ENRICHED HODGE STRUCTURES AND CYCLES ON COMPLEX ANALYTIC THICKENINGS.

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Abstract. In this article, we construct a version of the Bloch-Srinivas ([3]) category of enriched Hodge structures suitable for studying cycles on possibly singular quasi-projective varieties. We show that the cohomology of punctured links gives rise to a natural object of this category. These constructions are motivated by the study of cycles on analytic links.

Contents

1. Introduction

This paper is the first part of a project to associate Hodge theoretic invariants to algebraic K-groups in some new situations, arising from analytic geometry; a key example is the ring of convergent power series. In fact, the invariants we want to construct will involve (a version of) the Enriched Hodge Structures (abbreviated as "EHS") of [3], which we see as a sort of enhancement of Deligne's Mixed Hodge Structures (MHS), that also capture some phenomena which are not detected by MHS.

We give in this first part a self-contained construction of an enriched Hodge structure (see 2.1), and of the underlying mixed Hodge structure, associated to certain "triples" consisting of an analytic space and certain subspaces, and to a particular cohomological degree m, such that the MHS underlying the EHS will be on the m-th singular cohomology with

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integral coefficients associated to the triple.

To be more precise, let X denote a complex analytic variety, and $A, B \subset X$ be two complex analytic subvarieties such that A is a complete algebraic variety, $X \backslash B$ is smooth, and $A \backslash B$ is smooth. Let $i : A \hookrightarrow X$ and $j : X \backslash B \hookrightarrow X$ denote the natural inclusions, and consider the following singular cohomology group:

$$\mathbb{H}^m(A, i^{-1} Rj_* \mathbb{Z}).$$

The triples (X, A, B) form a category $\mathcal{T}'$ (see Definition 4.1), and the main result of this article is that the aforementioned cohomology groups have natural (in triples) enriched mixed Hodge structures, and therefore, in particular, mixed Hodge structures.

We do this by exhibiting certain (more or less) explicit Cohomological Enriched Hodge complexes (which are natural generalizations of Cohomological Mixed Hodge complexes) in the examples of relevance to us. Though similar constructions appear in the literature, they use somewhat different (possibly less explicit) methods, and make additional hypotheses which are not valid in our situation. For example, if X, B are algebraic, the theory of mixed Hodge modules due to M. Saito ([13]) gives a MHS on the aforementioned cohomology group. However, the theory of mixed Hodge modules does not give an underlying cohomological Mixed Hodge complex (giving rise to the relevant MHS), and the existence of such a complex is crucial for our construction of the enhancements to EHS.

On the other hand, for certain triples (X, A, B) the relevant mixed Hodge structures and the underlying mixed Hodge complexes were constructed previously by Durfee-Hain ([6]). More precisely, Durfee-Hain consider triples of the form $(X, A, B \cup A)$ where X is projective algebraic, A, B are closed subvarieties, and $X \backslash (A \cup B)$ is smooth. They then consider the cohomology of the 'link' $U \backslash (A \cup B)$, where U is a small enough neighborhood of A in X, and show that these cohomology groups carry a natural (real) mixed Hodge structure compatible with the cup-product pairing. Note that in *loc. cit.*, this link is denoted by $L(X, A \cup B, B)$. As the authors explain in *loc. cit.*, if $p \in V$ is an isolated singularity, then $L(X, V, p)$ is the link complement $L(X, p) \backslash L(V, p)$, where $L(X, p)$ (resp. $L(V, p)$) is the usual link of an isolated singularity (which is a real manifold of dimension $2 \dim(X) - 1$ (resp. $2 \dim(V) - 1$)).

The present work will provide the technical foundations for a subsequent work, with a more cycle theoretic flavor, where the constructions will be applied to study of the K-theory of convergent power series rings, and more generally cycles on punctured links. In particular, the development of K-theory in the context of triples and the construction and study of the related cycles class maps (including a projective bundle formula and Gysin maps in the setting of triples) will be left to a future paper. We note here that the K-theory of convergent power series appears naturally as the K-theory of a limit of certain triples, and the corresponding cohomology group (i.e. the target of the corresponding cycle class maps) appears in Theorem 5.3. Given the technical nature of the EHS construction, including the verification that it is well defined independent of choices, and appropriately functorial, we felt it makes sense to place this material in a single work, concentrating only on those aspects.

However, to give the reader some context for the constructions in this paper, we discuss in Section 6 one of the overall goals of the project. We discuss which K-groups we want to study, and sketch the construction of a version of Deligne-Beilinson cohomology which naturally arises from our EHS constructions (making use also of the Cohomological Enriched Hodge complexes underlying them, just as in the original constructions of Deligne, generalized by Beilinson). The (expected) properties of this theory of Enriched Deligne-Beilinson Cohomology groups will be developed carefully in subsequent work, but in particular we expect to construct a theory of Chern classes and Chern characters with values in these, and also to establish a description of these Enriched Deligne-Beilinson cohomologies as extensions of Hom and Ext^1 groups using our EHS groups constructed here. This is of course parallel to what we know for Deligne-Beilinson cohomology.

We also work out a few explicit examples "by hand", which give the flavor of the theory, and an indication as to what aspects of the K-theory may be reflected in our invariants. The computations even in simple cases exhibit a rich interplay with geometric features appearing in the blow ups (versions of log resolutions, for triples) which are needed to define our EHS structures. We hope that these computations, and a description of our (plausible) expectations, provide sufficient motivation for undertaking the specific constructions appearing in this paper.

We stress that even the construction of the underlying MHS, and in particular the cohomological Mixed Hodge complexes, are new in some situations, since we consider analytic spaces, though they are somewhat motivated by other such constructions in the literature in the algebraic case. We also note that our constructions are relatively elementary, made with explicit complexes of a geometric origin, and without recourse to D-modules etc. These are thus of independent interest.

Notation: In the following, MHS will denote the usual category of mixed Hodge structures.

2. Enriched Hodge structures

In this section, we define and study a close variant of the category of enriched Hodge structures studied by Bloch-Srinivas ([3]) and prove a basic proposition computing some Ext groups in this category. We give some natural examples of enriched Hodge structures arising from geometry, which will be useful for us later on.

2.1. (**The category of Enriched Hodge structures**) Let $\mathcal{C}$ denote the category whose objects are diagrams

$$\cdots \to V_h \xrightarrow{t_h} V_{h-1} \xrightarrow{t_{h-1}} V_{h-2} \to \cdots ,$$

where each V_i is a (not necessarily finite dimensional) complex vector space and such that there exist α and β with t_m an isomorphism for all $m > \alpha$ and $V_m = 0$ for all $m < \beta$. The morphisms in $\mathcal{C}$ are morphisms of diagrams. For an object $V \in \mathcal{C}$, it makes sense to define V_∞ by setting $V_\infty := V_m$ for large m. This is well defined upto (canonical) isomorphism. Given $M \in \mathrm{MHS}$ (recall, MHS denotes the usual category of mixed Hodge structures), let $M_p := M_{\mathbb{C}}/F^p M_{\mathbb{C}}$. Setting

$$s(M) := (\cdots \to M_{p+1} \to M_p \to M_{p-1} \to \cdots)$$

and noting that $F^\bullet$ is a decreasing filtration, gives rise to a functor $s : \mathrm{MHS} \to \mathcal{C}$.

Example 2.2. The image of the Tate object $s(\mathbb{Z}(-p))$ is given by the diagram with $s(\mathbb{Z}(-p))_m = \mathbb{C}$ for all $m \geqslant p+1$ and zero otherwise. By abuse of terminology, we shall refer to this as the Tate object of $\mathcal{C}$, and also denote it by $\mathbb{C}(-p)$.

Definition 2.3. An *enriched Hodge structure* is a triple (M, V, f), where $M \in \mathrm{MHS}$, $V \in \mathcal{C}$, and $f : V \to s(M)$ is a morphism such that f_m is an isomorphism for large m.

In particular, given an enriched Hodge structure one has an induced isomorphism $f_\infty : V_\infty \to s(M)_\infty$.

Definition 2.4. A morphism of enriched Hodge structures $\phi : (M_1, V_1, f_1) \to (M_2, V_2, f_2)$ is a pair (ϕ, ψ), with $\phi \in \mathrm{Hom}_{\mathcal{C}}(V_1, V_2)$ and $\psi \in \mathrm{Hom}_{\mathrm{MHS}}(M_1, M_2)$, such that ϕ and ψ are compatible with f_1 and f_2. In particular, the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{f_1} & s(M_1) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle s(\psi)} \\ V_2 & \xrightarrow{f_2} & s(M_2) \end{array}$$

We denote by EHS the category of enriched Hodge structures as defined above. It is easy to show that EHS is an abelian category, based on the fact that MHS and $\mathscr{C}$ are both abelian categories, and the functor s is exact.

Remark 2.5. We note that the above definition of EHS differs from the category of enriched Hodge structures define by Bloch-Srinivas in ([3]). For the moment we note that the current version is a full subcategory of the category defined by Bloch-Srinivas. We refer to 2.18 for more details.

Remark 2.6. Note that for any $(V, M, f) \in \mathrm{EHS}$, the linear maps f_m are onto for all m. This follows from the fact that $V_\infty \cong M_{\mathbb{C}}$ and $M_{p+1} \to M_p$ is onto for all p.

Example 2.7. Any object $M \in \mathrm{MHS}$ gives an object object $(M, s(M), Id) \in \mathrm{EHS}$. In particular, we have the Tate objects denoted $\mathbb{Z}(p) \in \mathrm{EHS}$.

The following are some easy examples of enriched Hodge structures.

Example 2.8. Let X be a smooth projective complex algebraic variety (or a compact Kahler manifold). Then we shall denote by $\mathrm{H}^i_E(X)$ the enriched Hodge structure associated to X given by the triple $(\mathrm{H}^i(X), s(\mathrm{H}^i(X)), \mathrm{Id})$ where $\mathrm{H}^i(X)$ denotes the usual Hodge structure on the singular cohomology of X.

Example 2.9. (Compactly supported cohomology) If X is a smooth (not necessarily projective) algebraic variety, then we can also associate an enriched Hodge structure $\mathrm{H}^i_{c,E}(X) := (\mathrm{H}^i_c(X), V, f)$ to compactly supported cohomology as follows. Here $\mathrm{H}^i_c(X)$ is the usual mixed Hodge structure on compactly supported cohomology of X constructed by Deligne. We set

$$V_p := \mathbb{H}^i_c(X, \mathcal{O}_X \to \cdots \to \Omega^{p-1}_X)$$

and let $f_p : \mathbb{H}^i_c(X, \mathcal{O}_X \to \cdots \to \Omega^{p-1}_X) \to \mathrm{H}^i_c(X)/F^p$ denote the natural map. Here, the compactly supported cohomology (i.e. the hypercohomology group above) is computed in the analytic topology.

Example 2.10. (Projective Variety) If X is a projective variety (not necessarily smooth), then we can associate an enriched Hodge structure $\mathrm{H}^i_E(X) := (\mathrm{H}^i(X), V, f)$ as follows. Again, $\mathrm{H}^i(X)$ is the usual mixed Hodge structure given by Deligne. Let $V'_p = \mathbb{H}^i(X, \mathcal{O}_X \to \cdots \to \Omega^{p-1}_X)$. This gives an object $V' \in \mathcal{C}$ with $V'_\infty := \mathbb{H}^i(X, \Omega^\bullet_X)$. If $\tilde{X}_\bullet \to X$ is a proper smooth (simplicial) hypercover, then one has a natural diagram:

$$\mathrm{H}^i(X) \to \mathbb{H}^i(X, \Omega^\bullet_X) \to \mathbb{H}^i(\tilde{X}_\bullet, \Omega^\bullet_{\tilde{X}_\bullet}) \to \mathrm{H}^i(X)$$

where the rightmost arrow is an isomorphism and the composition of all three maps is the identity. Let $K_\infty := \ker(V'_\infty \to \mathbb{H}^i(\tilde{X}, \Omega^\bullet_{\tilde{X}}))$ and set $V_p := V'_p/\mathrm{Im}(K_\infty)$. This gives an object $V \in \mathcal{C}$ such that $V_\infty \cong \mathrm{H}^i(X)$. Furthermore, there is a natural map $f : V \to s(\mathrm{H}^i(X))$ giving rise to the required enriched Hodge structure. It can be checked that this is independent of the chosen hypercover.

We note that the above constructions are not unique in any sense. For instance, any variety can be given an enriched Hodge structure by considering the enriched structure associated to the mixed Hodge structure on its cohomolgy as in Example 2.7.

2.11. (**Extensions of enriched Hodge structures**) Let $D = (M, V, f)$ be an enriched Hodge structure. For any m, let $W_m D = (M', V', f')$, where $M' := W_m M \in \mathrm{MHS}$,

$$V'_h := f_h^{-1}(W_m M_{\mathbb{C}}/F^h W_m M_{\mathbb{C}})$$

and f' is the natural induced morphism. Note that $M'_{\mathbb{Z}} := \ker(M_{\mathbb{Z}} \to M_{\mathbb{Q}}/W_m M_{\mathbb{Q}})$.

Remark 2.12. The previous construction gives rise to an increasing weight filtration $W.D$ (in EHS) on D. Moreover, this construction is functorial in D.

Proposition 2.13. *Let $D = (M, V, f) \in \mathrm{EHS}$, $D' = (M', V', f') := W_{2p}D$, and ψ denote the composite:*

$$M'_{\mathbb{Z}} \to M'_{\mathbb{C}} \xrightarrow{(f'_\infty)^{-1}} V'_\infty \to V'_p \xrightarrow{(2\pi i)^p} V'_p.$$

Then there are natural (in D) isomorphisms:

(1) $\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to \ker(\psi)$.
(2) $\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to \mathrm{coker}(\psi)$.

We begin with some lemmas.

Lemma 2.14. *Let D be as in the proposition. Then one has an exact sequence:*

$$0 \to \mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), M) \oplus \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}(-p), V) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}(-p), s(M))).$$

The second arrow is given by sending a pair (ϕ_1, ϕ_2) to $s(\phi_1) - f \circ \phi_2$.

Proof. This follows from the definition of morphisms of enriched Hodge structures. □

Lemma 2.15. *Let D and D' be as in Proposition 2.13.*

(1) *The natural map*
$$\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D') \to \mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D)$$
is an isomorphism.
(2) *The natural map*
$$\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D') \to \mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D),$$
is an isomorphism.

Proof. We first observe that the corresponding statements in MHS with D (resp D') replaced by M (resp. M') are true. In particular, $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), M/M') = 0$ and $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-p), M/M') = 0$. Consider the exact sequence of enriched Hodge structures

$$0 \to D' \to D \to D/D' \to 0.$$

Applying the Hom functor to this short exact sequence reduces us to showing that $\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D/D') = 0$ and $\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D/D') = 0$. By the previous observation and Lemma 2.14, we have an exact sequence:

$$0 \to \mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D/D') \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}(-p), V/V') \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}(-p), s(M/M')).$$

Therefore, the first statement will follow if the rightmost arrow in the above diagram is injective. On the other hand, any morphism $\mathbb{C}(-p) \to V/V'$ in $\mathcal{C}$ is determined by an element of $(V/V')_\infty \cong s(M/M')_\infty$ (which maps to zero in $(V/V')_p$). One can argue in a similar manner for the second part. We leave the details to the reader. □

Proof. (Proposition 2.13)
(1) Since $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), M') = M'_{\mathbb{Z}}$, an application of Lemma 2.15 gives a natural map $\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to M'_{\mathbb{Z}}$. Furthermore, since $\mathbb{Z}(-p)_p = 0$, it's composition with ψ is zero. This gives a natural injective map $\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to \ker(\psi)$, and it's easy to

construct an inverse.
(2) We begin by constructing a natural morphism: $\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to \mathrm{coker}(\psi)$. Let $[E] \in \mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D)$ be an extension class represented by an extension

$$0 \to D \to E \to \mathbb{Z}(-p) \to 0,$$

where $E = (W, N, g)$. Let $E' = (W', N', g') := W_{2p}E$. Then the extension above gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_{\mathbb{Z}} & \longrightarrow & N'_{\mathbb{Z}} & \longrightarrow & \mathbb{Z}(-p) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\ 0 & \longrightarrow & V'_p & \longrightarrow & W'_p & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

where the vertical maps are given by ψ. An application of the snake lemma, then gives a canonical element of $\mathrm{coker}(\psi)$. One can check that this is independent of the choice of representative for the extension, and hence we get a natural map

$$\theta : \mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}(-p), D) \to \mathrm{coker}(\psi).$$

Furthermore, an explicit computation shows that this is a group homomorphism, where the Ext group has the usual additive structure coming from the Baer sum.

Injectivity of θ: Suppose $\theta([E]) = 0$. Then there is a $c \in N'_{\mathbb{Z}}$ such that $\psi(c) = 0$ and the image of c in $\mathbb{Z}(-p)$ is $(2\pi i)^p \cdot 1$. It follows that c gives a splitting $\mathbb{C}(-p) \to W$ (in $\mathcal{C}$). It also gives rise to a splitting of the corresponding exact sequence of MHS.

Surjectivity of θ: By Lemma 2.15, it is sufficient to construct an extension

$$0 \to D' \to E' \to \mathbb{Z}(-p) \to 0$$

for any $\alpha \in V'_p$ such that $\theta([E]) = \bar{\alpha}$. Here $\bar{\alpha}$ is the image of α in $\mathrm{coker}(\psi)$. We begin by constructing a triple $E' = (W', N', g')$.
(a) *Construction of W':* We set $W'_m := V'_m \oplus \mathbb{C}$ if $m \geqslant p+1$ and $W'_m = V'_m$ otherwise. The transition morphisms for the diagram are the natural ones except $W'_{p+1} \to W'_p$ is given by sending $(0, 1)$ to α. This gives an object $W' \in \mathcal{C}$.
(b) *Construction of N':* We set $N'_{\mathbb{Z}} = M'_{\mathbb{Z}} \oplus \mathbb{Z}(-p)$. We define the weight filtration by setting $W_k N'_{\mathbb{Q}} := W_k M'_{\mathbb{Q}} \oplus W_k \mathbb{Z}(-p)_{\mathbb{Q}}$. Then we have $N'_{\mathbb{C}} = W'_{\infty}$. Furthermore, we set $F^k N'_{\mathbb{C}} := \ker(N'_{\mathbb{C}} \to s(N')_k)$. The morphism f' induces a morphism $g' : W' \to s(N')$. We leave it to the reader to check that these data define an enriched Hodge structure. The main thing is to check that the Hodge filtration is n-opposite. □

Example 2.16. Suppose X is a smooth projective variety. Then by Proposition 2.13 one has

$$\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}(-p), \mathrm{H}^i_E(X)) = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-p), \mathrm{H}^i_E(X))$$

and similarly for Ext^1.

Corollary 2.17. *For any $D \in \mathrm{EHS}$, one has $\mathrm{Ext}^i_{\mathrm{EHS}}(\mathbb{Z}(-p), D) = 0$ for all $i \geqslant 2$.*

Proof. By Proposition 2.13, it's enough to show that the endofunctor $D \mapsto W_k D$ is exact (for any k). The latter is a consequence of the surjectivity explained in Lemma 2.6. ☐

2.18. (**Comparison with the Bloch-Srinivas Category**) As remarked earlier, the category of enriched Hodge structures defined in Bloch-Srinivas ([3]) is slightly different than ours. We comment briefly on the difference.

Let us denote by EHS' the category of enriched Hodge structures defined by Bloch-Srinivas. An object in EHS' is a quadruple (M, V, ρ, π) where $M \in \mathrm{MHS}$, $V \in \mathcal{C}$, and $\pi : V \to s(M)$. Furthermore, one requires the existence of an a such that $s(M)_i \to s(M)_a$ and $V_i \to V_a$ are isomorphisms for all $i > a$. Therefore, it makes sense to speak of $s(M)_\infty$ and V_∞. Note that $s(M)_\infty = M_{\mathbb{C}}$. Finally, $\rho : s(M)_\infty \to V_a$ is a morphism such that $\pi \circ \rho$ is the identity (i.e. a splitting).

Note that we may view EHS as a full subcategory of EHS' in a natural way, since for any object (V, M, f) in EHS, the map $f_\infty : V_\infty \to s(M)_\infty$ is an isomorphism, and so may also be viewed as a split surjection, thus automatically determining the "extra" data of a splitting ρ.

On the other hand, EHS' has a full subcategory equivalent to $\mathscr{C}$, consisting of objects $(0, V, 0, 0)$ with trivial underlying MHS. It is clear that a general object in EHS' is the direct sum of two objects in the essential images of EHS and $\mathcal{C}$.

2.19. (**Further Remarks**) Let $D = (V, M, f), D' = (V', M', f') \in \mathrm{EHS}$. We want to define an object $D \otimes D'$ with underlying MHS $M \otimes M'$. We discuss here the existence of a monoidal structure on a certain full sub-category of EHS. Let $\mathrm{EHS}^{\mathrm{s}}$ denote the full subcategory consisting of objects (M, V, f) such that $V_h \to V_{h-1}$ is surjective for all h. We note that EHS^s is a full exact subcategory of the abelian category EHS.

Remark 2.20.
1. The natural inclusion $\mathrm{EHS}^{\mathrm{s}} \hookrightarrow \mathrm{EHS}$ has a natural right adjoint $\mathrm{EHS} \hookrightarrow \mathrm{EHS}^{\mathrm{s}}$. One replace V_p by the image of V_∞ in V_p.
2. The functor $s : \mathrm{MHS} \to \mathrm{EHS}$ factors through $\mathrm{EHS}^{\mathrm{s}}$. Moreover, the natural forgetful functor $ff : \mathrm{EHS}^{\mathrm{s}} \to \mathrm{MHS}$ is a left adjoint.

Proposition 2.21.
1. *The category $\mathrm{EHS}^{\mathrm{s}}$ has a natural symmetric monoidal structure.*
2. *The functors $s : \mathrm{MHS} \to \mathrm{EHS}^s$ and $ff : \mathrm{EHS}^{\mathrm{s}} \to \mathrm{MHS}$ are symmetric monoidal functors.*

Proof. Given two objects $D = (V, M, f)$ and $D' = (V', M', f')$, we define an object $D \otimes D' := (V \otimes V', M \otimes M', f \otimes f')$ as follows. We define $M \otimes M'$ using the usual monoidal structure on mixed Hodge structures. Let $L_k := \ker(V_\infty \to V_k)$ and similarly $L'_k := \ker(V'_\infty \to V_k)$. We set $(V \otimes V')_k := V_\infty \otimes V'_\infty / (\Sigma_{m+n=k} L_m \otimes L'_n)$. Note that $s(M \otimes M')_k =$

$(s(M)\otimes s(M'))_k$. It follows that one has induced maps $f_k\otimes f_{k'} : (V\otimes V')_k \to s(M\otimes M')_k$. Then we define the tensor product as $(V, M, f)\otimes(V', M', f') := (V\otimes V', M\otimes M', f\otimes f')$. We leave it to the reader to check that this gives a well defined symmetric monoidal structure on $\mathrm{EHS}^{\mathrm{s}}$. The second part of the proposition is a direct consequence of the construction.

□

Remark 2.22. For any object $(V, M, f) \in \mathrm{EHS}$, we can define Tate twists

$$(V, M, f)(-p) := (V(-p), M(-p), f(-p)) \otimes \mathbb{Z}(-p)$$

as follows. We set the underlying diagram of complex vector spaces, denoted by $V(-p)$, to be the original V shifted to the right by p (i.e. $V(-p)_k = V_{k+p}$). The mixed Hodge structure part $M(-p)$ is the usual Tate twist.

3. Preliminaries on Logarithmic forms

Let $f : X \to Y$ be a proper birational morphism of smooth complex analytic varieties (resp. smooth algebraic varieties) with $B \subset Y$ and $C := f^{-1}(B) \subset X$ simple normal crossings divisors. Suppose that $f|_{X\backslash C} : X\backslash C \to Y\backslash B$ is an isomorphism. In the algebraic case, it was shown in ([7] and [10]) that the natural adjunction map

$$\Omega^p_Y(\log B) \to Rf_*\Omega^p_X(\log C)$$

is an isomorphism. Moreover, the proof of this statement given in ([10]) also applies in the complex analytic situation. The aforementioned result will be applied to show the independence of the choice of resolutions in the construction of the enriched Hodge structures on thickenings of complex analytic varieties.

Below we give a short, independent, and self contained proof of the aforementioned result along with a slightly more general result (allowing for blow-ups along subvarieties not necessarily contained in B). The latter setting (i.e. allowing more general blow-ups) will be applied in a future paper in order to construct Gysin sequences in the EHS setting.

3.1. (The case of blow-ups) In this paragraph, let X be a smooth complex analytic variety, let B be a divisor with simple normal crossings on X, and let Z be a connected smooth subvariety of X. Following Hironaka ([8], Definition 2, page 141), we recall that Z has *normal crossings* with B if for each point $z \in Z$ there is a local coordinate system $(x_1, \cdots, x_n)$ on X such that Z is locally defined by $\{x_1 = 0, \ldots, x_i = 0\}$ and B is locally defined by $\{x_j x_{j+1} \cdots x_k = 0\}$, for some i, j, and k. We will henceforth assume that this is the case, in this section.

Remark 3.2. We could also start with X a smooth algebraic variety, B a simple normal crossings divisor (s.n.c.d), and Z a smooth connected complex analytic subvariety of X; we may similarly define the notion of Z having normal crossings with B. The discussion below in goes through in this setting as well.

Theorem 3.3. *Let $\pi : \widetilde{X} \to X$ be the blow-up of X along Z, E be the exceptional divisor of the blow-up, and let $\widetilde{B} = (E \cup \pi^{-1}B)_{red}$.*

(1) *Assume that Z is contained in B. Then*

$$\Omega^p_X(\log B) \to R\pi_*\Omega^p_{\tilde{X}}(\log \widetilde{B})$$

is a quasi-isomorphism for all $p \geqslant 0$.

(2) *Assume Z is not contained in B. Let s denote the codimension of Z in X. For all $p \geqslant 0$, in the derived category of sheaves on X, we have an exact triangle:*

$$i_*\Omega^{p-s}_Z(\log B|_Z)[-s] \to \Omega^p_X(\log B) \to R\pi_*\Omega^p_{\tilde{X}}(\log \widetilde{B}) \to i_*\Omega^{p-s}_Z(\log B|_Z)[1-s].$$

Here $i : Z \hookrightarrow X$ is the natural inclusion.

We first give a proof of the Theorem assuming the following Lemma.

Lemma 3.4. *Let $Q := \Omega^1_{\tilde{X}}(\log \widetilde{B})$, and let P denote the image of $\pi^*\Omega^1_X(\log B) \to \Omega^1_{\tilde{X}}(\log \widetilde{B})$. There are locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on Z and isomorphisms*

$$Q/P \cong \mathcal{O}_{\tilde{X}}(E) \otimes \pi^* i_*\mathcal{F} \text{ and } P/Q(-E) \cong \pi^* i_*\mathcal{G}$$

where $i : Z \hookrightarrow X$ denotes the inclusion. Furthermore, the rank of $\mathcal{F}$ is the codimension of Z in X minus the number of irreducible components of B that contain Z.

We now give a proof of Theorem 3.3.

Proof. (Theorem 3.3) We will appeal first to the standard fact that $L \to R\pi_*\pi^*L$ is an isomorphism for a locally free sheaf L on X. In particular, $\Omega^p_X(\log B) \to R\pi_* \bigwedge^p P$ is an isomorphism. Thus the first part is equivalent to the vanishing of $R\pi_*(\bigwedge^p Q/\bigwedge^p P)$, whereas the second requires an isomorphism of this with $i_*\Omega^{p-s}_Z(\log B|_Z)[1-s]$.

The other fact we appeal to is the vanishing of $R\pi_*\left(\mathcal{O}_{\tilde{X}}(iE) \otimes \pi^* i_* H\right)$ for every $0 < i < s = \mathrm{codim}(Z)$, and for every coherent sheaf H on Z. This is clear because E is a $\mathbb{P}^{s-1}$-bundle on Z and $\mathcal{O}_{\tilde{X}}(E)$ restricts to $\mathcal{O}(-1)$ of this projective bundle.

Every subsheaf $M \subset Q$ gives rise to a decreasing filtration

$$F^k_M \bigwedge^p Q = \mathrm{image}(\bigwedge^{p-k} Q \otimes \bigwedge^k M \to \bigwedge^p Q)$$

indexed by $0 \leqslant k \leqslant p$. In particular, we get two filtrations: $S^k = F^k_P$ and $T^k = F^k_{Q(-E)}$.

By Lemma 3.4, $Q(-E)$ is contained in P. It follows that S^k contains T^k, and this shows that $gr^a_S gr^b_T \bigwedge^p Q = 0$ when $a < b$. We are more concerned with $\bigwedge^p Q/\bigwedge^p P$, and therefore it suffices to concentrate of the (a, b)-th term for $0 \leqslant b \leqslant a < p$.

Again, by the Lemma 3.4, both Q/P and $P/Q(-E)$ are the push-forwards of locally free $\mathcal{O}_E$-modules. We deduce the isomorphism below for the terms $0 \leqslant b \leqslant a < p$:

$$gr_S^a gr_T^b \bigwedge^p Q \cong \mathcal{O}(-bE) \otimes \bigwedge^{a-b}(P/Q(-E)) \otimes \bigwedge^{p-(a-b)}(Q/P)$$

and employing the isomorphisms of Lemma 3.4, we get

$$gr_S^a gr_T^b \bigwedge^p Q \cong \mathcal{O}((p-a)E) \otimes \pi^* i_*(\bigwedge^{a-b} \mathcal{G} \otimes \bigwedge^{p-(a-b)} \mathcal{F}).$$

If the above (a,b)-th term is nonzero, then

$$0 < p - a \leqslant p - (a-b) \leqslant \operatorname{rank}(\mathcal{F}) \leqslant s.$$

Note that $0 < p - a < s$ gives the vanishing of $R\pi_*$ of the (a,b)-th term.

The only (a,b)-th term left to consider is when $(a,b) = (p-s, 0)$. Noting that $R^j\pi_*\mathcal{O}_E(sE)$ is zero when $0 \leqslant j \leqslant (s-1)$, we obtain the exact triangle

$$\Omega_X^p(\log B) \to R\pi_* \bigwedge^p Q \to i_*(\bigwedge^{p-s} \mathcal{G} \otimes \bigwedge^s \mathcal{F}) \otimes R^{s-1}\pi_*\mathcal{O}_E(sE)[1-s] \to \Omega_X^p(\log B)[1]$$

The first part of the Theorem follows because in that situation, the rank of $\mathcal{F}$ is less than s. The second part will be clear from the explicit description of $\mathcal{F}$ and $\mathcal{G}$ provided by the proof of the Lemma 3.4. □

Proof. (Lemma 3.4) Suppose $B = \sum_i B_i$, and let $\tilde{B}_i$ be the strict transform of B_i. Then $\tilde{B} = E + \sum_i \tilde{B}_i$. The standard commutative diagram of residue exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_{\tilde{X}} & \longrightarrow & \Omega^1_{\tilde{X}}(\log E) & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\ & & \downarrow Id & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1_{\tilde{X}} & \longrightarrow & \Omega^1_{\tilde{X}}(\log \tilde{B}) & \longrightarrow & \mathcal{O}_E \oplus (\oplus_i \mathcal{O}_{\tilde{B}_i}) & \longrightarrow & 0 \end{array}$$

gives an exact sequence

$$0 \to \Omega^1_{\tilde{X}}(\log E) \to \Omega^1_{\tilde{X}}(\log \tilde{B}) \to \oplus_i \mathcal{O}_{\tilde{B}_i} \to 0. \tag{1}$$

Consider now the commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\Omega^1_X & \longrightarrow & \Omega^1_{\tilde{X}} & \longrightarrow & \Omega^1_{E/Z} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\Omega^1_X & \longrightarrow & \Omega^1_{\tilde{X}}(\log E) & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_E & \xrightarrow{=} & \mathcal{O}_E & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

This gives an exact sequence of $\mathcal{O}_E$-modules:

$$0 \to \Omega^1_{E/Z} \to \mathrm{coker} \to \mathcal{O}_E \to 0.$$

On the other hand, one has the usual Euler sequence for the projective bundle $E \to Z$:

$$0 \to \Omega^1_{E/Z} \to \pi^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes \mathcal{O}_E(-1) \to \mathcal{O}_E \to 0.$$

We claim that both of these extensions are isomorphic. To see this, note that both extensions give elements of $\mathrm{H}^1(E, \Omega^1_{E/Z})$. Since the map $\mathrm{H}^1(E, \Omega^1_{E/Z}) \to \mathrm{H}^0(Z, R^1\pi_*\Omega^1_{E/Z})$ is an isomorphism, both extension give nowhere vanishing sections of the bundle $R^1\pi_*\Omega^1_{E/Z} \cong \mathcal{O}_Z$, and therefore must be isomorphic extensions. Since E is the exceptional divisor of π, the ideal $\mathcal{O}_{\tilde{X}}(-E)$ coincides with $\mathcal{O}_{\tilde{X}}(1)$ (for $\tilde{X} := \mathrm{Proj}(\oplus_{n\geqslant 0}\mathcal{I}^n_Z)$). The latter restricts to $\mathcal{O}_E(1)$ (for $E := \mathrm{Proj}(\oplus_{n\geqslant 0}\mathcal{I}^n/\mathcal{I}^{n+1})$). It follows that $\mathcal{O}_E(-1) \cong \mathcal{O}_{\tilde{X}}(E) \otimes \mathcal{O}_E$. By the previous remarks, we have an exact sequence:

$$0 \to \pi^*\Omega^1_X \to \Omega^1_{\tilde{X}}(\log E) \to \pi^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes \mathcal{O}_{\tilde{X}}(E) \to 0. \tag{2}$$

The exact sequences (1) and (2) give a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*\Omega^1_X & \longrightarrow & \Omega^1_{\tilde{X}}(\log E) & \longrightarrow & \pi^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes \mathcal{O}_{\tilde{X}}(E) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{\tilde{X}}(\log E) & \longrightarrow & \Omega^1_{\tilde{X}}(\log \tilde{B}) & \longrightarrow & \oplus_i \mathcal{O}_{\tilde{B}_i} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \oplus \mathcal{O}_{\tilde{B}_i} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
\tag{3}
$$

The standard residue exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log B) \to \oplus_i \mathcal{O}_{B_i} \to 0$$

gives after pullback an exact sequence:

$$0 \to \pi^*\Omega^1_X \to \pi^*\Omega^1_X(\log B) \to \oplus_i \mathcal{O}_{\pi^{-1}B_i} \to 0.$$

Note that the left-most morphism (between locally free sheaves) is injective since it is an injection at the generic point. It follows that one has a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & ker & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\Omega^1_X & \longrightarrow & \Omega^1_{\tilde{X}}(\log E) & \longrightarrow & \pi^*(\mathcal{I}_Z/\mathcal{I}_Z^2)\otimes\mathcal{O}_{\tilde{X}}(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
o & \longrightarrow & \pi^*\Omega^1_X(\log B) & \longrightarrow & \Omega^1(\log \tilde{B}) & \longrightarrow & Q/P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \oplus_i\mathcal{O}_{\pi^{-1}B_i} & \longrightarrow & \oplus_i\mathcal{O}_{\tilde{B}_i} & \longrightarrow & 0 & &
\end{array}$$

It follows that one has an exact sequence:

$$0 \to ker \to \oplus_i \mathcal{O}_{\pi^{-1}B_i} \to \oplus_i \mathcal{O}_{\tilde{B}_i} \to 0,$$

and therefore

$$ker \cong \oplus_{i, Z\subset B_i} \mathcal{O}_{\tilde{X}}(-\tilde{B}_i) \otimes \mathcal{O}_E.$$

(In particular this kernel is trivial is Z is not contained in B.) The previous commutative diagram combined with this exact sequence (and after twisting by $\mathcal{O}(-E)$) gives an exact sequence:

$$0 \to \oplus_{i,Z\subset B_i}\mathcal{O}_{\tilde{X}}(-\tilde{B}_i - E)\otimes\mathcal{O}_E \to \pi^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \to Q/P(-E) \to 0.$$

Since $\mathcal{O}_{\tilde{X}}(-\tilde{B}_i - E)\otimes\mathcal{O}_E \cong \pi^*\mathcal{O}_X(-B_i)\otimes\mathcal{O}_E$, we conclude that

$$(Q/P)(-E) \cong \pi^*(coker(\oplus_{Z\subset B_i}\mathcal{I}_{B_i} \to \mathcal{I}_Z/\mathcal{I}_Z^2)) = \pi^*(coker(\oplus_{Z\subset B_i}\mathcal{I}_{B_i}/\mathcal{I}_{B_i}\mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2)).$$

Let $\mathcal{F} := coker(\oplus_{Z\subset B_i}\mathcal{I}_{B_i}/\mathcal{I}_{B_i}\mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2)$. Then $Q/P \cong \mathcal{O}_{\tilde{X}}(E)\otimes\pi^*i_*\mathcal{F}$. Since Z has normal crossings with B_i, $\mathcal{I}_{B_i}/\mathcal{I}_{B_i}\mathcal{I}_Z$ are line bundles on Z. It follows that the map $\oplus_{Z\subset B_i}\mathcal{I}_{B_i}/\mathcal{I}_{B_i}\mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$ is an inclusion of vector bundles. This show that $\mathcal{F}$ has rank as stated in the Lemma. This proves the first part of the Lemma.

For the second part, note that one has a natural exact sequence:

$$0 \to (Q/P)(-E) \to P/P(-E) \to P/Q(-E) \to 0.$$

The result then follows by observing that $P/P(-E)$ is isomorphic to the restriction of $\pi^*\Omega^1_X(\log B)$ to E, and therefore is the pull-back of a bundle on Z. Since $E \to Z$ is a projective bundle, it follows that $P/Q(-E)$ must also be the pull back of a bundle on Z, and in particular, $P/Q(-E)$ also has the desired form. ☐

The general case (i.e. for arbitrary proper birational maps) of Theorem 3.3 Part (1) follows as a result of the previous theorem, the following categorical Lemma (whose proof is omitted), and an application of Hironaka's elimination of indeterminacies, as we now explain.

Lemma 3.5. *Let $\mathcal{D}$ be a category and u, v, w be morphisms in $\mathcal{D}$ such that both uv and vw are defined and are isomorphisms. Then u is an isomorphism.*

Corollary 3.6. *Let $f : X \to Y$ be a proper birational morphism of smooth complex analytic varieties with $B \subset Y$ and $C := f^{-1}(B) \subset X$ simple normal crossings divisors. Suppose that $f|_{X\backslash C} : X\backslash C \to Y\backslash B$ is an isomorphism. Then the natural adjunction map*

$$\Omega^p_Y(\log B) \to Rf_*\Omega^p_X(\log C)$$

is an isomorphism.

Proof. First, note that the question is local on the base, and fix a point $y \in Y$. If $y \in Y\backslash B$, then the statement is clear, so we may assume that $y \in B$. Consider the category $\mathcal{E}$ of triples (X, A, C) where X is a smooth complex analytic variety, A is a compact analytic subset, and C is an sncd. A morphism $\pi : (X_1, A_1, C_1) \to (X_2, A_2, C_2)$ in this category is a morphism of complex analytic spaces $\pi : X_1 \to X_2$ such that $\pi : X_1 \to \pi(X_1)$ is proper, $\pi(X_1)$ is an open neighborhood of A_2, $\pi^{-1}(C_2) = C_1$, $\pi^{-1}(A_2) = A_1$, $\pi|_{X_1\backslash C_1}$: $X_1\backslash C_1 \to \pi(X_1)\backslash C_2$ is an isomorphism. Let $\mathcal{D}$ denote the category of objects of $\mathcal{E}$ over (Y, y, B). Let $\mathrm{D}^b(\mathbb{C})$ denote the bounded derived category of complex vector spaces. Let $F : \mathcal{D} \to \mathrm{D}^b(\mathbb{C})$ denote the functor which sends a triple $(Y', A', C') \xrightarrow{f} (Y, y, B)$ over (Y, y, B) to the object $Rf_*\Omega^p_{Y'}(\log C')_y$. Consider now the set S_1 of morphisms in $\mathcal{D}$ given by blow-ups along smooth subvariety C' that has normal crossings with C'. More precisely, given a triple (Y', A', C') we consider morphisms obtained by blowing up along a smooth subvariety of C' that has normal crossings with C'. Given a triple $(X, A, C) \in \mathcal{D}$ and an open subset $U \subset X$ containing A, we may restrict everything to U and obtain a morphism $(U, A, C \cap U) \hookrightarrow (X, A, C) \in \mathcal{D}$. Let S_2 be the set of such morphisms. Finally, let S denote the set of morphisms obtained by composing morphisms in S_1 and S_2. We note that for morphisms in $f \in S$, $F(f)$ is an isomorphism. This is clear for morphisms in S_2, and for $f \in S_1$ it is a consequence of Theorem 3.3 (1). Note that given a morphism of triples $f \in \mathcal{D}$, we may find a morphism g such that fg exists and $fg \in S$. The existence of such a g follows from Hironaka's elimination of indeterminacies ([8], Section 7, Chapter 1).

Let h be a morphism in $\mathcal{D}$ such that $gh \in S$. Now $F(fg)$ and $F(gh)$ are both isomorphisms and, therefore by the previous lemma that $F(f)$ is also an isomorphism.

□

4. Preliminaries on Resolutions of Singularities

In this section, we recall some standard results from the theory of resolutions of singularities in a form convenient for our applications. We begin with some definitions of categories of *triples*.

Definition 4.1. (i) Let $\mathcal{T}$ denote the category whose objects are triples (X, A, B) such that X is a complex analytic space, $B \subset X$ is a closed complex-analytic subspace such that $X\backslash B$ is smooth, and A is a compact subset (not necessarily analytic) of X. A morphism $f : (X', A', B') \to (X, A, B)$ in $\mathcal{T}$ is a morphism of complex analytic varieties $f : X' \to X$ such that $f(A') \subset A$ and $f(X'\backslash B') \subset X\backslash B$.

(ii) Let $\mathcal{GT}$ denote the full-subcategory of $\mathcal{T}$ consisting of triples (X, A, B) such that X is smooth and B is a simple normal crossings divisor in X. We shall refer to such triples as *good triples.*

(iii) Let $\mathcal{T}'$ denote the full subcategory of $\mathcal{T}$ consisting of triples (X, A, B) such that A is a complex analytic sub-variety which is also a complete algebraic variety such that $A\backslash B$ is smooth.[1]

(iv) Let $\mathcal{GT}'$ denote the full subcategory of $\mathcal{T}$ whose objects are triples (X, A, B) that lie in both $\mathcal{GT}$ and $\mathcal{T}'$, and satisfy the following two additional conditions:
(1) $A \cap B$ is the union of irreducible components of B
(2) B has normal crossings with A', where A' is the closure of $A\backslash B$ in X. We shall refer to such triples as *very good triples.*

Remark 4.2. Given any triple (X, A, B) we can consider the triple, $(\overline{X\backslash B}, A \cap \overline{X\backslash B}, B \cap (\overline{X\backslash B})$. The invariants we consider in the sequel remain unchanged under this modification of triples. In particular, we shall often replace our original triple by this modified one.

We shall now define some classes of morphisms in the previously defined categories.

Definition 4.3. (i) For every triple $(X, A, B) \in \mathcal{T}$ and an open neighborhood $U(A)$ of A in X, one has an induced object $(U(A), A, U(A) \cap B) \in \mathcal{T}$ and an induced morphism $j : (U(A), A, U(A) \cap B) \to (X, A, B)$. Let S_1 denote the set of all such morphisms j. Note that if $(X, A, B) \in \mathcal{GT}$ is good, then so is $(U(A), A, U(A) \cap B)$ and j is a morphism in $\mathcal{GT}$. We let GS_1 denote the analogous set of such morphism in $\mathcal{GT}$.

[1]More precisely, there exists a complete algebraic variety Z such that A and Z_{an} are isomorphic as complex analytic spaces.

(ii) Let $(X, A, B) \in \mathcal{T}$ be a triple and let $I \subset \mathcal{O}_X$ be a coherent sheaf of ideals such that the support of $\mathcal{O}_X/I$ is reduced, and is a smooth closed analytic subvariety C contained in B. Let $f : X' \to X$ be the blow-up of I. Then one has an induced morphism of triples $(X', f^{-1}(A), f^{-1}(B)) \to (X, A, B)$. Let S_2 denote the collection of morphisms obtained in this manner. Note that any $f : (X', A', B') \to (X, A, B)$ in S_2 restricts to an isomorphism $X' \backslash B' \to X \backslash B$.

(iii) Let $(X, A, B) \in \mathcal{GT}$ be a good triple and let $I \subset \mathcal{O}_X$ be a coherent sheaf of ideals such that the support of $\mathcal{O}_X/I$ is a reduced smooth closed analytic subvariety C contained in B, such that C has normal crossings with B in the sense of Hironaka (see section 3). In particular, C is smooth though not necessarily connected. Let $f : X' \to X$ be the blow-up of I. Then one has an induced morphism of good triples $(X', f^{-1}(A), f^{-1}(B)) \to (X, A, B)$. Let GS_2 denote the collection of morphisms obtained in this manner. Note that any $f : (X', A', B') \to (X, A, B)$ in GS_2 restricts to an isomorphism $X' \backslash B' \to X \backslash B$.

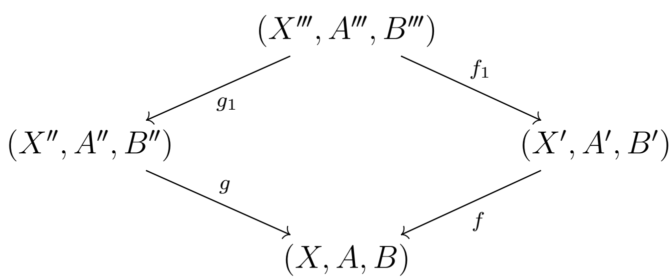
(iV) Let S denote the morphisms generated by compositions of elements of S_1 and S_2, and similarly for GS.

Remark 4.4. If $f : (X', A', B') \to (X, A, B)$ is a morphism in S, then the image of f is an open neighborhood of A, $f^{-1}(B) = B'$, the induced map $f : X' \to f(X')$ is proper, and f induces a biholomorphic map $X' \backslash B' \to f(X') \backslash B$. A similar remark applies to morphisms in GS.

Proposition 4.5. *With notation as above:*

(1) *Given $(X, A, B) \in \mathcal{T}$, there is a good triple $(X', A', B') \in \mathcal{GT}$ and a morphism $f : (X', A', B') \to (X, A, B)$ such that $f \in S$.*

(2) *Given another morphism from a good triple $g : (X'', A'', B'') \to (X, A, B)$ as in (1) above, there is a good triple (X''', A''', B''') and a commutative diagram*



such that

(i) *The image of f_1 (resp. g_1) is an open neighborhood of A' (resp. A'') and f_1 (resp. g_1) induces a proper biholomorphic map $X''' \backslash B''' \to f_1(X''') \backslash B'$ (resp. $X''' \backslash B''' \to g_1(X''') \backslash B''$).*

(ii) *g_1 is in GS (see Definition 4.3).*

(3) *For every triple $(X, A, B) \in \mathcal{T}'$, there is morphism $f : (X', A', B') \to (X, A, B)$ such that $f \in S$ and (X, A', B') is a very good triple.*

(4) *Given another* $g : (X'', A'', B'') \to (X, A, B)$ *as in Part (3), there is a very good triple* (X''', A''', B''') *and a commutative diagram*

$$\begin{array}{ccccc} & & (X''', A''', B''') & & \\ & \swarrow^{g_1} & & \searrow^{f_1} & \\ (X'', A'', B'') & & & & (X', A', B') \\ & \searrow^{g} & & \swarrow^{f} & \\ & & (X, A, B) & & \end{array}$$

such that f_1, g_1 *satisfy the properties analogous to* f *as specified in Part (1).*

Proof. (1) This is a direct application of the the canonical desingularization theorem ([14], Theorem 3.8.1, p. 41). Let (X, A, B) be an object of $\mathcal{T}$. Then the canonical desingularization theorem ([14], Theorem 3.8.1, p. 41) yields a desingularization $X' \to U$ (where U is a neighborhood of A in X) such that $f^{-1}B_{red}$ is a simple normal crossings divisor. Moreover, the morphism $(X', f^{-1}A, f^{-1}B) \to (U, A, B \cap U)$ is the composite of morphisms u_i with each $u_i \in S_2$. This proves the first part.

(2) First note that $\mathcal{T}$ has fiber products. Namely, given morphisms of triples $f_i : (X_i, A_i, B_i) \to (X, A, B)$ for $i = 1, 2$, the triple

$$(X_1 \times_X X_2, A_1 \times_X A_2, (B_1 \times_X X_2) \cup (B_1 \times_X X_2))$$

is the fiber product. Let $(X_3, A_3, B_3) := (X'', A', B') \times_{(X,A,B)} (X', A', B')$. Note that the natural projection maps $h : (X_3, A_3, B_3) \to (X'', A'', B'')$ and $l : (X_3, A_3, B_3) \to (X', A', B')$ are in S (being the base change of such maps). Therefore, it is enough to show that there is a good triple (X''', A''', B''') and a morphism $F : (X''', A''', B''') \to (X_3, A_3, B_3)$ such that $g_1 := h \circ F \in GS$. Then we can take $f_1 := l \circ F$. Note that $h \in S$, and we may in reduce to the case that $h \in S_1$ or $h \in S_2$ (by iterating the above procedure). If $f \in S_1$, then the fiber product is already a good triple and there is nothing to prove. So suppose that $h \in S_2$ and, in particular, X_3 is given by the blow-up of an ideal $I \subset \mathcal{O}_{X''}$ such that $\mathrm{supp}(\mathcal{O}_{X''}/I) \subset B''$. Since A'' is compact, we may apply 'resolution of marked ideals' ([14], Theorem 3.5.1, p. 40) to obtain:
(1) a neighborhood U of A'' in X'',
(2) a sequence of morphisms $g_1, \ldots, g_m \in GS_2$ such that the composite $g = g_1 \circ g_2 \circ \cdots \circ g_m$ is defined with domain (X''', A''', B''') and target $(U, A'', B'' \cap U)$
(3) a morphism $F : (X''', A''', B''') \to (X_3, A_3, B_3) \in \mathcal{T}$ such that $h \circ F = g \circ j$ where $j : (U, A'', B'' \cap U) \hookrightarrow (X''', A''', B'')$ is the inclusion.
It follows that $g \circ j \in GA$.

(3) We construct the required f as follows:
Step 1: We may desingularize X by blowing up in B ([14], 3.8.1; [9], 3.36 and

3.43). The resulting map f will be in S'.
Step 2: We apply principalization ([14], 3.6.1) to the product of the ideals of B and $B \cap A$. Here we only need to blow-up within B. We obtain a new triple (X, A, B) where X is smooth, B is a s.n.c.d. and $A \cap B$ is a s.n.c.d. contained in B.
Step 3: We apply embedded desingularization ([14], 3.7.1) to $A \cup B$. Here we only blow-up on a center contained in $A \cap B$. The exceptional divisor E of the blow-up is a s.n.c.d. and the strict transform of $A \cup B$ (which is the union of the strict transforms A' and B') is smooth, and has simple normal crossings with the closure of $A' \backslash B'$.

(4) This follows from the previous part by taking fiber products.

□

5. Enriched Hodge structures on thickenings

The main goal of this section is to give a construction of a mixed Hodge structure, which is enhanced to an Enriched Hodge structure, on certain complex analytic 'thickenings' (see Remark 5.2 below for what we mean by thickenings). In the next section, we shall give some relations between these constructions and the K-theory of convergent power series rings.

5.1. Enriched Hodge structure on triples: Statement of main theorem. Let X denote a complex analytic variety and let A, B denote two closed subvarieties such that $X \backslash B$ is smooth, A is a complete algebraic variety, and $Sing(A) \subset B$ (i.e. $(X, A, B) \in \mathcal{T}'$). The main goal of this section is to prove the following theorem. In the following, let $i : A \hookrightarrow X$ and $j : X \backslash B \hookrightarrow X$ denote the natural inclusions and consider

$$\varinjlim \mathrm{H}^m(U \backslash B) \cong \mathbb{H}^m(A, i^{-1} R j_* \mathbb{Z}).$$

Here the limit is over open (in the complex analytic topology) neighborhoods U of A.

Remark 5.2. Note that the we are really concerned with the system of open neighborhoods U of A in X, which are the "complex analytic thickenings" referred to in the title. In particular, we view any specific triple (X, A, B) as a representative of a "germ" of neighborhoods of A.

Theorem 5.3. *Let (X, A, B) be as above.*

(1) *Then $\mathbb{H}^m(A, i^{-1} R j_* \mathbb{Z})$ has a natural (in triples) mixed Hodge structure.*
(2) *There is a naturally defined object $\mathrm{EH}^m((X, A, B), \mathbb{Z})$ in the category* EHS *of Enriched Hodge Structures whose underlying mixed Hodge structure is the one given by (1) on $\mathbb{H}^m(A, i^{-1} R j_* \mathbb{Z})$*
(3) *(Functoriality) Given a morphism of triples $f : (X, A, B) \to (X', A', B') \in \mathcal{T}'$, the induced map*

$$\mathrm{EH}^m((X', A', B'), \mathbb{Z}) \to \mathrm{EH}^m((X, A, B), \mathbb{Z})$$

is a morphism of enriched Hodge structures.

The theorem will be proved in several steps by first considering some settings where the triples (X, A, B) satisfy some stronger hypotheses than those of the theorem. Before beginning the proof, we briefly outline the strategy.

Definition 5.4.
(1) A triple $(X, A, B) \in \mathcal{T}'$ is of type (H_1) if X is smooth, B is a s.n.c.d and A is the intersection of some components of B. Note that in this case A is smooth.
(2) A triple $(X, A, B) \in \mathcal{T}'$ is of type (H_2) if X is smooth, B is a s.n.c.d, A is not contained in any component of B and B has normal crossings with A. Note that this implies that A is smooth.
(3) A triple $(X, A, B) \in \mathcal{T}'$ is of type (H_3) if X is smooth, B is a s.n.c.d and $A = A' \cup A''$ where such that A' is a union of components of B, A'' is not contained in any component of B and B has normal crossings with A''. Note that this implies that A'' is smooth.

We will first prove Theorem 5.3 for triples of type (H_1). The construction of the mixed Hodge structure in this case is essentially due to Hain-Durfee ([6]). We use here the construction of Peters-Steenbrink ([11]); this construction will also yield the enhancement to EHS. The case (H_2) will follow directly from classical Hodge theory, since in this case B restricts to a s.n.c.d on A and one is simply looking at the cohomology of $A \backslash B$ (note A is smooth proper). The EHS will be that determined by the functor s applied to the MHS.

The case of triples of type (H_3) will then follow by replacing A in that case by the associated simplicial scheme $A.$ given by the irreducible components of A. The n-th component of that simplicial scheme is a disjoint union of intersections of the components of A, each of which has a natural associated triple of type (H_1) or (H_2). Finally, the general case will be reduced to the case of triples of type (H_3) using the resolution of singularities results of the previous section.

5.5. (**Triples of type** (H_1)) We begin by considering the special case of Theorem 5.3 where (X, A, B) is a triple of type (H_1). Note that A must be smooth in this case. Given such a triple, we can consider the log complex $\Omega^\bullet_X(\log B)$. This complex is equipped with the usual weight filtration W and Hodge filtration F defined by Deligne. In particular, the restriction $i^{-1}\Omega^\bullet_X(\log B)$ also comes equipped with filtrations W and F. On the other hand, it follows from ([11], 6.11) that, if $\mathcal{I}_A$ is the ideal of A, then $\mathcal{I}_A\Omega^\bullet(\log B)$ is a subcomplex of $\Omega^\bullet_X(\log B)$. It follows that $i^*\Omega^\bullet_X(\log B) := \Omega^\bullet_X(\log B) \otimes \mathcal{O}_A$ is a complex on A and we have a surjective map of bifiltered complexes on A:

$$(i^{-1}\Omega^\bullet_X(\log B), W, F) \to (i^*\Omega^\bullet_X(\log B), W, F).$$

Here the filtrations W and F on the image are by definition the image filtrations. Note that the image filtration F on the right hand term is the same as the intrinsically defined Hodge filtration on the right hand term. Moreover, by ([11], 6.12), the map of filtered complexes

$$(i^{-1}\Omega^\bullet_X(\log B), W) \to (i^*\Omega^\bullet_X(\log B), W)$$

is a filtered quasi-isomorphism.

Remark 5.6. In *loc. cit.* this quasi-isomorphism is only stated in the setting where X and B are algebraic varieties. However, the proof only makes use of residue exact sequences and goes through in our setting without change.

Note that this implies in particular that the complex on the RHS is quasi-isomorphic to $(i^{-1}Rj_*\mathbb{C}, \tau)$. This is since Deligne's theory already gives an isomorphism in the derived filtered category:

$$(i^{-1}Rj_*\mathbb{C}, \tau) \to (i^{-1}\Omega^\bullet_X(\log B), W)$$

We note that this quasi-isomorphism can be represented by a diagram of actual morphisms by choosing the canonical representative for $Rj_*\mathbb{Z}$ (and other derived complexes) given by the Godement resolution (see 7.13). In particular, it can be given by a pseudo-morphism (see 7.11) by restricting to A the following diagram of actual morphisms of filtered complexes:

$$(Rj_*\mathbb{C}, \tau) \to (Rj_*\Omega^\bullet_{X\backslash B}, \tau) \leftarrow (j_*\Omega^\bullet_{X\backslash B}, \tau) \leftarrow (\Omega^\bullet_X(\log B), \tau) \to (\Omega^\bullet_X(\log B), W).$$

Furthermore, one has the following description of the weight graded pieces of $i^*\Omega^\bullet_X(\log B)$. Suppose $B = B_1 \cup \cdots \cup B_k$ and $B_I := \cap_{i\in I} B_i$ where $I \subset \{1, \ldots, k\}$. Then the residue map induces an isomorphism:

$$\mathrm{Gr}^W_k(i^*\Omega^\bullet_X(\log B) \cong \oplus_{|I|=k}\Omega^\bullet_{B_I\cap A}[-k]$$

It follows that the data $(i^{-1}Rj_*\mathbb{Z}, (i^{-1}Rj_*\mathbb{Q}, \tau), (i^*\Omega^\bullet_X(\log B), W, F))$ gives rise to a (genuine) cohomological mixed Hodge complex. Moreover, it is easy to see that this data is functorial for a morphism of triples $f : (X, A, B) \to (X', A', B')$ of type (H_1) such that $f(A) \subset A'$, $f^{-1}(B') \subset B$, and $f^{-1}(B')$ is also a s.n.c.d. Finally, it is also clear that we can upgrade this to a cohomological enriched Hodge complex by considering the quadruple:

$$(i^{-1}Rj_*\mathbb{Z}, (i^{-1}Rj_*\mathbb{Q}, \tau), (i^*\Omega^\bullet_X(\log B), W, F), (i^{-1}\Omega^\bullet_X(\log B), F))$$

In particular, one has the following result:

Theorem 5.7. *Let (X, A, B) be a triple of type (H_1). Then the data*

$$(i^{-1}Rj_*\mathbb{Z}, (i^{-1}Rj_*\mathbb{Q}, \tau), (i^*\Omega^\bullet_X(\log B), W, F), (i^{-1}\Omega^\bullet_X(\log B), F))$$

gives rise to a (genuine) cohomological enriched Hodge complex on A. Moreover, it is functorial for a morphism of triples $f : (X, A, B) \to (X', A', B')$ of type (H_1) such that $f^{-1}(B')$ is a s.n.c.d. In particular, Theorem 5.3 is true for such triples.

Remark 5.8. By definition of the weight filtration, one has for triples of type (H_1):

$$W_k\mathbb{H}^k(A, i^{-1}Rj_*\mathbb{Q}) = \mathrm{Image}(\mathrm{H}^k(A, \mathbb{Q}) \to \mathbb{H}^k(A, i^{-1}Rj_*\mathbb{Q})).$$

Remark 5.9. There is an alternate approach to proving Theorem 5.7 which is also sometimes useful. Namely, consider the normal bundle $p : N_AX \to A$ and let $s : A \to N_AX$

denote the zero section. Then $Z := p^{-1}(D) \cup s(A)$ is a s.n.c.d. on $N_A X$ where D is the restriction of B to A. One can show that there is a natural filtered quasi-isomorphism

$$(s^{-1}\Omega^\bullet_{N_A X}(\log Z), W) \to (i^{-1}\Omega^\bullet_X(\log B), W).$$

One then notes that the left hand side is completely algebraic since A was assumed to be algebraic. In particular, one can now apply Deligne's mixed Hodge theory to obtain a mixed Hodge structure.

5.10. (**Triples of type** (H_2)) Let (X, A, B) denote a triple of type (H_2). In this case, the restriction of B to A, denoted $B|_A$, is a simple normal crossings divisor on A. Moreover, the complex $i^{-1}\Omega^\bullet_X(\log B)$ is naturally quasi-isomorphic to $\Omega^\bullet_A(\log B|_A)$ (with the weight filtration). It follows that the quadruple

$$(i^{-1}Rj_*\mathbb{Z}, (i^{-1}Rj_*\mathbb{Q}, \tau), (\Omega^\bullet_A(\log B|_A), W, F), (i^{-1}\Omega^\bullet_X(\log B), F))$$

gives rise to a (genuine) cohomological enriched Hodge complex on A. Moreover, these are functorial in morphisms of triples $f : (X, A, B) \to (X', A', B')$ of type (H_2) such that $f^{-1}(B')$ is an s.n.c.d. In particular, we have the following:

Theorem 5.11. *Let (X, A, B) be a triple of type (H_2). Then the data*

$$(i^{-1}Rj_*\mathbb{Z}, (i^{-1}Rj_*\mathbb{Q}, \tau), (\Omega^\bullet_A(\log B|_A), W, F), (i^{-1}\Omega^\bullet_X(\log B), F))$$

gives rise to a (genuine) cohomological enriched Hodge complex on A. Moreover, it is functorial for a morphism of triples $f : (X, A, B) \to (X', A', B')$ of type (H_2) such that $f^{-1}(B')$ is a s.n.c.d. In particular, Theorem 5.3 is true for such triples.

Before proceeding, we note that the functoriality statements in the previous theorems can be extended.

Lemma 5.12. *Let $f : (X', A', B') \to (X, A, B)$ be a morphism of triples where the domain is of type (H_1) and the range is of type (H_2) and $f^{-1}(B)$ is an s.n.c.d. Then one has an induced morphism of the corresponding cohomological enriched Hodge complexes.*

Proof. Since log de Rham complexes are functorial for such morphisms, the functoriality of the enriched part of the enriched Hodge complex is clear. Moreover, it is also not hard to see that the integral part and the rational part with the corresponding weight filtration is also functorial. Therefore we are reduced to showing that there is a natural morphism of filtered complexes:

$$f^{-1}\Omega^\bullet_A(\log B|A) \to \Omega^\bullet_{X'}(\log B') \otimes \mathcal{O}_{A'}.$$

On the other hand, one has natural maps:

$$f^{-1}(i^{-1}\Omega^\bullet_X(\log B)) \to i^{-1}\Omega^\bullet_{X'}(\log B') \to \Omega^\bullet_{X'}(\log B') \otimes \mathcal{O}_{A'},$$

where the first map results from the aforementioned functoriality. All the complexes in this diagram are equipped with the Hodge filtration, and the first map is clearly a morphism of filtered complexes. Moreover, the discussion in the beginning of 5.5 shows that the second map is also a map of filtered complexes. Finally, the discussion above shows that $i^{-1}\Omega^\bullet_X(\log B)$ is naturally quasi-isomorphic (as a bifiltered complex with both weight and Hodge filtration) to $\Omega^\bullet_A(\log B|A)$. ☐

5.13. (**Triples of type** (H_3)) Suppose now that we are given a triple (X, A, B) of type (H_3). In particular, $A = A' \cup A''$ where A' is a union of components of B, B has normal crossings with A'', and A'' is not contained in any component of B. Let $A' = \cup_{i=1}^k A_i$ where A_i are the irreducible components of A', and each A_i is a component of B. In the following, we let $A_0 := A''$. In this case, note that each (X, A_i, B) is a triple of type (H_1) for $1 \leqslant i \leqslant k$ and of type (H_2) for $i = 0$. Moreover, each triple (X, A_{ij}, B) (where $A_{ij} := A_i \cap A_j$) is again of type (H_1) if $i, j \neq 0$. On the other hand, consider the triple (X, A_{0j}, B) where $j \neq 0$. In this case, this triple is not in general of type (H_1) or (H_2). However, the triple $(A_0, A_{0j}, B|_{A_0})$ is again of type (H_1). Moreover, the following lemma shows that the natural morphism of triples $(A_0, A_{0j}, B|_{A_0}) \to (X, A_{0j}, B)$ induces an isomorphism on cohomology.

Lemma 5.14. *Let (X, A, B) be a triple of type (H_2), B' an intersection of components of B, and $A' := A \cap B'$. Let $i : A' \hookrightarrow A$, $j : A \backslash B|_A \hookrightarrow A$, $\tilde{i} : A' \hookrightarrow X$, and $\tilde{j} : X \backslash B \hookrightarrow X$, denote the natural inclusions. The induced morphism*

$$\tilde{i}^{-1} R\tilde{j}_* \mathbb{Z} \to i^{-1} R j_* \mathbb{Z}$$

is an isomorphism (in the derived category). In particular, it induces an isomorphism

$$\mathbb{H}^i(A', \tilde{i}^{-1} R\tilde{j}_* \mathbb{Z}) \to \mathbb{H}^i(A', i^{-1} R j_* \mathbb{Z}).$$

Since $(A, A', B|_A)$ is a triple of type (H_1), it follows that $\mathbb{H}^i(A', i^{-1} R j_* \mathbb{Z})$ has a canonical enriched Hodge structure.

Proof. First, consider the cartesian square:

$$\begin{array}{ccc} A\backslash B|_A & \longrightarrow & X\backslash B \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \tilde{j}} \\ A & \xrightarrow{\ i_A\ } & X \end{array}$$

We first note that, since A has normal crossings with B, the natural base change morphism

$$i_A^{-1} R\tilde{j}_* \mathbb{Z} \to R j_* \mathbb{Z}$$

is an isomorphism. For example, this can be checked locally, and therefore we are reduced to where X is a poly-disk. Moreover, if $(x_1, \cdots, x_n)$ denote the coordinates on X, then A is given by $\{x_1 = 0, \ldots, x_i = 0\}$, B is given by $\{x_j \cdots x_k = 0\}$ and $j > i$ (since A is not contained in any component of B). In this case, the result is a restatement of cohomological purity. To be more precise, if we write $X = D_1 \times \cdots \times D_n$ where D_i is the disk with coordinate x_i, then $A = D_{i+1} \times \cdots \times D_n$ and one has $\mathrm{H}^0(X\backslash B, \mathbb{Z}) = \mathbb{Z}$, $\mathrm{H}^1(X\backslash B, \mathbb{Z}) = \mathbb{Z}^{k-j+1}$, and $\mathrm{H}^j(X\backslash B, \mathbb{Z}) \cong \Lambda^j \mathrm{H}^1(X\backslash B, \mathbb{Z})$. One has the same computation for $\mathrm{H}^*(A\backslash B|_A, \mathbb{Z})$. Note that these cohomology groups are the stalks (at a point of A) of the sheaves appearing in the base change morphism above. Finally, applying i^{-1} to the aforementioned isomorphism gives the desired result.

□

We will now construct a simplicial 'triple' computing the cohomology of a triple of type (H_3) whose n-th term is built from triples of type (H_1) and (H_2). Let $\mathcal{A}_\bullet$ denote the simplicial scheme given by the components of A. In particular, $\mathcal{A}_0 := \coprod_i A_i$. In general, $\mathcal{A}_n$ is the disjoint union of terms $A_{i_0\cdots i_n} := A_{i_0} \cap \cdots \cap A_{i_n}$. The natural augmentation $\pi : \mathcal{A}_\bullet \to A$ is a proper hypercover. Moreover, each term $A_{i_0\cdots i_n}$ can be viewed as determining a triple $(X, A_{i_0\cdots i_n}, B)$, and the pull back (induced by the natural inclusion $A_{i_0\cdots i_n} \hookrightarrow A$) of the complex $i^{-1}Rj_*\mathbb{Z}$ to each such term is the corresponding complex on the triple $(X, A_{i_0\cdots i_n}, B)$. Therefore, by cohomological descent, it's enough to upgrade the complex $i^{-1}Rj_*\mathbb{Z}$ on $A_{i_0\cdots i_n}$ corresponding to the triple $(X, A_{i_0\cdots i_n}, B)$ to a cohomological enriched Hodge complex, and to note that these assemble to give a complex of sheaves on the simplicial scheme. An application of Lemma 7.16 will then show that the resulting total complex will have the structure of an cohomological enriched Hodge complex, and therefore its hypercohomology groups will have natural enriched Hodge structures.

For a given $A_{i_0\cdots i_n}$ if $i_j \neq 0$ for all $0 \leqslant j \leqslant n$, then $(X, A_{i_0\cdots i_n}, B)$ is a triple of type (H_1). On the other hand, if $i_j = 0$ for some j, then an application of the previous lemma still allows us to obtain a natural cohomological enriched Hodge complex by considered the associated triple of type (H_2). For example, we may assume that $i_0 = 0$ and that all other $i_j \neq 0$. Then we can consider the associated triple $(A_{i_0}, A_{i_0\cdots i_n}, B|_{A_{i_0}})$, and note that this is a triple of type (H_1). We have constructed a mixed Hodge complex on this triple. As for the enriched part, we shall take complex $i^{-1}\Omega^\bullet_X(\log B)$ where $i : A_{i_0\cdots i_n} \hookrightarrow X$ is a natural embedding. The previous lemma shows that this results in a cohomological enriched Hodge complex on $\mathcal{A}_n$, and the functoriality of the constructions of cohomological enriched Hodge complexes shows that these assemble to give a cohomological enriched Hodge complex on the simplicial scheme. Note that the face and degeneracy maps (on each component of X_n) either have same type triples as domain and range or have a type (H_1) triple mapping to one of type (H_2). In particular, the aforementioned 'functorialities' are sufficient for the construction of the desired cohomological enriched Hodge complex on the simplicial scheme $\mathcal{A}_\bullet$. In particular, one has the following theorem.

Theorem 5.15. *Let (X, A, B) be a triple of type (H_3). Then there is a natural mixed Hodge structure on $\mathbb{H}^m(A, i^{-1}Rj_*\mathbb{Z})$ and an enriched Hodge structure $\mathrm{EH}^m((X, A, B), \mathbb{Z})$ with underlying MHS given by the aforementioned MHS on $\mathbb{H}^m(A, i^{-1}Rj_*\mathbb{Z})$ such that:*

1. *(Functoriality) Given a morphism of triples $f : (X, A, B) \to (X', A', B')$ of type (H_3) such that $f^{-1}(B')$ is a s.n.c.d., the induced map*
$$\mathrm{EH}^m((X', A', B'), \mathbb{Z}) \to \mathrm{EH}^m((X, A, B), \mathbb{Z})$$
is a morphism of enriched Hodge structures.
2. *If there exist open sets $U \supset A$ and $U' \supset A'$ such that f restricts to an isomorphism $U\backslash B \to U'\backslash B'$, then the morphism from part (1) is an isomorphism of mixed Hodge structures.*

(3) *The object of* $\mathcal{C}$ *associated to* $\mathrm{EH}^m((X,A,B),\mathbb{Z})$ *is given by the following diagram in* $\mathcal{C}$*:*

$$\cdots \to \mathbb{H}^m(A, i^{-1}\Omega_X^{<p}(\log B)) \to \mathbb{H}^m(A, i^{-1}\Omega_X^{<p-1}(\log B)) \to \cdots$$

Proof. As noted above, the existence part follows from the preceding construction and an application of Lemma 7.16. We note that the 'enriched part' is simply the restriction to each of the components $A_{i_0\cdots i_n}$ of the (filtered by usual Hodge filtration) complexes $(\Omega^{\cdot}_X(\log B), F)$.

(1) Suppose we are given a morphism of triples of type (H_3) as in the Theorem. Then for each component A_i of A there is a component $A'_{j(i)}$ of A' such that f maps A_i to $A'_{j(i)}$. Note, there may be more than one choice for such a $A'_{j(i)}$, but we fix one choice here. This gives rise to a map of the associated simplicial schemes $f. : \mathcal{A}_\bullet \to \mathcal{A}'_\bullet$ which is compatible with f under the augmentation maps. It follows immediately that the induced map on cohomology

$$f : \mathbb{H}^m(A', i'^{-1}Rj'_*\mathbb{Z}) \to \mathbb{H}^m(A, i^{-1}Rj_*\mathbb{Z})$$

is a morphism of mixed Hodge structures, and moreover compatible with the underlying enriched structure (by the remark regarding the enriched part at the beginning of the proof) .

(2) One can argue as before (i.e. as a consequence of Deligne's mixed Hodge theory) since the induced map on the underlying abelian groups is an isomorphism.

(3) This follows from the definition of the 'enriched part'.

□

Remark 5.16. Let $(X,A,B) \in \mathcal{T}'$ be a triple, and suppose $A = \cup_{i=1}^k A_i$ where A_i are the irreducible components of A. We note that the constructions of this paragraph can be applied to any such triple where the intersections of the components of A are type (H_1) or (H_2). In particular, the analog of Theorem 5.15 holds in this case as well.

5.17. (**Proof of Theorem 5.3**) We shall now complete the proof of Theorem 5.3. We shall proceed in several steps.

(1) (Existence of EHS) Let $(X,A,B) \in \mathcal{T}'$. Then by Proposition 4.5 and Remark 4.4, there is a triple (X',A',B') of type (H_3) and a birational morphism of triples

$$\pi : (X',A',B') \to (X,A,B)$$

such that A' (resp. B') is the total transform of A (resp. B), $\pi_{X'\backslash B'} : X'\backslash B' \to f(X)\backslash B$ is an isomorphism, and $f(X')$ is an open neighborhood of A. As a result, the induced map on cohomology

$$\mathbb{H}^m(A', i'^{-1}Rj'_*\mathbb{Z}) \to \mathbb{H}^m(A, i^{-1}Rj_*\mathbb{Z})$$

is an isomorphism. We define the enriched Hodge structure on the RHS to be the one given by Theorem 5.15 on the LHS.

(2) (Independence of choices of MHS) Suppose we have triples (X_i, A_i, B_i), $i = 1, 2$, each of type (H_3) over the triple (X, A, B). By Proposition 4.5, there exists a diagram

$$\begin{array}{ccc} (X_3, A_3, B_3) & \longrightarrow & (X_1, A_1, B_1) \\ \downarrow & & \downarrow \\ (X_2, A_2, B_2) & \longrightarrow & (X, A, B) \end{array}$$

where $(X_3, A_3, B_3) \in \mathcal{GT}'$ and the induced morphism of triples

$$(X_3, A_3, B_3) \xrightarrow{f} (X, A, B)$$

is an isomorphism over $f(X_3)\backslash B$. Therefore, we may assume that we have a morphism of triples $g : (X_2, A_2, B_2) \to (X_1, A_1, B_1)$ over (X, A, B) (which is birational and an isomorphism over $g(X_2)\backslash B_1$, and $g(X_2)$ is an open neighborhood of A_1) , and in that situation, we want to show that g induces an isomorphism of mixed Hodge structures. Since g induces a morphism mixed Hodge structures and an isomorphism on the underlying cohomology groups, it must be an isomorphism of mixed Hodge structures.

(3) (Independence of choices of EHS) We continue with notation as above. It remains to show g induces an isomorphism of enriched Hodge structures. Note that we may assume that $g \in GS$. Given a triple (X, A, B) as above, let $\mathrm{H}^i_{\mathcal{C}}(X, A, B)$ denote the image in $\mathcal{C}$ of the enriched Hodge structure defined above. Recall, a morphism in GS is a composition of morphisms in GS_1 and GS_2. For morphisms in GS_2, the induced morphism on the corresponding objects of $\mathcal{C}$ is an isomorphism by Corollary 3.6, and the analogous statement for morphisms in GS_1 is clear.

(4) (Functoriality:) Let $f : (X_1, A_1, B_1) \to (X_2, A_2, B_2) \in \mathcal{T}'$ be a morphism of triples and $g_2 : (X'_2, A'_2, B'_2) \to (X_2, A_2, B_2)$ be a triple of type (H_3) over (X_2, A_2, B_2) which is an isomorphism over $g_2(X'_2)\backslash B_2$. Arguing as before, there is a triple of type (H_3) $(X'_1, A_{1'}, B'_1)$ and a commutative diagram:

$$\begin{array}{ccc} (X'_1, A_{1'}, B'_1) & \xrightarrow{f'} & (X'_2, A'_2, B'_2) \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ (X_1, A_1, B_1) & \xrightarrow{f} & (X_2, A_2, B_2) \end{array}$$

where $g_i \in S$. The result now follows from the functoriality for triples of type (H_3).

6. Future directions/projects

Here we will give an outline of certain applications of the machinery which has been developed in this article, which we expect to develop in a subsequent work.

As before, let (X, A, B) be a triple where X is a complex analytic space and A, B are closed analytic subvarieties of X such that A is a proper algebraic variety, and $X\backslash B$, $A\backslash B$ are both non-singular. Recall, we are concerned with the system of open neighbourhoods of A in X and we view any specific triple (X, A, B) as a representative of a "germ" of neighborhoods of A.

By the Main Theorem, associated to such a triple we have enriched Hodge structures $\mathrm{EH}^m((X, A, B), \mathbb{Z})$ corresponding to natural mixed Hodge structures on the cohomology groups $\mathrm{H}^m(X, i^{-1}Rj_*\mathbb{Z}_{X\backslash B})$, $m \geqslant 0$ (and their Tate twists). If X is smooth and B is an s.n.c.d., then the object of $\mathcal{C}$ associated to this EHS results by considering the hypercohomology of the (truncated) logarithmic de Rham complex on the ambient space X with log poles along B. For a general triple, the Main Theorem yields that, if we replace this triple by suitably blowing up within B to obtain a new triple with the extra smoothness and s.n.c.d. conditions, the resulting EHS is independent of that choice.

Our construction of the EHS, and the underlying MHS is in fact via a construction of a suitable cohomological Enriched Hodge complex (a natural generalization of a cohomological Mixed Hodge complex). As a result, this allows us to construct, via a mapping cone construction, a version of Deligne-Beilinson cohomology, which we call *Enriched Deligne-Beilinson cohomology* groups, with appropriate Tate twists, and denote by $\mathrm{EDB}^i((X, A, B), \mathbb{Z}(j))$ in the following. By construction, these which will then fit into short exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}, \mathrm{EH}^{i-1}((X, A, B), \mathbb{Z}(j)) \to \mathrm{EDB}^i((X, A, B), \mathbb{Z}(j)) \to \mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}, \mathrm{EH}^i(X(X, A, B), \mathbb{Z}(j)) \to 0$$

(for various indices i, j).

We indicate briefly the construction of EDB cohomology. Let (X, A, B) be a triple of type (H_3). Recall, in this case the object of $\mathcal{C}$ associated with $\mathrm{EH}^i((X, A, B), \mathbb{Z})$ results by considering the complex $i^{-1}\Omega^\bullet_X(\log B)$. The construction of EDB cohomology in this setting parallels that of the definition of Deligne-Beilinson cohomology in the open setting. More precisely, one has the natural morphisms

$$\iota : \Omega^{\geqslant p}_X(\log B) \to \Omega^\bullet_X(\log B))$$

and

$$\epsilon : Rj_*\mathbb{Z}(p) \to \Omega^\bullet_X(\log B)$$

in the derived category of complexes of sheaves. Consider the cone of the resulting morphism restricted to A:

$$C(p) := Cone(i^{-1}Rj_*\mathbb{Z}(p) \oplus i^{-1}\Omega_X^{\geqslant p}(\log B) \xrightarrow{\epsilon-\iota} i^{-1}\Omega_X^{\bullet}(\log B)),$$

and set $\mathrm{EDB}^i((X,A,B),\mathbb{Z}(j)) := \mathbb{H}^i(A, C(p))$. By definition, one obtains a long exact sequence:

$$\cdots \to \mathrm{EDB}^i((X,A,B),\mathbb{Z}(p)) \to \mathrm{H}^i(A, i^{-1}Rj_*\mathbb{Z}(p)) \to \mathrm{H}^i(A, i^{-1}Rj_*\mathbb{C})/F^p \to \cdots.$$

Here F^p is the filtration induced by the filtration $\Omega_X^{\geqslant p}(\log B)$ on hypercohomology. For an arbitrary triple, one proceeds by replacing it with a triple of type (H_3) as in the construction of EHS.

This is formally very similar to the construction of Deligne-Beilinson cohomology. We note however that the $\mathrm{Ext}^1_{\mathrm{EHS}}$ groups contain vector spaces, which are in general infinite dimensional. For example, consider the triple $(X,A,B) = (\mathbb{C}^n, \{0\}, \varnothing)$. Then the sections of the structure sheaf $\mathcal{O}_X$, when we then pass to the direct limit over neighbourhoods of $A = \{0\}$, yields the ring of convergent power series in n complex variables. While the corresponding, MHS will be more or less trivial (since A is a point), the EHS will involve terms of the de Rham complex of the ring of convergent power series.

The next step is to consider certain *algebraic K-groups* which can be associated to a triple (X,A,B) (which we as usual view as a representative for the "germ" of A in X), denoted by $\mathrm{K}_i((X,A,B))$, and defined as follows. Consider the abelian category of coherent analytic sheaves on X, which contains the Serre subcategory of coherent sheaves supported on B (equivalently, which have vanishing stalk at any point of $X\backslash B$). The quotient abelian category has a full exact subcategory consisting of objects obtained from coherent sheaves on X which are locally free when restricted to $X\backslash B$. Let $\mathcal{P}(X,B)$ denote the exact category obtained by this construction for a representative triple (X,A,B). We then define

$$\mathrm{K}_i((X,A,B)) = \varinjlim_{A\subset U\subset X} \mathrm{K}_i(\mathcal{P}(U, U\cap B)),$$

where the K_i denote the Quillen K-groups of an exact category (and the limit is over all open neighborhoods of A in X).

One can also use the Waldhausen framework in order to define the K-groups of a triple. In particular, for a representative triple (X,A,B), consider the bounded derived category of coherent analytic sheaves on X, with its usual Waldhausen structure. This contains the full triangulated subcategory $\mathcal{D}(X,B)$ of complexes which restrict to a perfect complex on $X\backslash B$; this in turn has a thick subcategory $\mathcal{D}_B(X)$ of complexes with cohomology supported in B. Then we may take

$$\mathrm{K}_i((X,A,B)) = \varinjlim_{A\subset U\subset X} \mathrm{K}_i(\mathcal{D}(U, U\cap B)/\mathcal{D}_{U\cap B}(U)).$$

It seems likely that there is a canonical identification between the groups $\mathrm{K}_i((X,A,B))$ given by either of the definitions; we thank Amnon Neeman for a suggestion about how to go about establishing this. We will assume this is the case, and use the notation $\mathrm{K}_i((X,A,B))$ to be the resulting groups. These will be contravariantly functorial for maps of triples $f : (X',A',B)' \to (X,A,B)$, which we take to be complex analytic maps $f : X' \to X$ (for suitable representatives) such that $f(A') \subset A$, and $f^{-1}(B) \subset B'$ (so that we have an induced restriction $X'\backslash B' \to X\backslash B$). We note also that the $\mathrm{EH}^r((X,A,B),\mathbb{Z}(s))$ are also contravariant functorial for the same morphisms of triples, as seen earlier.

Having discussed K-groups, we then expect to construct an appropriate theory of *Chern classes* and *Chern character* (with values in enriched Deligne-Beilinson cohomology), which exactly parallels the known constructions for such classes with domain algebraic K-theory of varieties and with values in Deligne-Beilinson cohomology (as in [1], and explained in detail in [12]).

Thus, we expect to construct Chern class maps

$$c_{i,j} : \mathrm{K}_i((X,A,B)) \to \mathrm{EDB}^{2j-i}((X,A,B),\mathbb{Z}(j)),$$

and similarly Chern character (component) maps

$$ch_{i,j} : \mathrm{K}_i((X,A,B)) \otimes \mathbb{Q} \to \mathrm{EDB}^{2j-i}((X,A,B),\mathbb{Q}(j),$$

where $ch_{i,j}$ are homomorphisms, while $c_{i,j}$ are homomorphisms for $i > 0$, and these satisfy the usual relations between Chern classes and components of the Chern character. We also expect the $ch_{i,j}$ to factor through an appropriate *weight j eigenspace* for the Adams operations, which are naturally defined on the $\mathrm{K}_i((X,A,B))$. The construction, properties and computations with these classes are the main goal of this project; in particular, we expect the Chern class maps to capture non-trivial and interesting information about the groups $\mathrm{K}_i((X,A,B))$.

We now discuss a few examples where this theory can be worked out directly by hand, which gives the flavour of what aspects of the K-groups our EHS's may describe. In particular, these examples show that interesting and subtle geometric features are naturally reflected in the computation and properties of the EHS's.

Example 6.1. Let $(X,A,B) = (\mathbb{C}^n,\{0\},\varnothing)$ be the germ at the origin of $\mathbb{C}^n$. We note that if R_n is the ring of convergent power series in n complex variables (so that R_n is the local ring $\mathcal{O}^{an}_{\mathbb{C}^n,0}$), then $\mathrm{K}_i((X,A,B))$ is just $\mathrm{K}_i(R_n)$.[2] As is well known, $\mathrm{K}_1(R_n) = R_n^\times$. Note that, in this case, the underlying MHS $\mathbb{H}^i(A,i^*\mathbb{Z}) = \mathrm{H}^i(A,\mathbb{Z}) = 0$ for $i > 0$, and $\mathbb{Z}$ for $i = 0$. Moreover, the object of $\mathcal{C}$ associated to the EHS is given by

[2] This is a consequence of the definition of the K-theory of a triple and the fact that K-theory commutes with direct limits of exact categories.

$V_p := \mathrm{H}^i(R_n \to \cdots \to \Omega^{p-1}_{R_n})$ (i.e. the truncated de Rham complex of converent power series).

Consider the map:

$$c_{1,1} : \mathrm{K}_1(R_n) \to \mathrm{EDB}^1((X,A,B),\mathbb{Z}(1)). \tag{4}$$

and the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^0((X,A,B),\mathbb{Z}(1))) \to \mathrm{EDB}^1((X,A,B),\mathbb{Z}(1)) \to \mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^1((X,A,B),\mathbb{Z}(1)) \to 0. \tag{5}$$

As an application of Proposition 2.13, one notes that

$$\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^1((X,A,B),\mathbb{Z}(1)) = 0$$

(since the underlying MHS is 0), and $\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^0((X,A,B),\mathbb{Z}(1)))$ is given by the cokernel of the map $\mathbb{Z} \xrightarrow{2\pi i} R_n$, which can be identified with the units $R_n^\times$ via the exponential map.

Example 6.2. Continuing with the previous example, consider now the map:

$$c_{2,2} : \mathrm{K}_2((X,A,B) \to \mathrm{EDB}^2((X,A,B),\mathbb{Z}(2)). \tag{6}$$

and the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^1((X,A,B),\mathbb{Z}(2))) \to \mathrm{EDB}^2((X,A,B),\mathbb{Z}(2)) \to \mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^2((X,A,B),\mathbb{Z}(2)) \to 0. \tag{7}$$

As in the previous example,

$$\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^2((X,A,B),\mathbb{Z}(2))) = 0$$

since A is a point. Since the underlying MHS of $\mathrm{EH}^1((X,A,B),\mathbb{Z}(2)) = 0$, we have that $\mathrm{EH}^1((X,A,B),\mathbb{Z}(2))$ is completely determined by a sequence of $\mathbb{C}$ vector spaces V_j, obtained from truncations of the de Rham complex of the germ $(\mathbb{C}^n,0)$ i.e. the de Rham complex of R_n. In particular, we find that

$$\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z},\mathrm{EH}^1((X,A,B),\mathbb{Z}(2)) \cong \mathrm{coker}\left(d : R_n \to \Omega^1_{R_n})\right).$$

Thus our map in (6) is of the form

$$\mathrm{K}_2(R_n) \to \mathrm{coker}\left(d : R_n \to \Omega^1_{R_n}\right).$$

Indentifying this cokernel with the *germs of closed 2-forms* $Z\Omega^2_{R_n}$, one finds that on Steinberg symbols $\{f,g\}$ with $f,g \in R_n^*$, we have

$$c_{2,2}(\{f,g\}) = \frac{1}{(2\pi i)^2}\mathrm{dlog}\,(f) \wedge \mathrm{dlog}\,(g).$$

It is thus easy to see that $c_{2,2}$ is surjective, and that elements of $\mathrm{K}_2(R_n)$ in the image of $\mathbb{C}^* \otimes R_n^* \to \mathrm{K}_2(R_n)$ lie in the kernel.

Remark 6.3. We may compare this with a computation of Bloch [2], showing that if $M_n \subset R_n$ is the maximal ideal, and $R_n(N) = R_n/M_n^N$, then

$$\ker(\mathrm{K}_2(R_n(N)) \to \mathrm{K}_2(\mathbb{C})) \cong \frac{\ker \Omega^1_{R_n(N)/\mathbb{Z}} \to \Omega^1_{\mathbb{C}/\mathbb{Z}}}{d(M_n^N)}.$$

Here we have the absolute Kahler differentials appearing on the right.

Example 6.4. Let $(\bar{X}, \bar{A}, \bar{B})$ be a representative of $(\mathbb{C}^2, \{0\}, \bar{C})$, where $\bar{C} = \bigcup_{i=1}^n \bar{C}_i$ are irreducible curve germs through the origin, contained in the germ at 0 of $\mathbb{C}^2$. Note that $\bar{A} = \{0\}$. Moreover, we can and do assume that for the given representative $\bar{B} = \bigcup_{i=1}^n \bar{B}_i$ where $\bar{B}_i$ is a representative for the germ $\bar{C}_i$, the $\bar{B}_i$ meet only at 0, and $\bar{B}_i\backslash 0$ is smooth. In general this is not a good triple, so we may perform a sequence of point blow ups at points lying over the origin, to obtain

(1) a nonsingular complex surface X with a proper birational map $\pi : X \to \bar{X}$ and with an exceptional divisor $E = \bigcup_{l=1}^t E_l$, which is a union of irreducible smooth rational curves E_l, whose dual graph forms a tree; $X\backslash E \to \bar{X}\backslash\{0\}$ is an isomorphism of surfaces
(2) the strict transform in X of $\bar{B} = \cup_{i=1}^n \overline{B_i}$ is a disjoint union or irreducible smooth curves $\bigcup_{i=1}^n B_i$, with $B_i \to \bar{B}_i$ being the normalization (which is bijective and birational)
(3) the divisor $B = E + \sum_i B_i$ has simple normal crossings, and $X\backslash B \cong \bar{X}\backslash\bar{B}$.

The triple (X, A, B) with $A = E$ is a good triple, which may be used to compute both the MHS and the EHS associated to $(\bar{X}, \bar{A}, \bar{B})$ in our Main Theorem.

We next note that the K-groups which we associate to this triple are $\mathrm{K}_i(R_f)$, where $R = R_2$ is the convergent power series ring in 2 variables, and fR is the ideal of the curve germ $\bar{C}$. Our goal is to make more explicit the map

$$c_{2,2} : \mathrm{K}_2((\bar{X}, \bar{A}, \bar{B}) \to \mathrm{EDB}^2((\bar{X}, \bar{A}, \bar{B})(2),$$

which is then a map

$$c_{2,2} : \mathrm{K}_2(R_f) \to \mathrm{EDB}^2((X, A, B), \mathbb{Z}(2)).$$

Now in the exact sequence (7) as above, the MHS associated to $\mathrm{EH}^i((X, A, B), \mathbb{Z})$ is generally nontrivial for both of $i = 1, 2$. We compute these explicitly using the geometry of the situation. This MHS is determined, as an abelian group, by the formula

$$\mathrm{H}^i(A, i^*Rj_*\mathbb{Z}) = \varinjlim_{A\subset U\subset X} \mathrm{H}^i(U\backslash B, \mathbb{Z}).$$

Here $j : X\backslash B \hookrightarrow X$ and $i : A \hookrightarrow X$ are the inclusion maps. Consider the Leray spectral sequence for j,

$$\mathrm{E}_2^{r,s} = \mathrm{H}^r(X, R^s j_*\mathbb{Z}) \Rightarrow \mathrm{H}^{r+s}(X\backslash B, \mathbb{Z}),$$

and then take the direct limit over U of the similar spectral sequences for the inclusions $j_U : U\backslash B \to U$. This leads to a limiting spectral sequence

$$\mathrm{E}_2^{r,s} = \varinjlim_{A\subset U\subset X} \mathrm{H}^r(U, R^s j_*\mathbb{Z}\mid_U) \Rightarrow \mathrm{H}^{r+s}(A, i^* Rj_*\mathbb{Z}).$$

Since we are interested in $\mathrm{H}^i(A, i^*Rj_*\mathbb{Z})$ with $i = 1, 2$, we want to determine $\mathrm{E}_\infty^{r,s}$ with $r + s = 1, 2$.

Working with this limiting spectral sequence, we first note that by the proper base change theorem, the direct limit of the cohomology groups $\mathrm{H}^r(U, \mathbb{Z})$ may be identified with $\mathrm{H}^r(E, \mathbb{Z})$, where $\mathrm{E} = \pi^{-1}(0)$ is the exceptional divisor for $\pi : X \to \bar{X}$. This gives a natural identification $\mathrm{E}_2^{r,0} \cong \mathrm{H}^r(E, \mathbb{Z})$.

Next we compute $R^1 j_*\mathbb{Z}\mid_U$, which is supported on $B \cap U$. By abuse of notation we shall denote by B the restriction $B \cap U$ to U (and similarly for E_l, B_i, and E). Note that for small enough U this is a simple normal crossing divisor which is the union of E and $\bigcup_{i=1}^n B_i$, where each B_i is isomorphic to a unit disc in $\mathbb{C}$. It follows that:

$$R^1 j_*\mathbb{Z}\mid_U \cong (\oplus_{i=1}^n \mathbb{Z}_{B_i}) \oplus (\oplus_{l=1}^t \mathbb{Z}_{E_l},$$
$$R^2 j_*\mathbb{Z}\mid_U \cong \oplus_{P\in S}\mathbb{Z}_P,$$
$$R^s j_*\mathbb{Z} = 0 \ \ \forall\ s \geqslant 3,$$

where S denotes the finite set of points of $B \cap U$ where pairs of components intersect (these are special cases of the general description of $R^s j_*\mathbb{Z}$ where j is the inclusion of the complement of a divisor with simple normal crossings). We note that S has cardinality $n + t - 1$, since the dual graph of $E = \bigcup_{l=1}^t E_l$ is a tree, and the B_i are pairwise disjoint and meet E transversally at smooth points.

From this, we obtain the following information about the limiting spectral sequence:

$$\mathrm{E}_2^{r,s} = 0 \text{ if } s \geqslant 3, \text{ or if } r \geqslant 3, \text{ or if } r > 2, s = 1, \text{ or if } r > 0, s = 2$$
$$\mathrm{E}_2^{2,0} \cong \mathrm{H}^2(E, \mathbb{Z}) \cong \oplus_{l=1}^t \mathrm{H}^2(E_l, \mathbb{Z}) \cong \mathbb{Z}^t$$
$$\mathrm{E}_2^{1,1} = 0 \text{ (since each } E_l \cong \mathbb{P}^1_{\mathbb{C}}, \text{ and each } B_i \text{ is isomorphic to a unit disc in } \mathbb{C})$$
$$\mathrm{E}_2^{0,2} \cong \oplus_{P\in S}\mathbb{Z} \cong \mathbb{Z}^{n+t-1}$$
$$\mathrm{E}_2^{2,1} \cong \oplus_{l=1}^t \mathrm{H}^2(E_l, \mathbb{Z}) \cong \mathbb{Z}^t$$
$$E_2^{1,0} = 0 \text{ (since } H^1(E, \mathbb{Z}) = 0, \text{ from the description of } E \text{ as a tree of smooth rational curves)}$$
$$\mathrm{E}_2^{0,1} \cong (\oplus_{i=1}^n \mathrm{H}^0(B_i, \mathbb{Z})) \oplus (\oplus_{l=1}^t \mathrm{H}^0(E_l, \mathbb{Z})) \cong \mathbb{Z}^{n+t}.$$

Now we consider the differentials. First note that $d_2^{0,2} : \mathrm{E}_2^{0,2} \to \mathrm{E}_2^{2,1}$ has kernel $\mathrm{E}_\infty^{0,2}$ and cokernel $\mathrm{E}_\infty^{2,1}$. But $\mathrm{H}^3(X\backslash B, \mathbb{Z}) \cong H^3(\bar{X}\backslash \bar{B}, \mathbb{Z}) = 0$. Hence $d_2^{0,2}$ is surjective, and we conclude that

$$\mathrm{E}_\infty^{0,2} \cong \mathbb{Z}^{n-1}.$$

Next, consider $d_2^{0,1} : \mathrm{E}_2^{0,1} \to \mathrm{E}_2^{2,0}$. Here $\mathrm{E}_2^{2,0} \cong \mathrm{H}^2(E, \mathbb{Z})$ using the proper base change isomorphism $\mathrm{H}^2(U, \mathbb{Z}) \to \mathrm{H}^2(E, \mathbb{Z})$ for small enough U. The differential $d_2^{0,1}$ then is a map

$$(\oplus_{i=1}^n \mathrm{H}^0(B_i \mathbb{Z}) \oplus (\oplus_{l=1}^t \mathrm{H}^0(E_l, \mathbb{Z}) \to \mathrm{H}^2(U, \mathbb{Z}),$$

which is known to be a sum of Gysin maps $\mathrm{H}^0(B_i, \mathbb{Z}) \to \mathrm{H}^2(U, \mathbb{Z})$ and $\mathrm{H}^0(E_l, \mathbb{Z}) \to \mathrm{H}^2(U, \mathbb{Z})$ associated to the inclusions $B_i \subset U$, $E_l \subset U$ (this is part of the description of the Leray spectral sequence for j, the inclusion of the complement of a divisor with simple normal crossings). Then composing with the isomorphism $\mathrm{H}^2(U, \mathbb{Z}) \cong \mathrm{H}^2(E, \mathbb{Z}) = \oplus_{l=1}^t \mathrm{H}^2(E_l, \mathbb{Z}) \cong \mathbb{Z}^t$, we see that the maps

$$\mathrm{H}^0(B_i, \mathbb{Z}) \to \oplus_{l=1}^t \mathrm{H}^0(E_l, \mathbb{Z})$$

are given by $1 \mapsto \sum_{l=1}^t (B_i \cdot E_l)$, which is a vector with exactly one nonzero component equal to 1 (since B_i meets E transversally at 1 point). Similarly on the summand $\mathrm{H}^0(E_l, \mathbb{Z})$ it is determined by the intersection numbers of that E_l with each of the components of E. In particular, we see that $d_2^{0,1}$ is surjective, with kernel isomorphic to $\mathbb{Z}^n$, since the intersection matrix of the components of E is a unimodular (negative definite) matrix (this is because $\pi : X \to \bar{X}$ has exceptional locus E supported over 0, and $\bar{X}$ is a nonsingular surface). Thus

$$\mathrm{E}_\infty^{0,1} \cong \mathbb{Z}^n.$$

Hence we have determined that $\mathrm{H}^1(A, i^* Rj_* \mathbb{Z}) \cong \mathbb{Z}^n$, $\mathrm{H}^2(A, i^* Rj_* \mathbb{Z}) \cong \mathbb{Z}^{n-1}$. If we then keep track more carefully of the Hodge structures, we find that in fact both these MHS's are pure, and we have

$$\mathrm{H}^1(A, i^* Rj_* \mathbb{Z}) \cong \mathbb{Z}(-1)^n,$$
$$\mathrm{H}^2(A, i^* Rj_* \mathbb{Z}) \cong \mathbb{Z}(-2)^{n-1}.$$

Here the Tate twists are determined by the (pure) Hodge structures on cohomology of certain smooth proper varieties (here $\mathrm{H}^0(\text{point})$ or $\mathrm{H}^2(\mathbb{P}^1_{\mathbb{C}})$) and also twists arising in Gysin maps.

Next we consider the $\mathrm{EH}^i((X, A, B), \mathbb{Z}(2))$. For $i = 2$ we find

$$\mathrm{Hom}_{\mathrm{EHS}}(\mathbb{Z}, \mathrm{EH}^2((X, A, B), \mathbb{Z}(2)) \cong \mathbb{Z}^{n-1},$$

while for $i = 1$ we see that

$$\mathrm{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}, \mathrm{EH}^1(X, A, B), \mathbb{Z}(2)) \cong$$
$$\mathrm{coker}\left(\mathbb{Z}(1)^n \to \varinjlim_U \mathbb{H}^1\left(U, \mathcal{O}_U \to \Omega^1_U(\log(B \cap U))\right)\right).$$

We analyze further the description of the latter. Note that there is an exact sequence

$$\mathrm{H}^0(U, \mathcal{O}_U) \to \mathrm{H}^0(U, \Omega^1_U(\log(B \cap U)) \to \mathbb{H}^1\left(U, \mathcal{O}_U \to \Omega^1_U(\log(B \cap U)\right) \to \mathrm{H}^1(U, \mathcal{O}_U)$$

and the last term is 0. We also have an exact sequence of (coherent analytic) sheaves

$$0 \to \Omega^1_U \to \Omega^1_U(\log(B \cap U)) \to (\oplus_{i=1}^n \mathcal{O}_{B_i}) \oplus (\oplus_{l=1}^t \mathcal{O}_{E_l}) \to 0,$$

Hence on taking cohomology, then passing to the limit over U, we obtain

$$0 \to \Omega^1_R \to \Omega^1_R(\log \bar{C}) \xrightarrow{\delta} (\oplus_{i=1}^n \widetilde{R/f_iR}) \oplus (\oplus_{l=1}^t \mathbb{C}) \xrightarrow{\theta} R^1\pi_*\Omega^1_X$$

where we regard the last term as the stalk of a skyscraper sheaf, and the second term is a notation for the stalk at 0 of $\pi_*\Omega^1_X(\log B)$. Here $\bar{C}$ has irreducible components $\bar{C}_i$ with ideal f_iR, for $1 \leqslant i \leqslant n$, and $\widetilde{R/f_iR}$ is the normalization of R/f_iR, which is also the stalk at 0 of $\pi_*\mathcal{O}_{B_i}$.

But the map θ is in fact easily seen to be essentially the differential $d_2^{0,1}$, expressed in de Rham cohomology; thus the R^1 skyscraper sheaf has stalk a $\mathbb{C}$-vector space with a basis indexed by the E_l, and the negative definite intersection pairing of the components of the exceptional divisor E implies that we have an exact sequence

$$0 \to \Omega^1_R \to \Omega^1(\log \bar{C}) \xrightarrow{\varphi} \oplus_{i=1}^n \widetilde{R/f_iR} \to 0$$

where φ is obtained by composing the similar earlier map δ with the projection to the direct summand. Here one sees also that

$$\Omega^1_R(\log \bar{C}) = \Omega^1_R \oplus \Sigma_i R\frac{df_i}{f_i}.$$

We note that this R-module is in general not free.

We thus obtain that the vector space associated to the EHS is

$$V = \operatorname{coker} R \to \Omega^1_R(\log \bar{C})$$

and we have an exact sequence

$$0 \to \frac{\Omega^1_R}{d(R)} \to V \to \oplus_{i=1}^n \widetilde{R/f_iR} \to 0.$$

Taking account now of the integral lattice of the MHS, this then yields an exact sequence

$$0 \to \frac{\Omega^1_R}{d(R)} \to \operatorname{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}, \mathrm{EH}^1((X, A, B), \mathbb{Z}(2)) \to (\oplus_{i=1}^n \widetilde{R/f_iR})/(2\pi i)\mathbb{Z} \to 0$$

which we may view, upon composing with an exponential mapping, as

$$0 \to \frac{\Omega^1_R}{d(R)} \to \operatorname{Ext}^1_{\mathrm{EHS}}(\mathbb{Z}, \mathrm{EH}^1((X, A, B), \mathbb{Z}(2)) \to \oplus_{i=1}^n \widetilde{R/f_iR}^* \to 0$$

where the last term is a sum of unit groups.

We may compare this with the localization sequence in K-theory

$$0 \to \mathrm{K}_2(R) \to \mathrm{K}_2(R[\frac{1}{\prod_i f_i}]) \to \mathrm{G}_1(R/(\prod_i f_i)R) \to 0$$

where G_1 is the Grothendieck group of finitely generated modules over the Noetherian ring $R/(\prod_i f_i)R$. We then further compute with a localization sequence for $R/(\prod_i f_i)R$ and its total quotient ring, which is just $\oplus_{i=1}^n Q(\widetilde{R/f_iR})$ where $Q(-)$ denotes the quotient field. This gives

$$0 \to \mathrm{G}_1(R/(\prod_i F_i)R) \to \oplus_{i=1}^n Q(\widetilde{R/f_iR})^* \xrightarrow{\partial} \mathbb{Z} \to 0,$$

where the last $\mathbb{Z}$ is G_0 of the residue field of $R/(\prod_i f_i)R$. We see easily that ∂ is the sum of the discrete valuations on each of the fields $Q(\widetilde{R/f_iR})$, and so we have an exact sequence

$$0 \to \oplus_{i=1}^n \widetilde{R/f_iR}^* \to \mathrm{G}_1(R/(\prod_i f_i)R) \to \mathbb{Z}^{n-1} \to 0$$

Thus we have constructed a surjective homomorphism

$$\alpha : \mathrm{K}_2(R[\frac{1}{\prod_i f_i}]) \to \mathbb{Z}^{n-1}$$

and its kernel fits into an exact sequence

$$0 \to \mathrm{K}_2(R) \to \ker(\alpha) \to \oplus_{i=1}^n \widetilde{R/f_iR}^* \to 0.$$

This is consistent with the above maps

$$c_{2,2} : \mathrm{K}_2(R) \to \frac{\Omega^1_R}{d(R)} = \mathrm{EDB}^2((X, A, \varnothing), \mathbb{Z}(2))$$

and

$$c_{2,2} : \mathrm{K}_2(R[\frac{1}{\prod_i f_i}]) \to \mathrm{EDB}^2((X, A, B), \mathbb{Z}(2)).$$

7. Appendix: Mixed hodge complexes and enriched mixed hodge complexes

We fix some notations and conventions regarding mixed Hodge structures and mixed Hodge complexes which will be used in the rest of the article. This section can be skipped and referred to as needed.

7.1. (**Filtered Derived Categories**) Let A be a commutative ring. In the following, $\mathrm{D}^+(A)$ will denote the bounded (below) derived category of A-modules. We let $\mathrm{D}^+\mathrm{F}(A)$ and $\mathrm{D}^+\mathrm{F}_2(A)$ denote the corresponding derived categories of filtered (resp. doubly filtered) complexes defined as in ([5], Section 7). Similarly, for a topological space X, we let $\mathrm{D}^+(X, A)$ (resp. $\mathrm{DF}^+(X, A), \mathrm{DF}_2^+(Y, A)$) denote the bounded below derived category of sheaves of (resp. filtered, doubly filtered) A-modules. One constructs these categories by first forming the usual homotopy category of filtered complexes (where homotopies of filtered chain complexes are required to be morphisms of filtered complexes), and then localizing with respect to filtered quasi-isomorphisms (i.e. morphisms such that the induced map on associated graded is a quasi-isomorphism).

If $f : X \to Y$ is a continuous map, then one can define in the usual way (for example via Godement resolutions) derived functors $Rf_* : \mathrm{DF}^+(X, A) \to \mathrm{DF}^+(X, A)$ and similarly for the derived category of doubly filtered complexes. Note that taking the associated graded also induces functors between the derived categories of filtered (or doubly filtered) modules and the usual derived category. The construction of the derived push-forward is compatible with taking the associated graded.

7.2. (**Mixed Hodge Structures**) In the following, MHS will denote the category of $\mathbb{Q}$-mixed Hodge structures. In particular, an object $M = (M_{\mathbb{Z}}, (M_{\mathbb{Q}}, W_\cdot), (M_{\mathbb{C}}, F^\cdot)) \in \mathrm{MHS}$ consists of a finitely generated abelian group $M_{\mathbb{Z}}$ with an increasing weight filtration $W_\cdot$ on $M_{\mathbb{Q}} \cong M_{\mathbb{Z}} \otimes \mathbb{Q}$, and a decreasing Hodge filtration $F^\cdot$ on $M_{\mathbb{C}} := M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that the triple $(M_{\mathbb{C}}, F^\cdot, W_\cdot)$ satisfies the usual conditions for a $\mathbb{C}$-mixed Hodge structure ([4]). Let $\mathbb{Z}(-p) \in \mathrm{MHS}$ denote the usual Tate Hodge structures of pure weight $2p$. If $M \in \mathrm{MHS}$, then let $W_k M \in \mathrm{MHS}$ denote the mixed Hodge structure with $W_k M_{\mathbb{Z}}$ given by pulling $W_k M_{\mathbb{Q}}$ to $M_{\mathbb{Z}}$ via the natural morphism $M_{\mathbb{Z}} \to M_{\mathbb{Q}}$, and the induced Hodge and weight filtrations.

7.3. (**Mixed Hodge Complexes**) We briefly recall some aspects of the theory of mixed Hodge complexes due to Deligne ([5], Section 8). A *$\mathbb{Z}$-Hodge complex* of weight n is a triple $(K, (K_{\mathbb{C}}, F), \alpha)$ such that $K \in \mathrm{D}^+(\mathbb{Z})$, $(K_{\mathbb{C}}, F) \in \mathrm{D}^+\mathrm{F}(\mathbb{C})$, and $\alpha : K_{\mathbb{C}} \to K \otimes \mathbb{C}$ is an isomorphism. These data are required to satisfy certain axioms, namely that the differential on $K_{\mathbb{C}}$ is strictly compatible with the filtration F, and that F induces a pure Hodge structure of weight $n + k$ on $\mathrm{H}^k(K_{\mathbb{C}})$. A *mixed Hodge complex* (MHC) is a 5-tuple $(K, (K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, F, W), \alpha, \beta)$, where $K \in \mathrm{D}^+(\mathbb{Z}), (K_{\mathbb{Q}}, W) \in \mathrm{D}^+\mathrm{F}(\mathbb{Q}), (K_{\mathbb{C}}, F, W) \in \mathrm{D}^+\mathrm{F}_2(\mathbb{C})$ and $\alpha : K_{\mathbb{Q}} \to K \otimes \mathbb{Q}$ (resp. $\beta : (K_{\mathbb{C}}, W) \to (K_{\mathbb{Q}}, W) \otimes \mathbb{C}$) is an isomorphism in $\mathrm{D}^+(\mathbb{Q})$ (resp. $\mathrm{D}^+\mathrm{F}(\mathbb{C})$). Furthermore, the induced triple $(Gr_n^{\mathrm{W}}(K_{\mathbb{Q}}), (Gr_n^W(K_{\mathbb{C}}), F), \beta)$ is required to be a hodge complex of weight n. The key result here, due to Deligne, states that the cohomology $\mathrm{H}^n(K)$ of a MHC with the induced filtrations $W[n]$ and F where

$$(W[n])_q(\mathrm{H}^n(K_{\mathbb{Q}})) := \mathrm{Im}(\mathrm{H}^n(W_{q-n}K_{\mathbb{Q}}) \to \mathrm{H}^n(K_{\mathbb{Q}}))$$
$$F^p(\mathrm{H}^n(K_{\mathbb{C}})) := \mathrm{Im}(\mathrm{H}^n(F^p K_{\mathbb{C}}) \to \mathrm{H}^n(K_{\mathbb{C}}))$$

gives rise to a mixed Hodge structure on $\mathrm{H}^n(K)$.

Remark 7.4. Here and in what follows, W is always an increasing filtration and F a decreasing filtration.

7.5. (**Cohomological Mixed Hodge Complexes**) A *cohomological Hodge complex* of weight n on a topological space X consists of a triple $(\mathcal{K}, (\mathcal{K}_{\mathbb{C}}, \mathcal{F}), \alpha)$ where $\mathcal{K} \in \mathrm{D}^+(X, \mathbb{Z})$, $(\mathcal{K}_{\mathbb{C}}, \mathcal{F}) \in \mathrm{D}^+\mathrm{F}(X, \mathbb{C})$, and $\alpha : \mathcal{K}_{\mathbb{C}} \to \mathcal{K} \otimes \mathbb{C}$ is an isomorphism. Furthermore, the induced triple $(R\Gamma(X, \mathcal{K}), R\Gamma(X, (\mathcal{K}_{\mathbb{C}}, \mathcal{F})), R\Gamma(\alpha))$ is required to be a Hodge complex of weight

n. A *cohomological mixed Hodge complex* is a 5-tuple $(\mathcal{K}, (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}), (\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}), \alpha, \beta)$, where $\mathcal{K} \in \mathrm{D}^+(X, \mathbb{Z}), (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}) \in \mathrm{D}^+\mathrm{F}(X, \mathbb{Q}), (\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}) \in \mathrm{D}^+\mathrm{F}_2(X, \mathbb{C})$ and $\alpha : \mathcal{K}_{\mathbb{Q}} \to \mathcal{K} \otimes \mathbb{Q}$ (resp. $\beta : (\mathcal{K}_{\mathbb{C}}, \mathcal{W}) \to (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}) \otimes \mathbb{C}$) is an isomorphism in $\mathrm{D}^+(X, \mathbb{Q})$ (resp. $\mathrm{D}^+\mathrm{F}(X, \mathbb{C})$). Furthermore, the associated graded under the weight filtrations should give rise to cohomological hodge complexes of weight n. An important fact is the following result of Deligne:

Theorem 7.6. *(Deligne, [5]) Let* $\mathrm{K} := (\mathcal{K}, (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}), (\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}), \alpha, \beta)$ *be a cohomological mixed Hodge complex on* X*. Then*

$$R\Gamma(\mathrm{K}) := (R\Gamma(\mathcal{K}), R\Gamma(\mathcal{K}_{\mathbb{Q}}, \mathcal{W}), R\Gamma(\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}), R\Gamma(\alpha), R\Gamma(\beta))$$

is a mixed Hodge complex. In particular, the cohomology groups $\mathbb{H}^i(X, \mathcal{K})$ *carries a mixed Hodge structure.*

7.7. (**Sheaves on simplicial spaces**) In the next paragraph, we recall some elements of the theory of mixed Hodge theory on simplicial spaces. Here we recall some element of the theory of sheaves on simplicial spaces. Again, we refer to [5] for details. In the following, $X.$ shall denote a simplicial space. In this setting, one has analogs of the categories defined in 7.1. In particular, let $\mathrm{D}^+(X_\bullet, A)$ (resp. $\mathrm{DF}^+(X_\bullet, A), \mathrm{DF}_2^+(X_\bullet, A)$) denote the bounded below category of sheaves of (resp. filtered, doubly filtered) A-modules. Recall that any space S defines the constant simplicial space $S_\bullet$ where $S_n = S$ for all n. Then, a sheaf on $S_\bullet$ is the same as giving a co-simplicial sheaf of S.

A morphism of simplicial spaces $f : X_\bullet \to Y_\bullet$ induces push-forward functors on the corresponding filtered derived categories. A augmentation of a simplicial spaces $X_\bullet$ over S is a morphism $\epsilon : X_\bullet \to S$ of simplicial spaces where S is considered as a constant simplicial space. In that case, the push-forward of a complex of sheaves (resp. filtered sheaves, doubly-filtered sheaves) on X_d gives rise to a complex of co-simplicial sheaves (resp. filtered, doubly filtered sheaves) on S. In particular, the global sections functor applied to a complex K of sheaves on $X_\bullet$ gives rise to a cosimplicial complex; we denote it by $R\Gamma^\bullet(X_\bullet, K)$ (and similarly for its filtered variations). We denote its associated total complex by $R\Gamma(X_\bullet, K) \in \mathrm{D}^+(A)$. We define the hyper-cohomology of K as $\mathbb{H}^i(X_\bullet, K) := \mathrm{H}^i(R\Gamma(X_\bullet, K))$.

Remark 7.8. In practice, one usually gets an object in $\mathrm{D}^+(X_\bullet, A)$ (or its filtered analogs) by first constructing a genuine complex on each X_n compatible with the simplicial structure. In particular, it is not enough to start with objects of $\mathrm{D}^+(X_n, A)$. In practice, we shall avoid this complication by simply choosing canonical Godement resolutions and, therefore, obtaining canonical complexes representing various derived objects.

7.9. (**Mixed Hodge complexes on simplicial schemes**) One can also define the analogs of cohomological MHC's in the simplicial setting We briefly recall these constructions. As before, the main reference for all results here is [5].

Given a commutative ring A as before, one can define the derived category $\mathrm{DG}^+(A)$ of differentially graded (DG) A-modules. A complex of of DG A-modules can be thought of as a double complex where the first degree is degree of the complex, and the second is given by the DG structure. One can also define filtered analogs $\mathrm{DG}^+F(A)$ and $\mathrm{DG}^+F_2(A)$ ([5], 8.1.10). In particular, one has a notion of DG mixed Hodge complex. This consists of objects $K \in \mathrm{DG}^+(\mathbb{Z})$, $(K_{\mathbb{Q}}, W) \in \mathrm{DG}^+F(\mathbb{Q})$, and $(K_{\mathbb{C}}, W, F) \in \mathrm{DG}^+F_2(\mathbb{C})$ with isomorphism α and β as before. Moreover, one requires that the induced data in each differential degree $(K^{\bullet,n}, W, F)$ gives rise to a mixed Hodge complex.

Similarly, we can define a cosimplicial MHC by using the cosimplicial degree instead of the garden module degree. We refer *loc. cit.* for the details. We note here that the usual functor which sends a cosimplicial A-module to the corresponding DG A-modules, sends cosimplicial MHC's to DG MHC's.

If $X_\bullet$ is a simplicial space, then a cohomological mixed Hodge complex on $X_\bullet$ consists of a 5-tuple $(\mathcal{K}, (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}), (\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}), \alpha, \beta)$, where $\mathcal{K} \in \mathrm{D}^+(X_\bullet, \mathbb{Z}), (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}) \in \mathrm{D}^+\mathrm{F}(X_\bullet, \mathbb{Q})$, $(\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}) \in \mathrm{D}^+\mathrm{F}_2(X_\bullet, \mathbb{C})$ and $\alpha : \mathcal{K}_{\mathbb{Q}} \to \mathcal{K} \otimes \mathbb{Q}$ (resp. $\beta : (\mathcal{K}_{\mathbb{C}}, \mathcal{W}) \to (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}) \otimes \mathbb{C}$) is an isomorphism in $\mathrm{D}^+(X_\bullet, \mathbb{Q})$ (resp. $\mathrm{D}^+\mathrm{F}(X_\bullet, \mathbb{C})$). Moreover, one requires that, for all n, the restriction of this data to X_n is a cohomological mixed Hodge complex.

Recall, applying the global sections functor gives (filtered) cosimplicial modules $R\Gamma^\bullet(X_\bullet, K)$, $R\Gamma^\bullet(X_\bullet, (K_{\mathbb{Q}}, W))$, and $R\Gamma^\bullet(X_\bullet, (K_{\mathbb{C}}, W, F))$. It follows that applying the global sections functor $R\Gamma^\bullet$ to the data of a cohomlogical MHC on $X_\bullet$ gives rise to a cosimplicial MHC (and therefore a DG MHC).

On the other hand, suppose K is a differentially graded MHC. Then the diagonal filtration $\delta(W, L)$ on the total complex $Tot(K)$ is defined as follows:

$$\delta(W, L)_n(Tot(K)^i) := \oplus_{p+q=i} W_{n+q} K^{p,q}.$$

Here q is the cosimplicial degree, and $L^r(Tot(K)) := \oplus_{q \geqslant r} K^{p,q}$ is the usual Leray filtration. One defines the Hodge filtration by setting $F^r(Tot(K)^i) := \oplus_{p+q=i} F^r K^{p,q}$. The following theorem of Deligne is the crucial result in the theory of differentially graded MHC's:

Theorem 7.10. *([5], 8.1.15) Let K be a differentially graded MHC defined by a cosimplicial MHC as above. Then $(Tot(K), \delta(W, L), F)$ give rise to a mixed Hodge complex.*

The theorem implies that, given a cohomological mixed Hodge complex K on a simplicial space $X_\bullet$, its hypercohomology $\mathbb{H}^i(X_\bullet, K)$ has a natural structure of a mixed Hodge structure. The weight filtration is the one induced by the diagonal filtration defined above on the associated total complex $R\Gamma(X_\bullet, K)$.

7.11. (**Pseudo-morphisms**) In order to construct, mixed Hodge complexes on simplicial schemes we require complexes (and not only objects of the derived category) and actual morphisms of complexes between these objects. In this context, it will be useful to consider the following notion of pseudo-morphisms of complexes.

Definition 7.12. Given two bounded complexes K and L in an abelian category, a pseudo-morphism between K and L is a chain of morphisms of complexes

$$K = K_0 \xrightarrow{f} K_1 \leftarrow K_2 \to \cdots \to K_{n+1} = L$$

where all the arrows except possibly f are quasi-isomorphisms. If f is also a quasi-isomorphism, then we say that the pseudo-morphism is a pseudo-isomorphism. A morphism of pseudo-morphisms consists of a sequence of morphisms $K_j \to L_j$ such that the obvious diagrams commute.

The notion of a pseudo-morphism allows one to lift the notion of mixed Hodge complex to actual complexes. We require the data to be defined at the level of complexes, where the role of quasi-isomorphisms is played by pseudo-isomorphisms. In fact, similar remarks apply to the cohomological mixed Hodge complexes as well as their simplicial and DG analogs. We leaves the details to the details (or see [11], Chapter 3). In the following, we shall use the prefix *genuine* in front of mixed Hodge complex (or cohomological mixed Hodge complex etc.) to emphasize that we are working with genuine complexes and pseudo-morphisms as above.

7.13. (**Remarks on constructing genuine MHC's**) In this article, we will construct mixed Hodge structures (and enriched Hodge structures) by first constructing cohomological mixed Hodge complexes on various (simplicial) spaces and then applying the results of Deligne recalled above to obtain a mixed Hodge structure. In practice, it will be important for us to work with genuine cohomological MHC's (i.e. objects in the category of chain complexes rather the derived categoryt) and pseudo-morphisms (see 7.11). Here we recall some examples which explain how we will in practice lift MHC's to genuine MHC's (in a canonical way).

Example 7.14. Let X be a smooth projective variety over $\mathbb{C}$ and $j : U \hookrightarrow X$ an open such that $D := X \backslash U$ is a simple normal crossings divisor. Then the data

$$(Rj_*\mathbb{Z}, (Rj_*\mathbb{Q}, \tau), (\Omega^\bullet_X(\log D), W, F))$$

gives rise to a cohomological mixed Hodge complex on X which gives the usual mixed Hodge structure on $\mathrm{H}^i(U, \mathbb{Z})$. Here W and F are the usual weight and Hodge filtrations, and τ is the usual truncation filtration. One can view this as a genuine cohomological MHC as follows. First, choose the canonical representative of $Rj_*\mathbb{Z}$ given by the Godement resolution, and similarly for $Rj_*\mathbb{Q}$. By abuse of notation, will still denote the corresponding complex by $Rj_*\mathbb{Z}$. We shall apply this to all derived functor appearing below (i.e. they will be represented by the associated Godement resolutions). This gives lifts of all our objects to actual (filtered) complexes. It remains to explain the construction of the pseudo-morphisms representing α and β. Since Godement resolutions are functorial, the construction of α is clear. We explain how to represent the isomorphism $\beta : (Rj_*\mathbb{Q}, \tau) \otimes \mathbb{C} \to (\Omega^\bullet_X(\log B), W)$ as a pseudo-isomorphism. One has a natural diagram:

$$(Rj_*\mathbb{Q}, \tau) \to (Rj_*\mathbb{C}, \tau) \to (Rj_*\Omega^\bullet_U, \tau) \leftarrow (j_*\Omega^\bullet_U, \tau) \leftarrow (\Omega^\bullet_X(\log D), \tau) \to (\Omega^\bullet_X(\log D), W)$$

All the arrows here are filtered quasi-isomorphisms except possibly the left most, which becomes one after tensoring with $\mathbb{C}$.

We note that the construction in the previous example is functorial in (X, U). In particular, in one has a simplicial scheme $X.$ and a simplicial simple normal crossings divisor on $X.$, then the above construction will allow one to obtain a cohomological MHC on $X.$

7.15. (**Enriched Hodge complexes**) In this paragraph, we slightly modify the theory of mixed Hodge complexes and cohomological mixed Hodge complexes to incorporate the theory of enriched Hodge structures.

An *enriched Hodge complex* (EHC) is a mixed Hodge complex plus an additional object $(K'_{\mathbb{C}}, \mathrm{F}') \in \mathrm{D}^+\mathrm{F}(\mathbb{C})$ and a morphism $\gamma : (K'_{\mathbb{C}}, F') \to (K_{\mathbb{C}}, F)$ such that the induced triples $(V, \mathrm{H}^i(K), f)$, where $V_p := \mathrm{H}^i(K'_{\mathbb{C}})/F'^p$ and f is induced via γ, are enriched Hodge structures. Note that if $\mathrm{H}^i(K'_{\mathbb{C}})$ are finite dimensional, then $V_p := \mathrm{H}^i(K'_{\mathbb{C}})/F'^p$ is automatically an object of $\mathcal{C}$. Therefore, checking the above condition amounts to checking that γ is a quasi-isomorphism. Here F'^p is the usual induced filtration on $\mathrm{H}^i(K)$.

Given an enriched Hodge complex K, its cohomology groups $\mathrm{H}^i(K)$ are naturally equipped with an enriched Hodge structure. The mixed Hodge structure part is simply the mixed Hodge structure on $\mathrm{H}^i(K)$ associated to the underlying mixed Hodge complex.

One can similarly define the notion of a cohomological enriched Hodge complex on a space X. This consists of data of filtered complexes $(\mathcal{K}, (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}), (\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}), (\mathcal{K}'_{\mathbb{C}}, \mathcal{F}'))$ where the triple $(\mathcal{K}, (\mathcal{K}_{\mathbb{Q}}, \mathcal{W}), (\mathcal{K}_{\mathbb{C}}, \mathcal{F}, \mathcal{W}))$ is a cohomological mixed Hodge complex such that $(R\Gamma(X, \mathcal{K}), R\Gamma(X, (\mathcal{K}_{\mathbb{Q}}, \mathcal{W})), R\Gamma(X, (\mathcal{K}_{\mathbb{C}}, \mathcal{W}, \mathcal{F})), R\Gamma(X, (\mathcal{K}'_{\mathbb{C}}, \mathcal{F}')))$ is an enriched Hodge complex. One can also define cohomological enriched Hodge complexes on simplicial spaces $X.$ in the same way as for usual cohomlogical MHC's on simplicial spaces. Finally, one can also define DG enriched Hodge complexes. We record the following observation for future use.

Lemma 7.16. *Suppose K is a DG enriched Hodge complex. Then the associated total complex*

$$(Tot(K), (Tot(K_{\mathbb{Q}}), \delta(W, L)), (Tot(K_{\mathbb{C}}), \delta(W, L), F), (Tot(K'_{\mathbb{C}}), F'))$$

is an enriched hodge complex.

Proof. At the level of mixed Hodge complexes this is the result of Deligne recalled in Theorem 7.10. Therefore, it is enough to show that the induced morphism $Tot(K_{\mathbb{C}}) \to$

$Tot(K'_{\mathbb{C}})$ is a quasi-isomorphism. But, this follows from the comparison of the spectral sequences associated to the double complexes $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$, and noting that the condition for being a DG enriched Hodge complex requires that the induced maps $K_{\mathbb{C}}^{\bullet,n} \to K_{\mathbb{C}}^{\prime\bullet,n}$ are quasi-isomorphisms for each n. □

We conclude this section by noting that there is an obvious notion of *genuine* enriched Hodge complexes (and its cohomological variants) analogous to the case of mixed Hodge complexes. One requires an actual filtered complex representing $(K'_{\mathbb{C}}, F')$ and that γ is a pseudo-isomorphism.

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