall the precise conjectures ([8], Remark 4.5). In the following $\mathrm{H}^i_{\mathcal{M}}(X, j) := \mathrm{K}_{2j-i}(X)^{(j)}_{\mathbb{Q}}$. In terms of Bloch's higher Chow groups one has $\mathrm{H}^i_{\mathcal{M}}(X, j) = \mathrm{CH}^j(X, 2j - i)_{\mathbb{Q}}$. Beilinson conjectures the existence of a decreasing filtration $F^{\cdot}$ on $\mathrm{H}^i_{\mathcal{M}}(X, j)$ such that:

B1 $F^0\mathrm{H}^i_{\mathcal{M}}(X, j) = \mathrm{H}^i_{\mathcal{M}}(X, j)$, $F^1\mathrm{CH}^j(X)_{\mathbb{Q}} = \mathrm{CH}^j(X)_{hom,\mathbb{Q}}$. Here $\mathrm{CH}^j(X)_{hom}$ is the subgroup of cycles homologically equivalent to zero.
B2 $F^r\mathrm{H}^{i_1}_{\mathcal{M}}(X, j_1) \cdot F^s\mathrm{H}^{i_2}_{\mathcal{M}}(X, j_2) \subset F^{r+s}\mathrm{H}^{i_1+i_2}_{\mathcal{M}}(X, j_1 + j_2)$
B3 F respects pull backs and push forwards for morphisms between smooth projective varieties $f : X \to Y$.
B4 There are functorial isomorphisms $Gr^{\nu}_F(\mathrm{H}^i_{\mathcal{M}}(X, j)) = \mathrm{Ext}^{\nu}_{\mathcal{M}(k)}(1, h^{i-\nu}(X)(j))$.
B5 $F^{\nu}\mathrm{H}^i_{\mathcal{M}}(X, j) = 0$ for $\nu >> 0$.

We now state the motivic Weak Lefschetz conjecture and derive it from the above conjectures.

Conjecture 2.1. *Let X be a smooth projective variety over k and $Y \subset X$ a smooth complete intersection. Then the natural restriction map*
$$\mathrm{H}^i_{\mathcal{M}}(X, j) \to \mathrm{H}^i_{\mathcal{M}}(Y, j)$$
is an isomorphism for all $i < \dim(Y)$.

Proof of Conjecture 2.1 assuming above conjectures. Recall that the usual weak Lefschetz theorem gives an isomorphism of Hodge structures
$$\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}(j)) \to \mathrm{H}^i(Y(\mathbb{C}), \mathbb{Q}(j))$$
for all $i < \dim(Y)$. By M3 and M4, this morphism is the realization of
$$h^i(X)(j) \to h^i(Y)(j).$$
On the other hand, the realization functor is fully faithful; in particular, a morphism in $\mathcal{M}(k)$ is an isomorphism if and only if its realization is an isomorphism. It follows that
$$h^i(X)(j) \to h^i(Y)(j)$$
is an isomorphism for all $i < \dim(Y)$. On the other hand, (B4) now implies that the natural restriction map
$$Gr^{\nu}_F(\mathrm{H}^i_{\mathcal{M}}(X, j)) \to Gr^{\nu}_F(\mathrm{H}^i_{\mathcal{M}}(Y, j))$$
is an isomorphism for all $i - \nu < \dim(Y)$. In particular, it is an isomorphism for all ν if $i < \dim(Y)$. □

Remark 2.2. One has $\mathrm{CH}^i(X)_{\mathbb{Q}} = \mathrm{H}^{2i}_{\mathcal{M}}(X, i)$. It follows that the conjecture implies
$$\mathrm{CH}^i(X)_{\mathbb{Q}} \to \mathrm{CH}^i(Y)_{\mathbb{Q}}$$
is an isomorphism for all $i < \dim(Y)/2$.

3. Preliminaries on Chow groups of formal schemes

Let X denote, as before, a smooth projective variety over an algebraically closed field k of characteristic zero. Furthermore, let $Y \subset X$ denote a smooth closed subvariety, and Y_n denote the n-th infinitesimal thickening of Y in X with $Y_0 = Y$. Finally, let $\mathfrak{X}$ denote the formal completion of X along Y. Recall, there are natural morphisms of ringed spaces:

$$Y \to Y_n \to \mathfrak{X} \xrightarrow{\pi} X.$$

In the following, we discuss the two possible definitions for the Chow groups of a formal scheme and state some basic properties. While we shall only work with one of these possible choices, we include a discussion of both here for the sake of completeness.

For any open $\mathfrak{U} \subset \mathfrak{X}$, let $\mathcal{K}_{p,\mathfrak{U}}$ denote the sheaf associated to the presheaf which sends $\mathfrak{U}$ to $\mathrm{K}_p(\mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}})$. Here $\mathrm{K}_p(\mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}})$ denotes the p-th Quillen K-group of the category of locally free $\mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}}$-modules, where $\mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}}$ denotes the usual topological restriction to $\mathfrak{U}$ of the structure sheaf of the formal scheme $\mathfrak{X}$. For any open $U \subset X$, we can consider the formal scheme $\mathfrak{U}$ given by taking the formal completion of U along $Y \cap U$. Then $\mathfrak{U}$ is an open formal subscheme of $\mathfrak{X}$. Furthermore, the functor which sends a locally free sheaf on U to its formal completion along $Y \cap U$, gives a morphism of K-groups:

$$\mathrm{K}_p(U) = \mathrm{K}_p(\mathcal{O}_U) \to \mathrm{K}_p(\mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}}).$$

This follows from the fact that the formal completion of a locally free sheaf is still locally free, and that this is an exact functor. It follows that one has a natural morphism of sheaves

$$\pi^{-1}(\mathcal{K}_{p,X}) \to \mathcal{K}_{p,\mathfrak{X}},$$

which gives rise to a natural morphism

$$\mathrm{H}^p(X, \mathcal{K}_{p,X}) \to \mathrm{H}^p(\mathfrak{X}, \mathcal{K}_{p,\mathfrak{X}}).$$

Similarly, for an open $V \subset Y$, one has restriction maps

$$\mathrm{K}_p(\mathcal{O}_{\mathfrak{X}}|_V) \to \mathrm{K}_p(\mathcal{O}_{Y_n}|_V),$$

giving rise to natural morphisms

$$\mathrm{H}^p(\mathfrak{X}, \mathcal{K}_{p,\mathfrak{X}}) \to \mathrm{H}^p(Y, \mathcal{K}_{p,Y_n}).$$

Furthermore, it follows by construction that the composition

$$\mathrm{H}^p(X, \mathcal{K}_{p,X}) \to \mathrm{H}^p(\mathfrak{X}, \mathcal{K}_{p,\mathfrak{X}}) \to \mathrm{H}^p(Y, \mathcal{K}_{p,Y_n})$$

is simply the natural restriction map

$$\mathrm{H}^p(X, \mathcal{K}_{p,X}) \to \mathrm{H}^p(Y, \mathcal{K}_{p,Y_n})$$

induced by the pullback map on K-groups

$$\mathrm{K}_p(U) \to \mathrm{K}_p(U \cap Y_n).$$

We set $\mathrm{CH}^p(\mathfrak{X}) := \mathrm{H}^p(\mathfrak{X}, \mathcal{K}_{p,\mathfrak{X}})$.

For the other definition of Chow groups of formal schemes using the continuous cohomology groups, we begin by recalling some preliminaries on continuous cohomology. If Z is a topological space, and $(\mathcal{F}_n)$ is a projective system of sheaves on Z, then following Jannsen [7], we can consider the continuous cohomology groups $\mathrm{H}^p_{cont}(Z, (\mathcal{F}_n))$. These satisfy the usual properties for a cohomology theory. In particular they are covariant in the $(\mathcal{F}_n)$ and short exact sequences of projective systems give rise to long exact sequences in continuous cohomology. Furthermore, one has the following standard exact sequence:

$$0 \to R^1\varprojlim \mathrm{H}^p(Z, \mathcal{F}_n) \to \mathrm{H}^{p+1}_{cont}(Z, (\mathcal{F}_n)) \to \varprojlim \mathrm{H}^{p+1}(Z, \mathcal{F}_n) \to 0.$$

Note that $\varprojlim$ is a left exact functor on the category of projective systems of abelian groups, and $R^1\varprojlim$ denotes the corresponding first right derived functor. We refer to [7] for the details. It is a standard fact that this $R^1\varprojlim$ vanishes on Mittag-Leffler systems. In particular, if the transition maps in the projective system $(\mathrm{H}^p(Z, \mathcal{F}_n))$ are surjective, one has an isomorphism:

$$\mathrm{H}^{p+1}_{cont}(Z, (\mathcal{F}_n)) \xrightarrow{\cong} \varprojlim \mathrm{H}^{p+1}(Z, \mathcal{F}_n).$$

It follows that if $(\mathcal{F})$ is the constant pro-system, then one has

$$\mathrm{H}^p_{cont}(Z, (\mathcal{F})) = \mathrm{H}^p(Z, \mathcal{F}).$$

The natural restriction maps

$$\mathcal{K}_{p,Y_n} \to \mathcal{K}_{p,Y_{n-1}}$$

gives rise to a pro-system $(\mathcal{K}_{p,Y_n})$ on Y, and so we can consider the continuous cohomology groups $\mathrm{H}^p_{cont}(Y, (\mathcal{K}_{p,Y_n}))$. The Bloch-Quillen formula then suggests the seond definition for the Chow groups of a formal scheme, and we let $\mathrm{CH}^p_{cont}(\mathfrak{X}) := \mathrm{H}^p_{cont}(Y, (\mathcal{K}_{p,Y_n}))$. Note that one has natural morphisms of pro-systems:

$$(\pi^{-1}(\mathcal{K}_{p,X})) \to (\mathcal{K}_{p,\mathfrak{X}}) \to (\mathcal{K}_{p,Y_n}) \to (\mathcal{K}_{p,Y}).$$

This gives rise to natural morphisms:

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p(\mathfrak{X}) \to \mathrm{CH}^p_{cont}(\mathfrak{X}) \to \mathrm{CH}^p(Y).$$

Furthermore, it is clear from the previous remarks that this composition is the usual restriction map

$$\mathrm{CH}^p(X) \to \mathrm{CH}^p(Y).$$

We conclude this section with the following proposition which shows that in the case of $p = 1$, both definitions agree with the usual Picard group of the formal scheme.

Proposition 3.1. *With X, Y as above, the natural restriction map*

$$\mathrm{CH}^1(\mathfrak{X}) \to \mathrm{CH}^1_{cont}(\mathfrak{X})$$

is an isomorphism. Furthermore, both are isomorphic to $\mathrm{Pic}(\mathfrak{X})$.

Proof. First, note that $\mathcal{K}_{1,Y_n} = \mathcal{O}^{\times}_{Y_n}$. It follows that $(\mathrm{H}^0(Y, \mathcal{O}^{\times}_{Y_n}))$ is Mittag-Leffler and, therefore, one has natural isomorphisms:

$$\mathrm{H}^1_{cont}(Y, (\mathcal{K}_{1,Y_n})) \to \varprojlim \mathrm{H}^1(Y, \mathcal{K}_{1,Y_n}) \cong \varprojlim \mathrm{Pic}(\mathrm{Y_n}).$$

Since $\mathrm{Pic}(\mathfrak{X}) \to \varprojlim \mathrm{Pic}(\mathrm{Y_n})$ is an isomorphism (see [5], page 200), we are done.

□

4. Cohomology of sheaves of differentials

Let X be a scheme over a field k (algebraically closed of characteristic zero) and $i : Y \hookrightarrow X$ a closed subscheme defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let Y_n denote the *n-th* infinitesimal neighborhood of Y in X with $Y_0 = Y$. We let $\Omega^1_{X/k}$ denote the usual sheaf of 1-forms on X relative to k. The goal of this section is to prove the following theorem:

Theorem 4.1. *Let X be a smooth projective variety over k, and Y a smooth complete intersection of multidegree $(d_1, \dots, d_r)$ such that $\dim(Y) \geq 3$. Then*

$$\mathrm{H}^i(Y_{n+1}, \Omega^1_{Y_{n+1}/k}) \to \mathrm{H}^i(Y_n, \Omega^1_{Y_n/k})$$

is an isomorphism for $0 \leq i < \dim(Y) - 2$, and an injection for $i = \dim Y - 2$.

We begin with some preliminary lemmas.

Lemma 4.2. *Let X be a smooth projective variety and $Y \subset X$ a smooth complete intersection of multidegree $(d_1, \dots, d_r)$. Then one has natural short exact sequences:*

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \to \Omega^1_{Y_{n-1}/k} \to 0, \text{ and}$$

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_{X/\mathbb{Z}} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \to \Omega^1_{Y_{n-1}/\mathbb{Z}} \to 0.$$

Proof. For any scheme V, let Ω^1_V denote $\Omega^1_{V/k}$ or $\Omega^1_{V/\mathbb{Z}}$. One has an exact sequence:

$$\mathcal{I}^n/\mathcal{I}^{2n} \xrightarrow{\delta} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \to \Omega^1_{Y_{n-1}} \to 0,$$

and by the local description of δ ([5], Proposition 8.4A, page 173), we have $\delta(\mathcal{I}^{n+1}/\mathcal{I}^{2n}) = 0$, and hence the left arrow factors as

$$\mathcal{I}^n/\mathcal{I}^{2n} \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}}.$$

By Lemma 1.7 in [10], the composite

$$\mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \to \Omega^1_{Y_n} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}}$$

is injective. Hence the first map is injective. □

Lemma 4.3. *Let X be a smooth projective variety and $Y \subset X$ a smooth complete intersection of multidegree $(d_1, \dots, d_r)$. Then one has*

$$\mathrm{H}^i(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0 \textit{ for all } 0 \leq i < \dim(Y).$$

Proof. Under our assumptions, Y is given to be the scheme-theoretic intersection of r hypersurfaces of degrees $d_1, \cdots, d_r$ in some projective space $\mathbb{P}^N$ with X. It follows that

$$\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_Y(-d_1) \oplus \cdots \oplus \mathcal{O}_Y(-d_r).$$

Since $\mathcal{I}^n/\mathcal{I}^{n+1} = Sym^n(\mathcal{I}/\mathcal{I}^2)$, $\mathrm{H}^i(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for $0 \leq i < \dim(Y)$ is now a direct consequence of Kodaira vanishing. □

We state the following corollary of the previous lemma for future reference. Although it will not be used in the rest of this section, it will be needed in § 4.

Corollary 4.4. *Let X be a smooth projective variety and $Y \subset X$ a smooth complete intersection of multidegree $(d_1, \dots, d_r)$. Then one has*

$$\mathrm{H}^i(Y, \mathcal{I}^j/\mathcal{I}^{n+1}) = 0 \textit{ for all } 0 \leq i < \dim(Y) \textit{ and } 1 \leq j \leq n.$$

Proof. First note that we have a descending filtration:

$$\mathcal{I}/\mathcal{I}^{n+1} \supset \mathcal{I}^2/\mathcal{I}^{n+1} \supset \cdots \supset \mathcal{I}^{n-1}/\mathcal{I}^{n+1} \supset \mathcal{I}^n/\mathcal{I}^{n+1} \supset 0.$$

The result now follows via descending induction on j, by an application of Lemma 4.3 to the graded pieces of this filtration. For the base case $j = n$, the result follows from Lemma 4.3 above. So assume that the result is true for some $j = j_0$ where $1 < j_0 \leq n$. Consider the short exact sequence

$$0 \to \mathcal{I}^{j_0}/\mathcal{I}^{n+1} \to \mathcal{I}^{j_0-1}/\mathcal{I}^{n+1} \to \mathcal{I}^{j_0-1}/\mathcal{I}^{j_0} \to 0.$$

Taking cohomology, we get a long exact sequence

$$\cdots \to \mathrm{H}^i(Y, \mathcal{I}^{j_0}/\mathcal{I}^{n+1}) \to \mathrm{H}^i(Y, \mathcal{I}^{j_0-1}/\mathcal{I}^{n+1}) \to \mathrm{H}^i(Y, \mathcal{I}^{j_0-1}/\mathcal{I}^{j_0}) \to \cdots.$$

Now the extreme terms vanish by induction and Lemma 4.3 respectively, and thus we are done. □

Proof of Theorem 4.1. Consider the following commutative diagram:

(2)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[d] & & \\
0 \arrow[r] & \mathcal{I}^{n+1}/\mathcal{I}^{n+2} \arrow[r] \arrow[d] & \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n} \arrow[r] \arrow[d] & \Omega^1_{Y_n/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r] & \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & \Omega^1_{Y_{n-1}/k} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

The bottom two rows are exact by the Lemma 4.2. The middle column is given by tensoring the standard exact sequence

(3) $$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{Y_n} \to \mathcal{O}_{Y_{n-1}} \to 0$$

with the locally free sheaf $\Omega^1_{X/k}$ and hence is exact. We consider this as a diagram of sheaves on Y.

We first claim that the leftmost vertical arrow is zero. This is because the composite

$$\mathcal{I}^{n+1}/\mathcal{I}^{n+2} \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}}$$

is zero by [5], Proposition 8.4A, page 173. Since the second map is an injection, this implies that the first map is zero.

Next, we note that $\mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for all $0 \leq i < \dim(Y) - 1$. To see this, consider the short exact sequence

$$0 \to \mathcal{I}^{n+1}/\mathcal{I}^{n+2} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1} \to 0, \tag{4}$$

obtained by tensoring with $\mathcal{I}^n/\mathcal{I}^{n+1}$, the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^1_{Y/k} \to 0.$$

Taking cohomology, we get an exact sequence

$$\cdots \to \mathrm{H}^i(Y, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}) \to \mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1}) \to \mathrm{H}^i(Y, \Omega^1_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1}) \to \cdots .$$

By Kodaira vanishing, we see that $\mathrm{H}^i(Y, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}) = 0$ for $0 \leq i < \dim Y$, and by Kodaira-Akizuki-Nakano vanishing, we have $\mathrm{H}^i(Y, \Omega^1_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for $0 \leq i < \dim Y - 1$. It follows that $\mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for all $0 \leq i < \dim(Y) - 1$.

Therefore, taking the long exact sequence in cohomology associated to the middle vertical column in the diagram above gives isomorphisms

$$\mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n}) \to \mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}})$$

for all $0 \leq i < \dim(Y) - 2$ and an injection when $i = \dim(Y) - 2$. Taking cohomology of the horizontal exact sequences in diagram (2) above, gives the following diagram of long exact sequences:

$$\begin{tikzcd}
\mathrm{H}^i(Y, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}) \arrow[r] \arrow[d] & \mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n}) \arrow[r] \arrow[d] & \mathrm{H}^i(Y, \Omega^1_{Y_n/k}) \arrow[r] \arrow[d] & \mathrm{H}^{i+1}(Y, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}) \arrow[d] \\
\mathrm{H}^i(Y, \mathcal{I}^{n}/\mathcal{I}^{n+1}) \arrow[r] & \mathrm{H}^i(Y, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}}) \arrow[r] & \mathrm{H}^i(Y, \Omega^1_{Y_{n-1}/k}) \arrow[r] & \mathrm{H}^{i+1}(Y, \mathcal{I}^{n}/\mathcal{I}^{n+1})
\end{tikzcd}$$

By the vanishings obtained above, it follows that the restriction maps

$$\mathrm{H}^i(Y, \Omega^1_{Y_n/k}) \to \mathrm{H}^i(Y, \Omega^1_{Y_{n-1}/k})$$

are isomorphisms for all $0 \leq i < \dim(Y) - 2$, and an injection for $i = \dim Y - 2$.

□

Let $\Omega^1_{(Y_n, Y_{n-1})/k}$ denote the kernel of the natural surjection $\Omega^1_{Y_n/k} \to \Omega^1_{Y_{n-1}/k}$. Then we have the following corollary.

Corollary 4.5. *Let X and Y be as above. Then*

$$\mathrm{H}^i(Y, \Omega^1_{(Y_n, Y_{n-1})/k}) = 0$$

for $0 \leq i < \dim(Y) - 1$.

Proof. This follows directly from the theorem by looking at the long exact sequence in cohomology associated to following short exact sequence:

$$0 \to \Omega^1_{(Y_n, Y_{n-1})/k} \to \Omega^1_{Y_n/k} \to \Omega^1_{Y_{n-1}/k} \to 0.$$

□

5. Stability for K-Cohomology

In this section, we prove a stability theorem for K-cohomology under infinitesimal thickenings of $Y \subset X$, and use this to prove the main theorem. In the following, X and Y will be satisfy the same assumptions as in §3. As before, Y_n will denote the n-th infinitesimal neighborhood of Y in X.

We begin by recalling a theorem of Bloch allowing us to compute the fiber of K-theory along a nilpotent thickening. As before, we denote by $\mathrm{K}_i(X)$ the i-th Quillen K-theory group of the exact category of locally free sheaves on X. Recall, $\mathcal{K}_{i,X}$ denotes the sheaf on X associated to the presheaf given by sending $U \subset X$ to $\mathrm{K}_i(U)$.

Theorem 5.1 ([1], Theorem (0.1)). *Let B be a local $\mathbb{Q}$-algebra, A an augmented, commutative B-algebra and assume that J is the kernel of the augmentation homomorphism satisfying $J^N = 0$, $N >> 0$. If $\Omega^1_{A/\mathbb{Z}}$ denotes the module of absolute Kähler differentials, then there is a canonical isomorphism*

$$\tau : \frac{\Omega^1_{(A,J)/\mathbb{Z}}}{d(J)} \cong \mathrm{K}_2(A, J),$$

where $\Omega^1_{(A,J)/\mathbb{Z}} := \ker(\Omega^1_{A/\mathbb{Z}} \to \Omega^1_{B/\mathbb{Z}})$, and $\mathrm{K}_2(A, J) := \ker(\mathrm{K}_2(A) \to \mathrm{K}_2(B))$.

One has the following sheafified version of Bloch's theorem.

Corollary 5.2 ([9]). *Let V be a $\mathbb{Q}$-scheme, Z an infinitesimal extension of V. Suppose that $\mathcal{O}_Z \to \mathcal{O}_V$ is locally split. Define*

$$\Omega^1_{(Z,V)/\mathbb{Z}} = \ker(\Omega^1_{Z/\mathbb{Z}} \to \Omega^1_{V/\mathbb{Z}}).$$

Let $\mathcal{I}$ be the ideal sheaf of V in Z. Then there is a natural isomorphism of sheaves

$$\frac{\Omega^1_{(Z,V)/\mathbb{Z}}}{d(\mathcal{I})} \cong \mathcal{K}_{2,(Z,V)}.$$

We begin with some preliminary lemmas. Unless otherwise mentioned, for any scheme V, we will let Ω^1_V denote either $\Omega^1_{V/\mathbb{Z}}$ or $\Omega^1_{V/k}$.

Lemma 5.3. *With notation as above, we have*

$$\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \xrightarrow{\cong} \Omega^1_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}}. \tag{5}$$

Proof. By Lemma 4.2, we have an exact sequence

$$0 \to \mathcal{I}^{n+1}/\mathcal{I}^{n+2} \xrightarrow{\delta} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n} \to \Omega^1_{Y_n} \to 0.$$

Once again by the local description of δ ([5], Proposition 8.4A, page 173), we see that the left arrow vanishes on restricting to Y_{n-1}, and hence we have an isomorphism of the last two terms. $\square$

Lemma 5.4. *There is an exact sequence of sheaves on Y:*

$$0 \to \Omega^1_Y \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1} \to \Omega^1_{Y_n} \to \Omega^1_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}} \to 0 \tag{6}$$

Proof. We claim that we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{I}^{n+1}/\mathcal{I}^{n+2} \arrow[r, equal] \arrow[d] & \mathcal{I}^{n+1}/\mathcal{I}^{n+2} \arrow[d] & & \\
0 \arrow[r] & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r] & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n} \arrow[r] \arrow[d] & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d, "\cong"] & 0 \\
 & & \Omega^1_{Y_n} \arrow[r] \arrow[d] & \Omega^1_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

The middle row is the sequence obtained by tensoring the exact sequence (3), over $\mathcal{O}_X$, with the locally free sheaf Ω^1_X, and hence is exact. The middle column is exact by Lemma 4.2, and the isomorphism in the right column is by Lemma 5.3. The square in the bottom right corner is commutative as all the maps are restriction maps. The isomorphism in the diagram implies that the injection in the middle column factors as

$$\mathcal{I}^{n+1}/\mathcal{I}^{n+2} \hookrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1} \hookrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n}.$$

This proves our claim.

The statement in the lemma now follows by noting that in the diagram above, the kernel of the map in the bottom row is isomorphic, by the snake lemma, to the cokernel of the left vertical map which in turn by the short exact sequence (4) is isomorphic to $\Omega^1_Y \otimes_{\mathcal{O}_X} \mathcal{I}^n/\mathcal{I}^{n+1}$. This yields the desired exact sequence. □

Lemma 5.5. *With notation as above, we have a short exact sequence*

$$0 \to \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_n} \to \Omega^1_{Y_n/\mathbb{Z}} \to \Omega^1_{Y_n/k} \to 0. \tag{7}$$

Proof. We only need to show that the first map is injective. When $n = 0$, this is true since $Y = Y_0$ is assumed to be smooth. For general n, we will prove this by induction. Consider the commutative square

$$\begin{tikzcd}
 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d, equal] & \Omega^1_{Y_n/\mathbb{Z}} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & \Omega^1_{Y_n/k} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_{n-1}} \arrow[r] & \Omega^1_{Y_{n-1}/\mathbb{Z}} \arrow[r] & \Omega^1_{Y_{n-1}/k} \arrow[r] & 0
\end{tikzcd} \tag{8}$$

The bottom row is exact by the inductive hypothesis. The vertical arrows are just the restriction maps. It follows now that the top row is left exact as well. This in fact, proves more: namely that the sequence

$$0 \to \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_m} \to \Omega^1_{Y_n/\mathbb{Z}} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_m} \to \Omega^1_{Y_n/k} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_m} \to 0$$

is also exact for any $m < n$. To complete the proof, now consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r] \arrow[d] & \Omega^1_{Y/\mathbb{Z}} \otimes_{\mathcal{O}_{Y_n}} \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r] \arrow[d] & \Omega^1_{Y/k} \otimes_{\mathcal{O}_{Y_n}} \mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r] \arrow[d] & 0 \\
 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_n} \arrow[r] \arrow[d] & \Omega^1_{Y_n/\mathbb{Z}} \arrow[r] \arrow[d] & \Omega^1_{Y_n/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & \Omega^1_{Y_n/\mathbb{Z}} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & \Omega^1_{Y_n/k} \otimes_{\mathcal{O}_{Y_n}} \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exactness of the top row follows from the fact that Y is smooth, and that of the bottom row was proved in the preceding paragraph. By Lemma 5.4, the middle and right columns are exact. The left column is exact as it is obtained by tensoring an exact sequence with a vector space. The exactness of the middle row now follows. □

Lemma 5.6. *One has a short exact sequence:*

$$0 \to \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}^{n+1} \to \Omega^1_{(Y_n,Y)/\mathbb{Z}} \to \Omega^1_{(Y_n,Y)/k} \to 0.$$

Proof. One has a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}^{n+1} \arrow[r] \arrow[d] & \Omega^1_{(Y_n,Y)/\mathbb{Z}} \arrow[r] \arrow[d] & \Omega^1_{(Y_n,Y)/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y_n} \arrow[r] \arrow[d] & \Omega^1_{Y_n/\mathbb{Z}} \arrow[r] \arrow[d] & \Omega^1_{Y_n/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{O}_{Y} \arrow[r] \arrow[d] & \Omega^1_{Y/\mathbb{Z}} \arrow[r] \arrow[d] & \Omega^1_{Y/k} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{9}$$

The exactness of the middle and rightmost columns follow by definition. The left column is given by tensoring

$$0 \to \mathcal{I}/\mathcal{I}^{n+1} \to \mathcal{O}_{Y_n} \to \mathcal{O}_Y \to 0$$

with the vector space $\Omega^1_{k/\mathbb{Z}}$ and is therefore exact. The exactness of the middle row follows from Lemma 5.5 and that of bottom row from the fact that Y is smooth. It follows that the top row is exact. $\square$

Lemma 5.7. *Let X be a smooth projective variety and $Y \subset X$ a smooth complete intersection as before. Then*

$$\mathrm{H}^i(Y, \Omega^1_{(Y_n,Y)/\mathbb{Z}}) = 0 \textit{ for all } 0 \leq i < \dim(Y) - 1.$$

Proof. Taking the cohomology long exact sequence associated to the short exact sequence in the top row of the commutative diagram (9) in the previous lemma gives:

$$\cdots \to \mathrm{H}^i(Y, \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}^{n+1}) \to \mathrm{H}^i(Y, \Omega^1_{(Y_n,Y)/\mathbb{Z}}) \to \mathrm{H}^i(Y, \Omega^1_{(Y_n,Y)/k}) \to \cdots.$$

If V is a finite dimensional k-vector space, then by Lemma 4.4,

$$\mathrm{H}^i(Y, V \otimes_k \mathcal{I}/\mathcal{I}^{n+1}) = \mathrm{H}^i(Y, \mathcal{I}/\mathcal{I}^{n+1}) \otimes V = 0, \text{ for } 0 \leq i < \dim(Y).$$

On the other hand, any k-vector space is a direct limit of finite dimensional subspaces. Since tensor product and cohomology commute with direct limits, it follows that same result holds for $\mathrm{H}^i(Y, \Omega^1_{k/\mathbb{Z}} \otimes_k \mathcal{I}/\mathcal{I}^{n+1})$.

We next claim that the cohomology group $\mathrm{H}^i(Y, \Omega^1_{(Y_n,Y)/k})$ vanishes in the desired range. To see this, consider the commutative diagram

(10)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \Omega^1_{(Y_{n-1},Y)/k} \arrow[d] & \\
0 \arrow[r] & \Omega^1_{(Y_n,Y_{n-1})/k} \arrow[r] \arrow[d] & \Omega^1_{Y_n/k} \arrow[r] \arrow[d, equal] & \Omega^1_{Y_{n-1}/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{(Y_n,Y)/k} \arrow[r] & \Omega^1_{Y_n/k} \arrow[r] & \Omega^1_{Y/k} \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

By applying the snake lemma to this diagram, we get a short exact sequence

$$0 \to \Omega^1_{(Y_n,Y_{n-1})/k} \to \Omega^1_{(Y_n,Y)/k} \to \Omega^1_{(Y_{n-1},Y)/k} \to 0.$$

Taking cohomology, we get an exact sequence

$$\cdots \to \mathrm{H}^i(Y, \Omega^1_{(Y_n,Y_{n-1})/k}) \to \mathrm{H}^i(Y, \Omega^1_{(Y_n,Y)/k}) \to \mathrm{H}^i(Y, \Omega^1_{(Y_{n-1},Y)/k}) \to \cdots.$$

We use induction on n to prove the claim above. When $n = 1$, this follows by Corollary 4.5. So suppose that $\mathrm{H}^i(Y, \Omega^1_{(Y_{n-1},Y)/k}) = 0$ for $0 \leq i < \dim Y - 1$. Then the leftmost term in the above long exact sequence of cohomologies vanishes, once again by Corollary 4.5. The result now follows. $\square$

Theorem 5.8. *The cohomology groups $\mathrm{H}^i(Y, \mathcal{K}_{2,(Y_n,Y)})$ vanish for $0 \leq i < \dim(Y) - 1$.*

Proof. There is a short exact sequence:

$$0 \to \mathcal{I}/\mathcal{I}^{n+1} \xrightarrow{d} \Omega^1_{(Y_n,Y)/\mathbb{Z}} \to \mathcal{K}_{2,(Y_n,Y)} \to 0. \tag{11}$$

The sequence is exact everywhere, except possibly on the left, by Corollary 5.2. Left exactness follows from the fact that the composite

$$\mathcal{I}/\mathcal{I}^{n+1} \xrightarrow{d} \Omega^1_{(Y_n,Y)/\mathbb{Z}} \to \Omega^1_{(Y_n,Y)/k} \hookrightarrow \Omega^1_{Y_n/k},$$

is the composite

$$\mathcal{I}/\mathcal{I}^{n+1} \to \mathcal{O}_{Y_n} \xrightarrow{d} \Omega^1_{Y_n/k}.$$

As explained in the proof of Lemma 4.2, the first map is injective, and the kernel of the second map contains scalars which miss the image of the first map. Hence the composite is injective.

The exact sequence (11) above gives a long exact sequence in cohomology

$$\cdots \to \mathrm{H}^i(Y, \Omega^1_{(Y_n,Y)/\mathbb{Z}}) \to \mathrm{H}^i(Y, \mathcal{K}_{2,(Y_n,Y)}) \to \mathrm{H}^{i+1}(X, \mathcal{I}/\mathcal{I}^{n+1}) \to \cdots.$$

The result now follows by an application of Corollary 4.4 and the previous lemma. □

Lemma 5.9. *Let X be a smooth projective variety, $Y \subset X$ as above, and Y_n the n-th infinitesimal thickening as before. Then the natural morphism of sheaves $\mathcal{K}_{i,Y_n} \to \mathcal{K}_{i,Y}$ is surjective.*

Proof. It is enough to show that the morphism on stalks

$$\mathcal{K}_{i,Y_n,y} \to \mathcal{K}_{i,Y,y}$$

is surjective. In particular, it is enough to show that the morphism

$$\mathrm{K}_i(\mathcal{O}_{Y_n,y}) \to \mathrm{K}_i(\mathcal{O}_{Y,y})$$

is surjective. Since the map on algebras splits, so does the resulting map on K-theory. The splitting of the map of algebras is a consequence of *formal smoothness*. For example, after restricting to some small enough open neighborhood U of $y \in Y$, and some $U_n \subset Y_n$ extending U, we have that the natural morphism

$$\mathrm{Hom}(U_n, U) \to \mathrm{Hom}(U, U)$$

is surjective. In particular, the embedding $Y \to Y_n$ has a section in some neighborhood of y. □

Theorem 5.10. *Let $Y \subset X$ be a smooth complete intersection such that* $\dim(Y) \geq 3$. *Then one has:*

1. $\mathrm{H}^i(Y, \mathcal{K}_{2,Y_n}) \cong \mathrm{H}^i(Y, \mathcal{K}_{2,Y_{n-1}})$ *for all* $0 \leq i < \dim(Y) - 2$.
2. $\mathrm{H}^i(Y, \mathcal{K}_{2,Y_n}) \to \mathrm{H}^i(Y, \mathcal{K}_{2,Y_{n-1}})$ *is injective for* $i = \dim(Y) - 2$.

Proof. We prove the first statement, the proof of the second is similar. First note that it is enough to show the result for the restriction maps

$$\mathrm{H}^i(Y, \mathcal{K}_{2,Y_n}) \cong \mathrm{H}^i(Y, \mathcal{K}_{2,Y}).$$

It follows from the previous lemma that we have an exact sequence:

$$0 \to \mathcal{K}_{2,(Y_n,Y)} \to \mathcal{K}_{2,Y_n} \to \mathcal{K}_{2,Y} \to 0.$$

Applying Theorem 5.8 to the associated cohomology long exact sequence gives the result. □

The following is a direct consequence of the above theorem.

Corollary 5.11. *Let X and Y be as in the theorem and suppose that* $\dim(Y) \geq 5$*. Then one has a natural isomorphism*

$$\varprojlim \mathrm{H}^2(Y, \mathcal{K}_{2,Y_n}) \to \mathrm{H}^2(Y, \mathcal{K}_{2,Y}).$$

We now prove the main result of this note.

Proof of Theorem 1.4. We need to show that the restriction map

$$\mathrm{H}^2_{cont}(Y, (\mathcal{K}_{2,Y_n})) \to \mathrm{H}^2(Y, \mathcal{K}_{2,Y})$$

is an isomorphism. Recall, we have an exact sequence [7]:

$$0 \to R^1\varprojlim \mathrm{H}^1(Y, \mathcal{K}_{2,Y_n}) \to \mathrm{H}^2_{cont}(Y, (\mathcal{K}_{2,Y_n})) \to \varprojlim \mathrm{H}^2(Y, \mathcal{K}_{2,Y_n}) \to 0.$$

By Theorem 5.10, $(\mathrm{H}^1(Y, \mathcal{K}_{2,Y_n}))$ is Mittag-Leffler. In particular, the left term in the exact sequence vanishes. Finally, by Corollary 5.11, the right term is $\mathrm{H}^2(Y, \mathcal{K}_{2,Y})$. □

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