Lecture Notes on Random Walks in Random Environments

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This lecture notes arose out of a mini-course I taught in January 2013 at Instituto Nacional de Matemática pura e Aplicada (IMPA) in Rio de Janeiro, Brazil. In these lecture notes I do not always give all the details of the proofs, nor do I prove all the results in their greatest generality. A more detailed treatment of most of these topics can be found in Zeitouni's lecture notes on RWRE [Zei04].

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1 Introduction to RWRE

We begin with a very brief introduction into the model of RWRE. For simplicity, we will begin by describing the model of nearest-neighbor RWRE on $\mathbb{Z}$. Once that model is understood it is easy for the reader to understand how to define RWRE on other graphs such as multi-dimensional integer lattices, trees, and other random graphs.

The case of one-dimensional RWRE is the simplest to describe since in that case an environment is an elment $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$. For any environment ω and any $x \in \mathbb{Z}$, we can construct a Markov chain X_n on $\mathbb{Z}$ with distribution given by P_ω^x defined by $P_\omega^x(X_0 = x) = 1$ and

$$P_\omega^x(X_{n+1} = z \,|\, X_n = y) = \begin{cases} \omega_y & z = y+1 \\ 1-\omega_y & z = y-1 \\ 0 & \text{otherwise.} \end{cases}$$

Since we will often be concerned with RWRE starting at $x = 0$ we will use the notation P_ω for P_ω^0.

Considering a random walk in an arbitrary environment is obviously too general, and so we wish to give some additional structure to the environment by assuming that the environment ω is an Ω-valued random variable with distribution P on the space of environments Ω. Then, since for any fixed event G for the random walk, $P_\omega^x(G)$ is a $[0,1]$-valued random variable since ω is random. Thus, we can define another probability measure $\mathbb{P}^x$ on X_n by

$$\mathbb{P}^x(\cdot) = E_P[P_\omega^x(\cdot)].$$

Again, for simplicity we will use the notation $\mathbb{P}$ for $\mathbb{P}^0$. In general, distirbution on environments is assumed to be such that the sequence $\{\omega_x\}_{x\in\mathbb{Z}}$ is stationary and ergodic. However, as an introduction to the model it is often best to consider the simplest example where the environment $\{\omega_x\}$ is an i.i.d. sequence.

Since there are two different sources of randomness in the model of RWRE (the environment and the walk), there are two different types of probabilistic questions that can be asked.

- **Quenched** The distribution P_ω of the RWRE for a fixed environment is called the quenched law of the RWRE. Under the quenched law X_n is a Markov chain, and so all the tools of Markov chains are available. However, the challenge is typically to prove a result that is true under the quenched law P_ω for P-a.e. environment ω.
- **Averaged/Annealed** The distribution $\mathbb{P}$ is called the averaged law for the RWRE (some prefer the term "annealed" over "averaged," but we will use averaged in these notes). Under the averaged law the RWRE is no longer a Makov chain since the past history gives information about the environment. For instance, note that $\mathbb{P}(X_1 = 1) = E_P[\omega_0]$ but

$$\begin{aligned} \mathbb{P}(X_3 = 1 \,|\, X_1 = 1,\, X_2 = 0) &= \frac{\mathbb{P}(X_1 = 1,\, X_2 = 0,\, X_3 = 1)}{\mathbb{P}(X_1 = 1,\, X_2 = 0)} \\ &= \frac{E_P[\omega_0^2(1-\omega_1)]}{E_P[\omega_0(1-\omega_1)]}. \end{aligned}$$

  On the other hand, due to the averaging over all environments the averaged law has homogeneity that the quenched law is lacking. For instance, due to the stationarity of the environment ω it is true that $\mathbb{P}(X_n = X_0) = \mathbb{P}^x(X_n = X_0)$ for any starting location $x \in \mathbb{Z}$.

To make sure one understands the model of RWRE, it is helpful to consider a specific example.

Example 1.1. Suppose that the environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}}$ is i.i.d. with distribution

$$P(\omega_0 = 3/4) = p, \quad P(\omega_0 = 1/3) = 1-p, \qquad \text{for some } p \in [0,1].$$

An example of part of such an environment is shown in Figure 1.1 where sites with $\omega_x = 3/4$ are colored red and sites with $\omega_x = 1/3$ are colored blue.

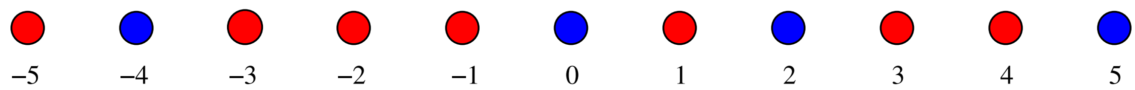

Figure 1: An example of an environment from Example 1.1. Sites colored red are such that $\omega_x = 3/4$ and blue sites are such that $\omega_x = 1/3$.

Thus far we have explained the model of RWRE only in the nearest-neighbor case on $\mathbb{Z}$. However, it is easy to see that the model can be expanded to other graphs besides $\mathbb{Z}$ and that the distribution on environments does not need to be i.i.d. We now give some examples, leaving the details of making the model precise to the reader.

Example 1.2 (Random walk among random conductances)**.** For any graph G (common examples would be $\mathbb{Z}$ or $\mathbb{Z}^d$), assign a conductance $c_{xy} = c_{yx}$ to every edge (x,y) of the graph. Given these conductances, the random walk then chooses an adjacent edge to move along with probability proportional to the conductance of the edge. That is,

$$P_\omega(X_{n+1} = y \,|\, X_n = x) = \frac{c_{xy}}{\sum_{z\sim x} c_{xz}}.$$

Typically the conductances are chosen to be i.i.d., but this does not make the environment i.i.d. in the sense that ω_x and ω_y are dependent if x and y are connected by an edge.

Example 1.3 (Random walk on Galton-Watson trees)**.** In this example, part of the randomness of the environment is the choice of the graph on which the process evolves. That is, we first choose a random Galton-Watson tree. Then we can assign transition probabilities ω_x to every vertex x of the tree in some deterministic or random manner. For instance, possible choices are

- Simple random walk - choose one of the neighboring vertices with equal porobability.
- Biased random walk - Fix a parameter $\beta > 0$. If the vertex x has k “descendants” then move to a descendant of x with probability $\beta/(1+\beta k)$ and to the ancestor of x with probability $1/(1+\beta k)$.
- Choose transition probabilities randomly in some way. For instance do a biased random walk but with a different bias factor $\beta_x > 0$ at each vertex, where the β_x are i.i.d.

Example 1.4 (Random walk on super-critical percolation clusters)**.** Let $p > p_c(d)$ be fixed, where $p_c(d)$ is the critical value for edge percolation on $\mathbb{Z}^d$. Choose an instance of p-edge percolation on $\mathbb{Z}^d$, conditioned on 0 being in the unique infinite component. Then perform a simple random walk on the remaining edges. Note that this is a special case of the random conductance example where the conductances on the edges of $\mathbb{Z}^d$ are Bernoulli(p).

2 One-dimensional RWRE - First Order Asymptotics

Having introduced the model of RWRE, we now turn our study to one-dimensional nearest neighbor RWRE. Recall that for a RWRE on $\mathbb{Z}$, the environment $\omega = \{\omega_x\} \in [0,1]^{\mathbb{Z}}$. To avoid certain degeneracy complications, and to make the proofs easier we will make the following assumptions.

Assumption 1. *There exists a $c > 0$ such that $P(\omega_0 \in [c, 1-c]) = 1$.*

Assumption 2. *The distribution P is such that $\{\omega_x\}_x$ is an i.i.d. sequence.*

In this section, we will study the first order asymptotics of the behavior of the RWRE: criterion for recurrence/transience and a law of large numbers.

2.1 Recurrence/Transience

In Solomon's seminal paper on RWRE [Sol75], Solomon gave an explicit criterion for recurrence or transience. While a naive guess might be that the RWRE is transient to $+\infty$ if and only if $\mathbb{P}(X_1 = 1) = E_P[\omega_0] > 1/2$ this is not the case. In fact, the recurrence or transience of the RWRE is determined by the quantity $E_P[\log \rho_0]$, where

$$\rho_x = \frac{1-\omega_x}{\omega_x}, \quad x \in \mathbb{Z}. \tag{1}$$

Theorem 2.1. *Let Assumptions 1 and 2 hold. Then,*

$$\begin{aligned}
E_P[\log \rho_0] < 0 \quad &\Longrightarrow \quad \lim_{n\to\infty} X_n = +\infty, \quad \mathbb{P}-a.s.\\
E_P[\log \rho_0] > 0 \quad &\Longrightarrow \quad \lim_{n\to\infty} X_n = -\infty, \quad \mathbb{P}-a.s.\\
E_P[\log \rho_0] = 0 \quad &\Longrightarrow \quad \liminf_{n\to\infty} X_n = -\infty, \limsup_{n\to\infty} X_n = +\infty, \quad \mathbb{P}-a.s.
\end{aligned}$$

Remark 2.2. Note that the statement of Theorem 2.1 is under the averaged measure $\mathbb{P}$, but that it also holds quenched. For instance, if $E_P[\log \rho_0] < 0$ then

$$1 = \mathbb{P}(\lim_{n\to\infty} X_n = \infty) = E_P[P_\omega(\lim_{n\to\infty} X_n = \infty)],$$

and so we can conclude that $P_\omega(\lim_{n\to\infty} X_n = \infty) = 1$ for P-a.e. environment ω.

Example 2.3. If the distribution on environments is as in Example 1.1 then X_n is transient to $+\infty$ if and only if $p > \log(2)/\log(6) \approx 0.3869$. Note that $E_P[\omega_0] > 1/2$ if and only if $p > 0.4$, which demonstrates the gap between the true criterion for transience and the naive guess.

Proof. The key to the proof of Theorem 2.1 is an explicit formula for hitting probabilities. To this end, we introduce some notation. For a fixed environment ω, we define the *potential* V of the environment by

$$V(k) = \begin{cases} \sum_{i=0}^{k-1} \log \rho_i & k \geq 1 \\ 0 & k = 0 \\ -\sum_{i=k}^{-1} \log \rho_i & k \leq -1. \end{cases} \tag{2}$$

Also, for any $x \in \mathbb{Z}$ define the hitting time T_x by

$$T_x = \inf\{n \geq 0 : X_n = x\}. \tag{3}$$

Then, since under the quenched law P_ω the random walk is simply a birth-death Markov chain, for any fixed $a \leq x \leq b$ we have the following formula for hitting probabilities.

$$P_\omega^x(T_a < T_b) = \frac{\sum_{i=x+1}^b e^{V(i)}}{\sum_{i=a+1}^x e^{V(i)} + \sum_{i=x+1}^b e^{V(i)}}. \tag{4}$$

To see this, it is enough to note that if we denote the right side by $h(x)$ then $h(a) = 1$, $h(b) = 0$ and

$$h(x) = \omega_x h(x+1) + (1-\omega_x)h(x-1), \quad a < x < b.$$

We will prove that the RWRE is transient to $+\infty$ when $E_P[\log \rho_0] < 0$ and leave the remaining cases to the reader. First, note that if $E_P[\log \rho_0] < 0$ then since the environment is i.i.d. it follows that $V(i) \sim E_P[\log \rho_0] i$ as $i \to \pm\infty$. In particular this implies that $\sum_{i=1}^\infty e^{V(i)} < \infty$ and $\sum_{i=-\infty}^0 e^{V(i)} = \infty$. Therefore, from the hitting probability formula in (4) we obtain that

$$P_\omega(T_n < \infty) = \lim_{a \to -\infty} P_\omega(T_n < T_a) = \lim_{a \to \infty} \frac{\sum_{i=a+1}^0 e^{V(i)}}{\sum_{i=a+1}^0 e^{V(i)} + \sum_{i=1}^n e^{V(i)}} = 1,$$

and

$$\begin{aligned}
\lim_{a \to -\infty} P_\omega(T_a < \infty) &= \lim_{a \to -\infty} \lim_{b \to \infty} P_\omega(T_a < T_b) \\
&= \lim_{a \to -\infty} \lim_{b \to \infty} \frac{\sum_{i=1}^b e^{V(i)}}{\sum_{i=a+1}^0 e^{V(i)} + \sum_{i=1}^b e^{V(i)}} \\
&= \lim_{a \to -\infty} \frac{\sum_{i=1}^\infty e^{V(i)}}{\sum_{i=a+1}^0 e^{V(i)} + \sum_{i=1}^\infty e^{V(i)}} \\
&= 0.
\end{aligned}$$

The first of these implies that $\limsup_{n \to \infty} X_n = \infty$, P_ω-a.s. The second can be used to show that $\liminf_{n \to \infty} X_n = \infty$, P_ω-a.s. as well. Indeed otherwise the random walk would return infinitely often to some vertex, and by uniform ellipticity each time there would be a positive probability of reaching site a before returning to x. Thus, if any site is visited infinitely often then $T_a < \infty$ for all a. □

2.2 Law of Large Numbers

Having established a criterion for recurrence/transience we now turn toward a law of large numbers. That is, we wish to show that the limit $\lim_{n \to \infty} X_n/n$ exists and doesn't depend on the environment ω.

Theorem 2.4 ([Sol75]). *If Assumptions 1 and 2 hold, then*

$$\lim_{n \to \infty} \frac{X_n}{n} = \begin{cases} \frac{1 - E_P[\rho_0]}{1 + E_P[\rho_0]} & E_P[\rho_0] < 1 \\ 0 & E_P[\rho_0] \geq 1 \quad \text{and} \quad E_P[\rho_0^{-1}] \geq 1 \qquad \mathbb{P}\text{-a.s.} \\ -\frac{1 - E_P[\rho_0^{-1}]}{1 + E_P[\rho_0^{-1}]} & E_P[\rho_0^{-1}] < 1, \end{cases}$$

Remark 2.5. Jensen's inequality implies that $1/E_P[\rho_0^{-1}] \leq E_P[\rho_0]$, and thus it cannot happen that $E_P[\rho_0] < 1$ and $E_P[\rho_0^{-1}] < 1$. Also, Jensen's inequality implies that it is possible to have $E_P[\log \rho_0] < 0$ and $E_P[\rho_0] \geq 1$ (see the example below) so that the RWRE can be transient but with asymptotically zero speed.

Example 2.6. Again, if the distribution on environments is as in Example 1.1 then the speed is positive if $p > 0.6$. Thus, the RWRE is transient with asymptotically zero speed if $p \in (0.3689..., 0.6]$.

We will give the proof of Theorem 2.4 when $E_P[\log \rho_0] \leq 0$ (that is, when the random walk is recurrent or transient to the right). The formula for the limiting speed when the walk is transient to the left is obtained by symmetry.

The starting point for the proof of Theorem 2.4 is the following lemma.

Lemma 2.7. *Suppose that* $\limsup_{n\to\infty} X_n = \infty$ *and* $\lim_{n\to\infty} T_n/n = c \in [1,\infty]$. *Then,*

$$\lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1}{c} & \textit{if } c < \infty \\ 0 & \textit{if } c = \infty. \end{cases}$$

Proof. Let $X_n^* = \max_{k\leq n} X_k$ denote the maximum distance to the right that the random walk has reached by time n. Then, $T_{X_n^*} \leq n < T_{X_n^*+1}$ so that

$$\frac{T_{X_n^*}}{X_n^*} \leq \frac{n}{X_n^*} \leq \frac{T_{X_n^*+1}}{X_n^*+1}\frac{X_n^*+1}{X_n^*}.$$

Since $X_n^* \to \infty$, the fact that $T_k/k \to c$ implies that

$$\lim_{n\to\infty} \frac{X_n^*}{n} = \begin{cases} \frac{1}{c} & \text{if } c < \infty \\ 0 & \text{if } c = \infty. \end{cases}$$

It remains to show that X_n/n has the same limit as X_n^*/n. Since $X_n \leq X_n^*$ this is trivial when $c = \infty$ (that is, when $X_n^*/n \to 0$), and so it is enough to show that $\lim_{n\to\infty}(X_n^* - X_n)/n = 0$ when $c < \infty$. Since the step sizes are at most 1 we have that $X_n^* - X_n \leq n - T_{X_n^*}$, and thus

$$\limsup_{n\to\infty} \frac{X_n^* - X_n}{n} \leq \lim_{n\to\infty} 1 - \lim_{n\to\infty} \left(\frac{T_{X_n^*}}{X_n^*}\right)\left(\frac{X_n^*}{n}\right) = 1 - c\left(\frac{1}{c}\right) = 0.$$

□

Next, we introduce some notation. For any $k \geq 1$ let $\tau_k := T_k - T_{k-1}$. *(Recall that we are assuming the random walk is recurrent or transient to the right so that* $\tau_k < \infty$ *for all* $k \geq 1$.*)*

Lemma 2.8. *Under the averaged measure* $\mathbb{P}$, *the sequence* $\{\tau_k\}_{k\geq 1}$ *is ergodic.*

Proof. Let $\{\xi_{k,j}\}_{k\in\mathbb{Z},\, j\geq 0}$ be a i.i.d. collection of $U(0,1)$ random variables that is independent of ω. Then, given an environment ω we can use the random variables $\xi_{k,j}$ to construct the random walk. If $X_n^* = k$ and $n - T_{X_n^*} = j$ then

$$X_{n+1} = \mathbf{1}_{\{\xi_{k,j} < \omega_{X_n}\}} - \mathbf{1}_{\{\xi_{k,j} \geq \omega_{X_n}\}} \quad \text{if } k = X_n^* \text{ and } j = n - T_{X_n^*}.$$

It is clear that the random walk constructed this way has the same distribution as the averaged law for the RWRE. Note that to construct the path of the RWRE up until time T_1, only the random variables $\{\xi_{0,j}\}_{j\geq 0}$ are needed. Similarly, the path of the random walk on the time interval $[T_k, T_{k+1}]$ only depends on $\{\xi_{k,j}\}_{j\geq 1}$.

Now, denote $\Xi_k = \{\xi_{k,j}\}_{j\geq 0}$ and let θ be the left shift operator on environments so that $(\theta^k\omega)_n = \omega_{k+n}$. Then it is clear from the above construction of the random walk that there is a deterministic function f such that $\tau_k = f(\theta^{k-1}\omega, \Xi_{k-1})$. Since the environment is i.i.d. and the sequence $\{\Xi_k\}_{k\in\mathbb{Z}}$ is independent of ω, it follows that $\{(\theta^k\omega, \Xi_k)\}_{k\in\mathbb{Z}}$ is ergodic and therefore $\tau_k = f(\theta^{k-1}\omega, \Xi_{k-1})$ is ergodic as well. □

The final ingredient we need before giving the proof of Theorem 2.4 is a formula for the quenched mean of T_1.

Lemma 2.9. *If $E_P[\log \rho_0] < 0$, then for P-a.e. environment ω*

$$\begin{aligned} E_\omega[\tau_1] &= \frac{1}{\omega_0} + \sum_{k=1}^{\infty} \frac{1}{\omega_{-k}} \rho_{-k+1}\rho_{-k+2}\cdots\rho_0 \\ &= 1 + 2\sum_{k=0}^{\infty} \rho_{-k}\rho_{-k+1}\cdots\rho_0. \end{aligned} \tag{5}$$

Proof. First we give the idea of the proof. By conditioning on the first step of the random walk we obtain that

$$\begin{aligned} E_\omega[\tau_1] &= \omega_0 + (1-\omega_0)E_\omega^{-1}[1+T_1] \\ &= 1 + (1-\omega_0)E_\omega^{-1}[T_1] \\ &= 1 + (1-\omega_0)\left(E_{\theta^{-1}\omega}[\tau_1] + E_\omega[\tau_1]\right). \end{aligned}$$

Then, solving for $E_\omega[\tau_1]$ we obtain that

$$E_\omega[\tau_1] = \frac{1}{\omega_0} + \rho_0 E_{\theta^{-1}\omega}[\tau_1]. \tag{6}$$

Iterating this formula we obtain that for any $m < \infty$

$$E_\omega[\tau_1] = \frac{1}{\omega_0} + \sum_{k=1}^{m}\left(\frac{1}{\omega_{-k}}\rho_{-k+1}\rho_{-k+2}\cdots\rho_0\right) + \rho_{-m}\rho_{-m+1}\cdots\rho_0 E_{\theta^{-m-1}\omega}[\tau_1]. \tag{7}$$

Finally, taking $m\to\infty$ we obtain the first equality in (5). There are two difficulties in the above argument. First of all, in order to solve for $E_\omega[\tau_1]$ as in (6) we need $E_\omega[\tau_1] < \infty$, and to iterate this we need $E_{\theta^{-k}\omega}[\tau_1] < \infty$ for any $k \geq 1$ as well. Secondly, even if all these quenched expectations are finite we need to prove that the last term in (7) vanishes as $m \to 0$.

Both of these difficulties can be handled by truncating the hitting times. For a fixed $M < \infty$ it is easy to see that

$$\begin{aligned} E_\omega[\tau_1 \wedge M] &= 1 + (1-\omega_0)E_\omega^{-1}[(1+T_1)\wedge M] \\ &\leq 1 + (1-\omega_0)\left(E_{\theta^{-1}\omega}[\tau_1\wedge M] + E_\omega[\tau_1\wedge M]\right), \end{aligned}$$

and since now all expectations are finite we obtain that

$$E_\omega[\tau_1 \wedge M] \leq \frac{1}{\omega_0} + \rho_0 E_{\theta^{-1}\omega}[\tau_1 \wedge M].$$

Iterating this gives

$$\begin{aligned} E_\omega[\tau_1 \wedge M] = \frac{1}{\omega_0} + \sum_{k=1}^{m} \left(\frac{1}{\omega_{-k}} \rho_{-k+1}\rho_{-k+2}\cdots\rho_0 \right) \\ + \rho_{-m}\rho_{-m+1}\cdots\rho_0 E_{\theta^{-m-1}\omega}[\tau_1 \wedge M] \end{aligned}$$

The assumption that $E_P[\log \rho_0] < \infty$ implies that $\rho_{-m}\rho_{-m+1}\cdots\rho_0 \to 0$ as $m \to \infty$ and last quenched expectation is bounded above by M. Thus, can take $m \to \infty$ to obtain that

$$E_\omega[\tau_1 \wedge M] \leq \frac{1}{\omega_0} + \sum_{k=1}^{\infty} \left(\frac{1}{\omega_{-k}} \rho_{-k+1}\rho_{-k+2}\cdots\rho_0 \right).$$

Taking $M \to \infty$, the monotone convergence theorem then gives

$$E_\omega[\tau_1] \leq \frac{1}{\omega_0} + \sum_{k=1}^{\infty} \left(\frac{1}{\omega_{-k}} \rho_{-k+1}\rho_{-k+2}\cdots\rho_0 \right). \tag{8}$$

To obtain the corresponding lower bound to (5), note that the sum on the right side of (5) is finite P-a.s. since $E_P[\log \rho_0] < 1$. Therefore, $E_\omega[\tau_1] < \infty$ for almost every environment ω, and since the environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}}$ is stationary it follows that $E_{\theta^{-k}\omega}[\tau_1] < \infty$ for all $k \in \mathbb{Z}$ for almost every environment ω. Thus, the argument leading to (7) is valid and by omitting the last term we obtain that

$$E_\omega[\tau_1] \geq \frac{1}{\omega_0} + \sum_{k=1}^{m} \left(\frac{1}{\omega_{-k}} \rho_{-k+1}\rho_{-k+2}\cdots\rho_0 \right).$$

Finally, taking $m \to \infty$ proves a matching lower bound to (8).

We have thus proved the first equality in (5). The second equality follows easily from the fact that $1/\omega_x = 1 + \rho_x$. □

We are now ready to give the proof of Theorem 2.4.

Proof. Since the sequence $\{\tau_k\}_{k\geq 1}$ is ergodic under $\mathbb{P}$, Birkhoff's ergodic theorem implies that

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \tau_k = \mathbb{E}[\tau_1].$$

Using the second formula for $E_\omega[\tau_1]$ in (5) and the fact that the environment is i.i.d., we obtain that

$$
\begin{aligned}
\mathbb{E}[\tau_1] &= E_P[E_\omega[\tau_1]] \\
&= 1 + 2\sum_{k=0}^{\infty} E_P[\rho_{-k}\rho_{-k+1}\cdots\rho_0] \\
&= 1 + 2\sum_{k=0}^{\infty} E_P[\rho_0]^{k+1} \\
&= \begin{cases} \frac{1+E_P[\rho_0]}{1-E_P[\rho_0]} & \text{if } E_P[\rho_0] < 1 \\ \infty & \text{if } E_P[\rho_0] \geq 1. \end{cases}
\end{aligned}
$$

This gives a formula for $\lim_{n\to\infty} T_n/n$. The formula for $\lim_{n\to\infty} X_n/n$ follows from Lemma 2.7. □

2.3 Notes

The results in this section are true under much weaker assumptions.

- Theorem 2.1 holds as long as the environment is ergodic and $E_P[\log\rho_0]$ exists (including $+\infty$ or $-\infty$). The only part of the proof that is more difficult without the i.i.d. assumption is proving recurrence when $E_P[\log\rho_0] = 0$. For this what is needed is that $\sum_{j=0}^{n-1}\log\rho_j$ changes sign infinitely many times as $n\to\infty$. Zeitouni uses a Lemma of Kesten to show that this is indeed the case [Zei04].
- The law of large numbers also holds under the weaker assumptions of ergodic environments and $E_P[\log\rho_0]$ being well defined. However, if the environment is not i.i.d. then the formula for the speed v_P does not simplify as much. Instead, the best we can do is

$$
\mathrm{v}_P = \left(1 + 2\sum_{k=0}^{\infty} E_P[\rho_{-k}\rho_{-k+1}\cdots\rho_0]\right)^{-1}.
$$

3 Limiting Distributions - Central Limit Theorems

Having given a characterization of recurrence/transience and a formula for the limiting velocity, the next natural step is to consider fluctuations from the deterministic velocity - that is, limiting distributions. In this section we focus on the case when the limiting distributions are Gaussian. We will see in the next section that this is certainly not always the case.

3.1 Limiting Distributions for Hitting Times

As was the case with the proof of the law of large numbers, we will deduce limiting distributions for X_n by first proving limiting distributions for T_n. We begin with the following quenched CLT for the hitting times.

Theorem 3.1. *If Assumptions 1 and 2 hold and $E_P[\rho_0^2] < 1$ then*

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma_1\sqrt{n}} \leq t\right) = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz =: \Phi(t), \qquad \forall t \in \mathbb{R}, \quad P\text{-a.s.}, \tag{9}$$

where $\sigma_1^2 = E_P[\,Var_\omega(T_1)] < \infty$.

Remark 3.2. As stated, the convergence in (9) is true for P-a.e. environment and any fixed t. However, since $\Phi(t)$ is a continuous function and both sides are monotone in t it follows that the convergence is uniform in t. That is,

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left| P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma_1\sqrt{n}} \leq t\right) - \Phi(t)\right| = 0, \quad P\text{-a.s.}$$

A key element in the proof of Theorem 3.1 will be the following Lemma.

Lemma 3.3. *If $E_P[\rho_0^2] < 1$ then $\mathbb{E}[\tau_1^2] < \infty$.*

Proof. we first derive a formula for $E_\omega[\tau_1^2]$ in a similar manner to the derivation of the formula for $E_\omega[\tau_1]$ in Lemma 2.9. By conditioning on the first step of the random walk,

$$\begin{aligned} E_\omega[\tau_1^2] &= \omega_0 + (1-\omega_0)E_\omega^{-1}[(1+T_1)^2] \\ &= 1 + (1-\omega_0)\left\{2E_{\theta^{-1}\omega}[\tau_1] + 2E_\omega[\tau_1] + 2(E_{\theta^{-1}\omega}[\tau_1])(E_\omega[\tau_1]) + E_{\theta^{-1}\omega}[\tau_1^2] + E_\omega[\tau_1^2]\right\}. \end{aligned}$$

Then we can solve for $E_\omega[\tau_1^2]$ to obtain

$$E_\omega[\tau_1^2] = \frac{1}{\omega_0} + \rho_0\left\{2E_{\theta^{-1}\omega}[\tau_1] + 2E_\omega[\tau_1] + 2(E_{\theta^{-1}\omega}[\tau_1])(E_\omega[\tau_1]) + E_{\theta^{-1}\omega}[\tau_1^2]\right\}.$$

At this point, we can simplify things by noting that $\rho_0 E_{\theta^{-1}\omega}[\tau_1] = E_\omega[\tau_1] - \frac{1}{\omega_0}$. Combining this with the above formula for $E_\omega[\tau_1^2]$ and doing a little bit of algebra one obtains that

$$E_\omega[\tau_1^2] = 2(E_\omega[\tau_1])^2 - \frac{1}{\omega_0} + \rho_0 E_{\theta^{-1}\omega}[\tau_1^2].$$

Iterating this m times and then taking $m \to \infty$ we can arrive at the following formula for $E_\omega[\tau_1^2]$.

$$E_\omega[\tau_1^2] = 2(E_\omega[\tau_1])^2 + 2\sum_{n=1}^{\infty} \left\{ (\rho_{-n+1}\rho_{-n+2}\cdots\rho_0)\,(E_{\theta^{-n}\omega}[\tau_1])^2 \right\} - E_\omega[\tau_1]. \tag{10}$$

We remark that the argument leading to (10) we have ignored some technical difficulties that arise. However, as in the proof of Lemma 2.9 the formula in (10) can be justified by repeating the above argument for the truncated second moment $E_\omega[(\tau_1 \wedge M)^2]$. We leave the details to the interested reader.

Having proved the formula (10), we now note that since the environment is i.i.d. that

$$\mathbb{E}[\tau_1^2] = 2E_P[(E_\omega\tau_1)^2]\sum_{n=0}^{\infty} E_P[\rho_0]^n - \mathbb{E}[\tau_1].$$

(Note that here we have used that $E_{\theta^{-n}\omega}[\tau_1]$ depends only on ω_x for $x \le n$.) Since $E_P[\rho_0^2] < 1$ implies that $E_P[\rho_0] < 1$ as well, it will follow that $\mathbb{E}[\tau_1^2] < \infty$ if we can show that $E_P[(E_\omega\tau_1)^2] < \infty$. To this end, from the second formula for $E_\omega[\tau_1]$ in (5) it follows that

$$\begin{aligned}
E_P[(E_\omega[\tau_1])^2] &\le 4E_P\left[\left(\sum_{k=0}^{\infty}\rho_{-k}\rho_{-k+1}\cdots\rho_0\right)^2\right] \\
&= 4E_P\left[\sum_{k\ge 0}(\rho_{-k}\rho_{-k+1}\cdots\rho_0)^2 + 2\sum_{0\le k<n}(\rho_{-n}\cdots\rho_{-k-1})(\rho_{-k}\rho_{-k+1}\cdots\rho_0)^2\right] \qquad (11)\\
&= 4\left\{\sum_{k\ge 0}(E_P[\rho_0^2])^{k+1} + \sum_{0\le k<n}(E_P[\rho_0])^{n-k}(E_P[\rho_0^2])^{k+1}\right\},
\end{aligned}$$

and these last sums are finite when $E_P[\rho_0^2] < 1$. □

Remark 3.4. Since $\sigma_1^2 = E_P[E_\omega[\tau_1^2] - (E_\omega[\tau_1])^2] = \mathbb{E}[\tau_1^2] - E_P[(E_\omega[\tau_1])^2]$, it follows from Lemma 3.3 that $\sigma_1^2 < \infty$ if $E_P[\rho_0^2] < 1$. In fact, by being more careful with the argument in the proof of Lemma 3.3 one can derive the following formula for σ_1^2 in terms of $E_P[\rho_0]$ and $E_P[\rho_0^2]$.

$$\sigma_1^2 = \frac{4(1+E_P[\rho_0])(E_P[\rho_0]+E_P[\rho_0^2])}{(1-E_P[\rho_0])^2(1-E_P[\rho_0^2])}.$$

Proof of Theorem 3.1. Under the quenched measure, $T_n - E_\omega T_n = \sum_{k=1}^n(\tau_k - E_\omega[\tau_k])$ is the sum of n independent zero mean random variables (note that the random variables are not identically distributed). The main idea is to use the Lindberg-Feller criterion to prove a central limit theorem. That is, the statement of the theorem will follow if we can check that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_\omega[(\tau_k - E_\omega[\tau_k])^2] = \sigma_1^2, \quad P\text{-a.s.}, \tag{12}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E_\omega\left[(\tau_k - E_\omega\tau_k)^2\mathbf{1}_{\{|\tau_k - E_\omega[\tau_k]|\ge\varepsilon\sqrt{n}\}}\right] = 0, \quad \forall\varepsilon>0, \quad P\text{-a.s.} \tag{13}$$

To prove (12), note that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E_\omega[(\tau_k - E_\omega[\tau_k])^2] = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{Var}_{\theta^{k-1}\omega}\tau_1 = E_P[\mathrm{Var}_\omega\tau_1],$$

where the last equality follows from Birkohff's ergodic Theorem. The proof of (13) is similar. Fix $\varepsilon > 0$ and $M < \infty$. Then,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E_\omega\left[(\tau_k - E_\omega\tau_k)^2\mathbf{1}_{\{|\tau_k - E_\omega[\tau_k]|\geq\varepsilon\sqrt{n}\}}\right] &\leq \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E_\omega\left[(\tau_k - E_\omega\tau_k)^2\mathbf{1}_{\{|\tau_k - E_\omega[\tau_k]|\geq M\}}\right]\\ &= E_P\left[E_\omega\left[(\tau_k - E_\omega\tau_k)^2\mathbf{1}_{\{|\tau_k - E_\omega[\tau_k]|\geq M\}}\right]\right],\end{aligned}$$

where again the last equality follows from Birkhoff's ergodic Theorem. Since $\sigma_1^2 = E_P[E_\omega[(\tau_1 - E_\omega[\tau_1])^2] < \infty$, it follows that the right side can be made arbitrarily small by taking $M \to \infty$. This finishes the proofs of (12) and (13), and thus also the proof of the theorem. □

Having proved the quenched central limit theorem for hitting times, we next give an limiting distribution under the averaged measure.

Theorem 3.5. *If Assumptions 1 and 2 hold and $E_P[\rho_0^2] < 1$ then*

$$\lim_{n\to\infty}P_\omega\left(\frac{T_n - n/\mathrm{v}_P}{\sigma\sqrt{n}} \leq t\right) = \Phi(t), \quad \forall t\in\mathbb{R},$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2$, with σ_1^2 defined as in Theorem 3.1 and

$$\sigma_2^2 = Var(E_\omega[\tau_1]) + 2\sum_{k=1}^{n-1}Cov(E_\omega[\tau_1], E_{\theta^k\omega}[\tau_1]) < \infty.$$

Remark 3.6. Using the second formula in (5) for $E_\omega[\tau_1]$, it is not too difficult to compute $\mathrm{Var}(E_\omega[\tau_1])$ and $\mathrm{Cov}(E_\omega[\tau_1], E_{\theta^k\omega}[\tau_1])$. By doing this, one can derive the following formula for σ_2^2.

$$\sigma_2^2 = \frac{4(1+E_P[\rho_0])\mathrm{Var}_P(\rho_0)}{(1-E_P[\rho_0])^3(1-E_P[\rho_0^2])}.$$

A first step in proving the averaged CLT is to prove the following CLT for the quenched mean of the hitting times.

Theorem 3.7. *If Assumptions 1 and 2 hold and $E_P[\rho_0^2] < 1$ then*

$$\lim_{n\to\infty}P\left(\frac{E_\omega[T_n] - n/\mathrm{v}_P}{\sigma_2\sqrt{n}} \leq t\right) = \Phi(t), \quad \forall t\in\mathbb{R},$$

where $\sigma_2^2 < \infty$ is defined as in Theorem 3.5.

Proof. Note that $E_\omega[T_n] - n/\mathrm{v}_P = \sum_{k=1}^{n}(E_\omega[\tau_k] - 1/\mathrm{v}_P) = \sum_{k=0}^{n-1}(E_{\theta^k\omega}[\tau_1] - 1/\mathrm{v}_P)$ is the sum of an ergodic, zero-mean sequence. Then the proof of the CLT for $E_\omega[T_n]$ will follow if we can check the condition for the CLT for sums of ergodic sequences in [Dur96, p. 417]. That is, we need to show that

$$\sum_{n=0}^{\infty} \sqrt{E_P\left[\left(E_P\left[E_\omega[\tau_1]-1/\mathrm{v}_P\right|\mathcal{F}_{-n}]\right)^2\right]} < \infty, \qquad \text{where } \mathcal{F}_{-n} = \sigma(\omega_x : x \leq -n). \tag{14}$$

However, it is clear from the second formula for $E_\omega[\tau_1]$ in (5) that

$$\begin{aligned} E_P\left[E_\omega[\tau_1]-1/\mathrm{v}_P\right|\mathcal{F}_{-n}] &= 1+2\sum_{k=1}^{n} E_P[\rho_0]^k + 2E_P[\rho_0]^n \sum_{k\geq n} \rho_{-k}\rho_{-k+1}\cdots\rho_{-n} - \frac{1}{\mathrm{v}_P} \\ &= E_P[\rho_0]^n \left\{ -\frac{1}{1-E_P[\rho_0]} + 2\sum_{k\geq n} \rho_{-k}\rho_{-k+1}\cdots\rho_{-n} \right\}, \end{aligned}$$

where the second inequality follows from the fact that $1/\mathrm{v}_P = \mathbb{E}[\tau_1] = (1+E_P[\rho_0])/(1-E_P[\rho_0])$ and a little bit of algebra. Therefore,

$$\begin{aligned} &\sum_{n=0}^{\infty} \sqrt{E_P\left[\left(E_P\left[E_\omega[\tau_1]-1/\mathrm{v}_P\right|\mathcal{F}_{-n}]\right)^2\right]} \\ &\qquad = \sum_{n=0}^{\infty} E_P[\rho_0]^n \sqrt{E_P\left[\left(-\frac{1}{1-E_P[\rho_0]} + 2\sum_{k\geq n} \rho_{-k}\rho_{-k+1}\cdots\rho_{-n}\right)^2\right]} \\ &\qquad = \sqrt{E_P\left[\left(-\frac{1}{1-E_P[\rho_0]} + 2\sum_{k\geq 0} \rho_{-k}\rho_{-k+1}\cdots\rho_{0}\right)^2\right]} \sum_{n=0}^{\infty} E_P[\rho_0]^n \end{aligned}$$

where the last equality follows from the fact that the environment is a stationary sequence. Finally, the computation in (11) shows that the expectation under the square rootis finite, and since $E_P[\rho_0] < 1$ the sum is finite as well. This completes the proof of (14) and thus also of the theorem. ☐

Proof of Theorem 3.5. The proof of the averaged CLT for hitting times follows easily from Theorems 3.1 and 3.7. The idea is that

$$\frac{T_n - n/\mathrm{v}_P}{\sqrt{n}} = \frac{T_n - E_\omega[T_n]}{\sqrt{n}} + \frac{E_\omega[T_n] - n/\mathrm{v}_P}{\sqrt{n}},$$

and Theorems 3.1 and 3.7 imply that the terms on the right side are asymptotically zero mean Gaussian random variables with variance σ_1^2 and σ_2^2 respectively. Moreover, the second term on the right depends only on the environment, while the first term is asymptotically independent of the environment (since the limiting distribution in Theorem 3.1 doesn't depend on ω). Therefore, we expect that right side should be asymptotically the sum of two independent mean zero Gaussians with varianes σ_1^2 and σ_2^2.

To make the proof precise, we first write

$$\begin{aligned} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{\sigma\sqrt{n}} \leq t\right) &= \mathbb{P}\left(\frac{T_n - E_\omega[T_n]}{\sigma\sqrt{n}} \leq t - \frac{E_\omega[T_n]-n/\mathrm{v}_P}{\sigma\sqrt{n}}\right) \\ &= E_P\left[P_\omega\left(\frac{T_n - E_\omega[T_n]}{\sigma_1\sqrt{n}} \leq \frac{\sigma t}{\sigma_1} - \frac{\sigma_2}{\sigma_1}\frac{E_\omega[T_n]-n/\mathrm{v}_P}{\sigma_2\sqrt{n}}\right)\right]. \end{aligned}$$

Since, as noted in Remark 3.2, the convergence in the quenched CLT is uniform in t it follows that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{\sigma\sqrt{n}} \le t\right) &= \lim_{n\to\infty} E_P\left[\Phi\left(\frac{\sigma t}{\sigma_1} - \frac{\sigma_2}{\sigma_1}\frac{E_\omega[T_n] - n/\mathrm{v}_P}{\sigma_2\sqrt{n}}\right)\right] \\ &= E\left[\Phi\left(\frac{\sigma t}{\sigma_1} - \frac{\sigma_2}{\sigma_1} Z\right)\right], \quad \text{with } Z \sim N(0,1). \end{aligned} \tag{15}$$

Note that the last equality above follows from Theorem 3.7. Finally, note that

$$\Phi\left(\frac{\sigma t}{\sigma_1} - \frac{\sigma_2}{\sigma_1} Z\right) = P\left(Z' \le \frac{\sigma t}{\sigma_1} - \frac{\sigma_2}{\sigma_1} Z\right) = P\left(\frac{\sigma_1}{\sigma} Z' + \frac{\sigma_2}{\sigma} Z \le t\right),$$

where Z' is a $N(0,1)$ random variable that is independent of Z. Since $\frac{\sigma_1}{\sigma}Z' + \frac{\sigma_2}{\sigma}Z \sim N(0,1)$ (recall that $\sigma^2 = \sigma_1^2 + \sigma_2^2$) it follows that the last line of (15) is equal to $\Phi(t)$. ☐

3.2 Limiting Distributions for the Position of the RWRE

We now show how to deduce quenched and averaged CLTs for X_n from the corresponding CLTs for the hitting times T_n.

Theorem 3.8. *If Assumptions 1 and 2 hold and $E_P[\rho_0^2] < 1$, then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n\mathrm{v}_P}{\mathrm{v}_P^{3/2}\sigma\sqrt{n}} < t\right) = \Phi(t), \quad \forall t \in \mathbb{R},$$

where as in Theorem 3.5 $\sigma^2 = \sigma_1^2 + \sigma_2^2 < \infty$.

Proof. Recall the definition of $X_n^* = \max_{k\le n} X_k$. We will first prove the averaged CLT for X_n^* in place of X_n and then show that X_n is close enough to X_n^* for the same limiting distribution to hold. Note that $\{X_n^* < k\} = \{T_k > n\}$. Then, for any $t \in \mathbb{R}$ and $n \ge 1$ let $x(n) := \lceil n\mathrm{v}_P + \mathrm{v}_P^{3/2}\sigma\sqrt{n}t\rceil$ so that

$$\begin{aligned}\mathbb{P}\left(\frac{X_n^* - n\mathrm{v}_P}{\mathrm{v}_P^{3/2}\sigma\sqrt{n}} < t\right) &= \mathbb{P}\left(X_n^* < n\mathrm{v}_P + \mathrm{v}_P^{3/2}\sigma\sqrt{n}t\right) \\ &= \mathbb{P}\left(T_{x(n,t)} > n\right) \\ &= \mathbb{P}\left(\frac{T_{x(n,t)} - x(n,t)/\mathrm{v}_P}{\sigma\sqrt{x(n,t)}} > \frac{n - x(n,t)/\mathrm{v}_P}{\sigma\sqrt{x(n,t)}}\right).\end{aligned}$$

It follows from the above definition of $x(n,t)$ that

$$\lim_{n\to\infty} \frac{n - x(n,t)/\mathrm{v}_P}{\sigma\sqrt{x(n,t)}} = -t.$$

Thus, we can conclude from Theorem 3.5 that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n^* - n\mathrm{v}_P}{\mathrm{v}_P^{3/2}\sigma\sqrt{n}} < t\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{T_{x(n,t)} - x(n,t)/\mathrm{v}_P}{\sigma\sqrt{x(n,t)}} > -t\right) = 1 - \Phi(-t) = \Phi(t).$$

It remains to show that X_n is close enough to X_n^* to have the same limiting distribution. To this end, the following Lemma is more than enough to finish the proof of the CLT for X_n.

Lemma 3.9. *If Assumptions 1 and 2 hold and* $\mathbb{E}_P[\log \rho_0] < \infty$*, then*

$$\lim_{n\to\infty} \frac{X_n^* - X_n}{(\log n)^2} = 0, \quad \mathbb{P}\text{-}a.s.$$

Proof. By the Borel-Cantelli Lemma, it is enough to show that

$$\sum_{n\geq 1} \mathbb{P}(X_n^* - X_n \geq \delta(\log n)^2) < \infty, \quad \forall \delta > 0. \tag{16}$$

To this end, note that the event $\{X_n^* - X_n \geq \delta(\log n)^2\}$ implies that after first hitting some $k \leq n$ the random walk then backtracks to $k - \lceil \delta(\log n)^2 \rceil$. Thus, by the strong Markov property (using the quenched law)

$$P_\omega(X_n^* - X_n \geq \delta(\log n)^2) \leq \sum_{k=1}^{n} P_\omega^k(T_{k-\lceil \delta(\log n)^2\rceil} < \infty).$$

Taking expectations with respect to P we obtain

$$\mathbb{P}(X_n^* - X_n \geq \delta(\log n)^2) \leq \sum_{k=1}^{n} E_P\left[P_\omega^k(T_{k-\lceil \delta(\log n)^2\rceil} < \infty)\right] = n\mathbb{P}(T_{-\lceil \delta(\log n)^2\rceil} < \infty),$$

where in the last equality we used the stationarity of the distribution on environments. Then, the proof of (16) will be completed if we can show that there exist constants $C_1, C_2 > 0$ such that

$$\mathbb{P}(T_{-k} < \infty) \leq C_1 e^{-C_2 k}, \quad \forall k \geq 1. \tag{17}$$

To this end, note that from the formula for hitting probabilities (4) we can see that

$$P_\omega(T_{-k} < \infty) = \frac{\sum_{j\geq 1} e^{V(j)}}{\sum_{j\geq -k+1} e^{V(j)}} \leq \sum_{j\geq 1} e^{V(j)-V(-k+1)}.$$

Typically, $V(j)-V(-k+1)$ is close to $(j+k-1)E_P[\log \rho_0]$ and so for k large we expect $P_\omega(T_{-k} < \infty)$ to be exponentially small. To this end, fix $c > 0$ and note that

$$\begin{aligned}
\mathbb{P}(T_{-k} < \infty) = E_P[P_\omega(T_{-k} < \infty)] &\leq \frac{e^{-kc}}{1-e^{-c}} + P\left(\sum_{j=1}^{\infty} e^{V(j)-V(-k+1)} > \frac{e^{-kc}}{1-e^{-c}}\right) \\
&\leq \frac{e^{-kc}}{1-e^{-c}} + \sum_{j\geq 1} P\left(e^{V(j)-V(-k+1)} > e^{-c(j+k-1)}\right) \\
&= \frac{e^{-kc}}{1-e^{-c}} + \sum_{j\geq 1} P\left(V(j) - V(-k+1) > -c(j+k-1)\right) \\
&= \frac{e^{-kc}}{1-e^{-c}} + \sum_{j\geq k} P\left(V(j) > -cj\right),
\end{aligned} \tag{18}$$

where the last equality follows from the fact that $V(j) - V(-k+1)$ has the same distribution as $V(j+k-1)$ since the environment ω is stationary. Now since $V(j) = \sum_{i=0}^{j-1} \log \rho_i$ is the sum of i.i.d. bounded random variables, it follows from Cramer's Theorem [DZ98, Theorem 2.2.3] that

$P(V(j) > -cj)$ decays exponentially in j if $c < -E_P[\log \rho_0]$. That is, for $0 < c < -E_P[\log \rho_0]$ there exists a $\delta > 0$ (depending on c) such that $P(V(j) > -cj) \leq e^{-\delta j}$ for all j sufficiently large. Applying this to (18) we obtain that

$$\mathbb{P}(T_{-k} < \infty) \leq \frac{e^{-kc}}{1 - e^{-c}} + \frac{e^{-\delta k}}{1 - \varepsilon^{-\delta}}$$

for all k sufficiently large. This proves (17) and thus also the lemma. □

□

We can also prove a quenched CLT for X_n. However, since the centering is random (depending on the environment) instead of deterministic in the the quenched CLT for T_n, determining the proper centering for X_n is more difficult.

Theorem 3.10. *If Assumptions 1 and 2 hold and* $E_P[\rho_0^2] < 1$*, then*

$$\lim_{n\to\infty} P_\omega \left(\frac{X_n - n\mathrm{v}_P + Z_n(\omega)}{\mathrm{v}_P^{3/2}\sigma_1\sqrt{n}} < t \right) = \Phi(t), \quad \forall t \in \mathbb{R}, \tag{19}$$

where σ_1^2 *is defined as in Theorem 3.1 and* $Z_n(\omega) = \mathrm{v}_P \sum_{k=1}^{\lfloor n\mathrm{v}_P \rfloor} (E_\omega[\tau_k] - 1/\mathrm{v}_P)$.

Sketch of the proof. The idea of the proof is essentially the same as the proof of Theorem 3.8. As mentioned above, the main difficulty is determining a proper quenched centering. Let $c_n(\omega)$ be some possible environment-dependent centering scheme. Then, denoting $y(n,t,\omega) = \lceil c_n(\omega) + \mathrm{v}_P^{3/2}\sigma_1\sqrt{n}t \rceil$

$$\begin{aligned} P_\omega \left(\frac{X_n^* - c_n(\omega)}{\mathrm{v}_P^{3/2}\sigma_1\sqrt{n}} < t \right) &= P_\omega \left(X_n^* < c_n(\omega) + \mathrm{v}_P^{3/2}\sigma_1\sqrt{n}t \right) \\ &= P_\omega \left(T_{y(n,t,\omega)} > n \right) \\ &= P_\omega \left(\frac{T_{y(n,t,\omega)} - E_\omega\left[T_{y(n,t,\omega)}\right]}{\sigma_1\sqrt{y(n,t,\omega)}} > \frac{n - E_\omega\left[T_{y(n,t,\omega)}\right]}{\sigma_1\sqrt{y(n,t,\omega)}} \right). \end{aligned}$$

We wish to choose the centering scheme $c_n(\omega)$ so that

$$\lim_{n\to\infty} \frac{y(n,t,\omega)}{n} = \mathrm{v}_P, \quad \text{and} \quad \lim_{n\to\infty} \frac{n - E_\omega\left[T_{y(n,t,\omega)}\right]}{\sigma_1\sqrt{n\mathrm{v}_P}} = -t, \quad \forall t, \quad P\text{-a.s.}, \tag{20}$$

in which case it would follow from Theorem 3.1 that

$$\lim_{n\to\infty} P_\omega \left(\frac{X_n^* - c_n(\omega)}{\mathrm{v}_P^{3/2}\sigma_1\sqrt{n}} < t \right) = \lim_{n\to\infty} P_\omega \left(\frac{T_{y(n,t,\omega)} - E_\omega\left[T_{y(n,t,\omega)}\right]}{\sigma_1\sqrt{y(n,t,\omega)}} > -t \right) = 1 - \Phi(-t) = \Phi(t).$$

It remains to check that the conditions in (20) are satisfied for $c_n(\omega) = n\mathrm{v}_P - Z_n(\omega)$. We will not give a completely rigorous proof of these facts, but instead explain why they indeed hold and leave the details to the reader. It will be crucial below to note that Theorem 3.7 and the definition of $Z_n(\omega)$ imply that $Z_n(\omega)/\sqrt{n}$ converges in distribution to a zero-mean Gaussian random variable.

Informally, this implies that $Z_n(\omega)$ is typically of size $\mathcal{O}(\sqrt{n})$. The first condition in (20) is easily checked since

$$\lim_{n\to\infty} \frac{y(n,t,\omega)}{n} = \lim_{n\to\infty} \frac{\lceil n\mathrm{v}_P - Z_n(\omega) + \mathrm{v}_P^{3/2}\sigma_1\sqrt{n}t\rceil}{n} = \mathrm{v}_P - \lim_{n\to\infty}\frac{Z_n(\omega)}{n} = \mathrm{v}_P, \quad P\text{-a.s.}$$

Checking the second condition in (20) is more difficult. First note that

$$n - E_\omega[T_{y(n,t,\omega)}] = n - E_\omega[T_{\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor}] - \sum_{k=\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor+1}^{\lceil n\mathrm{v}_P - Z_n(\omega) + \mathrm{v}_P^{3/2}\sigma_1\sqrt{n}t\rceil} E_\omega[\tau_k]. \tag{21}$$

Since the last sum on the right is the sum of $\mathrm{v}_P^{3/2}\sigma_1\sqrt{n}t$ ergodic random variables with mean $1/\mathrm{v}_P$ it should be true that

$$\lim_{n\to\infty}\frac{1}{\sigma_1\sqrt{n\mathrm{v}_P}} \sum_{k=\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor+1}^{\lceil n\mathrm{v}_P - Z_n(\omega) + \mathrm{v}_P^{3/2}\sigma_1\sqrt{n}t\rceil} E_\omega[\tau_k] = t, \quad P\text{-a.s.} \tag{22}$$

Next, note that

$$\begin{aligned}
E_\omega[T_{\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor}] - n &= \sum_{k=1}^{\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor}\left(E_\omega[\tau_k] - \frac{1}{\mathrm{v}_P}\right) + \frac{\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor}{\mathrm{v}_P} - n \\
&= \sum_{k=1}^{\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor}\left(E_\omega[\tau_k] - \frac{1}{\mathrm{v}_P}\right) - \frac{Z_n(\omega)}{\mathrm{v}_P} + \delta_n(\omega) \\
&= -\sum_{k=\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor+1}^{\lfloor n\mathrm{v}_P\rfloor}\left(E_\omega[\tau_k] - \frac{1}{\mathrm{v}_P}\right) + \delta_n(\omega)
\end{aligned}$$

where $|\delta_n(\omega)| < 1/\mathrm{v}_P$ is an error term coming from the integer effects of the floor function, and the last equality follows from the definition of $Z_n(\omega)$. Since this last sum is the sum of $Z_n(\omega)$ zero-mean ergodic terms and $Z_n(\omega)/\sqrt{n}$ converges in distribution, it should be the case that

$$\lim_{n\to\infty}\frac{n - E_\omega[T_{\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor}]}{\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=\lfloor n\mathrm{v}_P - Z_n(\omega)\rfloor+1}^{\lfloor n\mathrm{v}_P\rfloor}\left(E_\omega[\tau_k] - \frac{1}{\mathrm{v}_P}\right) = 0, \quad P\text{-a.s.} \tag{23}$$

Combining (21), (22) and (23) verifies the second condition in (20). □

Remark 3.11. As mentioned above, the above justification of the centering scheme $c_n(\omega) = n\mathrm{v}_P - Z_n(\omega)$ skips some technical details (in particular we are trying to apply Birkhoff's ergodic theorem with ω-dependent endpoints of the summands). For more details and other centering schemes that can be used see [Gol07].

3.3 Notes

The above proofs of the quenched and averaged CLTs differ from those given at other places in the literature.

- Kesten, Kozlov, and Spitzer [KKS75] also deduce limiting distributions for X_n from corresponding limiting distributions for T_n. However, since they are primarily interested in the non-Gaussian limits when $E_P[\rho_0^2] < 1$ (see the next section) they prove much more than is needed to obtain a CLT in the case when $E_P[\rho_0^2] < 1$. Also, in [KKS75] only averaged limiting distributions are proved for X_n and T_n.

- Zeitouni [Zei04] proves an averaged CLT for X_n using a method known as the *environment viewed from the point of view of the particle*. With this method he is able to prove a CLT for certain ergodic, non-i.i.d. laws on environments. As a byproduct he comes very close to proving a quenched CLT, but with the limit (19) holding only in P-probability instead of P-a.s.

- The idea of using the Lindberg-Feller criterion for proving a quenched CLT for T_n first used by Alili [Ali99]. However, it wasn't until later that Goldsheid [Gol07] and independently Peterson [Pet08] showed how to obtain a quenched CLT for X_n by choosing an appropriate environment-dependent centering scheme. Goldsheid is able to prove the quenched CLT for certain uniformly ergodic environment. Peterson's proof on the other hand proves a functional CLT (convergence to Brownian motion) for both T_n and X_n and is valid for environments satifying a certain technical mixing condition.

4 Limiting Distributions - The non-Gaussian Case

In the previous section we proved quenched and averaged central limit theorems under the assumption that $E_P[\rho_0^2] < 1$. In this section we will examine the (quenched and averaged) limiting distributions when this assumption is removed. We will, however, continue to assume that $E_P[\rho_0] < 0$ so that the RWRE is transient to the right. The recurrent case is very different, but at the end of the section we will make some remarks about the limiting distributions in the recurrent case.

The reader should be warned that this section begins a change in the notes where we will omit the proofs of certain technical arguments. Many of the remaining results are quite technical, and to aid the reader we will instead try to give a hueristic understanding of the technical parts and only give the full arguments for the less technical sections.

Throughout this section, we will always be assuming Assumptions 1 and 2 and that $E_P[\log \rho_0] < 0$. If in addition we have $P(\omega_0 \geq 1/2) = 1$, then $P(\rho_0 \leq 1) = 1$ and $P(\rho_0 < 1) > 0$. In this case $E_P[\rho_0^2] < 1$ and so the central limit theorems from the previous section apply. Thus, we will assume instead that $P(\omega_0 < 1/2) > 0$ and we claim that in this case there exists a unique $\kappa = \kappa(P) > 0$ such that

$$E_P[\rho_0^\kappa] = 1. \tag{24}$$

To see this, note that $\phi(\gamma) = E_P[\rho_0^\gamma] = E_P[e^{\gamma \log \rho_0}]$ is the moment generating function for $\log \rho_0$. Therefore, $\phi(\gamma)$ is a convex function in γ with slope $\phi'(0) = E_P[\log \rho_0] < 0$ at the origin. Therefore, $\phi(\gamma) < r(0) = 1$ for some $\gamma > 0$. On the other hand, since $P(\omega_0 < 1/2) = P(\rho_0 > 1) > 0$ then it follows that $\phi(\gamma) \to \infty$ as $\gamma \to \infty$ (note that Assumption 1 implies that $\phi(\gamma) < \infty$ for all $\gamma \in \mathbb{R}$). Since $\phi(\gamma)$ is convex there is thus a unique $\kappa > 0$ satisfying (24).

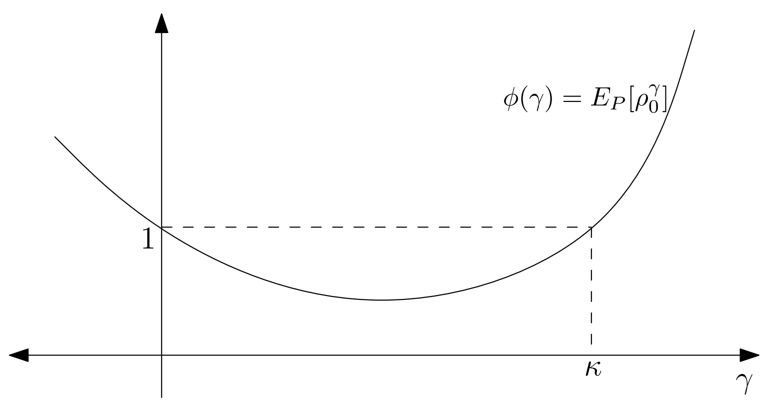

Figure 2: A visual depiction of the parameter $\kappa = \kappa(P)$. Note that the derivative of the curve at the origin is $E_P[\log \rho_0] < 0$. Also, it is clear from the picture that $E_P[\rho_0^\gamma] < 1 \iff \gamma \in (0, \kappa)$.

Note that some of the results in the previous sections can be stated in terms of κ.

- The random walk is transient with zero speed if $\kappa \in (0, 1]$ and with positive speed if $\kappa > 1$.
- The central limit theorems and the moment bounds on τ_1 in Section 3 all hold if and only if $\kappa > 2$.

The main results in this section will also need the following technical assumption.

Assumption 3. *The distribution of* $\log \rho_0$ *is non-arithmetic. That is, the support of* $\log \rho_0$ *is not contained in* $a + b\mathbb{Z}$ *for any* $a, b \in \mathbb{R}$.

The key place that this assumption is used is in the following Lemma.

Lemma 4.1. *Let Assumptions 1, 2 and 3 hold and let* $\kappa > 0$ *be defined as in* (24). *Then, there exists a constant* $C > 0$ *such that*

$$\lim_{t\to\infty} P(E_\omega[\tau_1] > t) \sim Ct^{-\kappa}, \quad \text{as } t \to \infty. \tag{25}$$

Proof. This is essentially a direct application of [Kes73, Theorem 5]. □

Remark 4.2. The proof of Lemma 4.1 is rather technical and so we will content ourselves with only giving a reference to the paper [Kes73]. However, to give some intuition of the result note that Lemma 4.1 implies that $E_P[(E_\omega[\tau_1])^\gamma] < \infty$ if and only if $\gamma < \kappa$. Since $E_\omega[\tau_1] = 1 + 2\sum_{k=0}^\infty (\rho_{-k}\cdots\rho_0)$ it is reasonable to expect that $E_P[(E_\omega[\tau_1])^\gamma] < \infty$ if and only if $E_P[\rho_0^\gamma] < 1$, but the definition of the parameter κ implies that $E_P[\rho_0^\gamma] < 1$ if and only if $\gamma \in (0, \kappa)$.

4.1 Background - Stable Distributions and Poisson Point Processes

Before discussing the limiting distributions when $\kappa \in (0, 2)$ we need to review some facts about stable distributions and Poisson point processes.

4.1.1 Stable Distribution

Recall that a (non-degenerate) distribution F is a stable distribution if for any $n \geq 2$ there exist constants $c_n \in \mathbb{R}$ and $a_n > 0$ such that if $X_1, X_2, \dots X_n$ are i.i.d. with common distribution F, then $(X_1 + X_2 + \cdots + X_n - c_n)/a_n$ also has distribution F.

The stable distributions are characterized first of all by their *index* $\alpha \in (0, 2]$. We will refer to a stable distribution with index α as an α-stable distribution. The 2-stable distributions are the two-parameter family of Normal/Gaussian distributions $N(\mu, \sigma^2)$. The family of α-stable distributions with $\alpha \in (0, 2)$ are a characterized by three parameters:

$$\text{centering } \mu \in \mathbb{R}, \quad \text{scaling } b > 0, \quad \text{and skewness } \gamma \in [-1, 1].$$

The centering and scaling parameters have the same roles as the mean and variance of the normal family of distributions. However, note that α-stable random variables have infinite variance when $\alpha < 2$ and infinite mean when $\alpha < 1$. Unlike the Normal distributions, the α-stable distributions are symmetric only when the skewness parameter $\gamma = 0$. When the skewness parameter $\gamma = 1$ or $\gamma = -1$, the distribution is said to be totally skewed to the right or left, respectively.

In general, there are not explicit formulas for the stable distributions. However, there are a few special cases where the densities are known.

- The standard Cauchy distribution has density $f(x) = \frac{1}{\pi(1+x^2)}$. This is a 1-stable distribution with $\mu = 0$, $b = 1$, and $\gamma = 0$.
- The Lévy distribution has density $f(x) = (2\pi)^{-1/2} x^{-3/2} e^{-\frac{1}{2x}} \mathbf{1}_{\{x>0\}}$. This is a $\frac{1}{2}$-stable distribution with $\mu = 0$, $b = 1$, and $\gamma = 1$.

Aside from the normal distributions and the above two examples, the stable distributions are generally defined by their characteristic function.

Example 4.3. The stable distributions that we will be interested in with regard to RWRE are the α-stable distributions that are totally skewed to the right. We will denote by $L_{\alpha,b}$ the α-stable distribution with scaling parameter b, centering $\mu = 0$, and skew $\gamma = 1$. These are the distributions with characteristic functions

$$\begin{aligned}\hat{L}_{\alpha,b}(u) &= \int_{\mathbb{R}} e^{ixu} \, L_{\alpha,b}(dx) \\ &= \exp\left\{ -b|u|^\alpha \left(1 - i\frac{u}{|u|}\phi_\alpha(u) \right) \right\} \quad \text{where } \phi_\alpha(u) := \begin{cases} \tan\left(\frac{\alpha\pi}{2}\right) & \alpha \neq 1 \\ \frac{2}{\pi}\log|u| & \alpha = 1. \end{cases}\end{aligned}$$

Stable distributions arise naturally as limiting distributions of sums of i.i.d. random variables. If the i.i.d. random variables have finite mean, then the central limit theorem implies that (after centering and scaling properly) the limiting distribution is Gaussian. On the other hand, stable distributions with $\alpha < 2$ arise as limits of sums of i.i.d. random variables with infinite variance. However, while the central limit theorem is robust in the sense that only a finite moment is needed, to obtain α-stable limiting distributions more precise information on the tail asymptotics is needed.

Example 4.4. Let $\xi_1, \xi_2, \xi_3, \dots$ be a sequence of non-negative i.i.d. random variables, and suppose that there exists some $b > 0$ and $\alpha \in (0,2)$ such that

$$P(\xi_1 > t) \sim bt^{-\alpha}, \quad \text{as } t \to \infty. \tag{26}$$

Recall that $L_{\alpha,b}$ are the distribution functions for the totally skewed to the right α-stable distributions. Then,

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^n \xi_i - C_\alpha(n)}{n^{1/\alpha}} \leq x \right) = L_{\alpha,b}(x),$$

where the centering term

$$C_\alpha(n) = \begin{cases} 0 & \alpha \in (0,1) \\ nE[\xi_1 \mathbf{1}_{\{\xi_1 \leq n\}}] & \alpha = 1 \\ nE[\xi_1] & \alpha \in (1,2). \end{cases}$$

Note that when $\alpha = 1$ the tail asymptotics (26) imply that $C_1(n) \sim bn\log n$, but that in general we cannot replace the centering term by $bn\log n$ in this case since it may be that

$$\limsup_{n\to\infty} \frac{|C_1(n) - bn\log n|}{n} = \limsup_{n\to\infty} \left| E[\xi_1 \mathbf{1}_{\{\xi_1 \leq n\}}] - \log n \right| > 0.$$

Example 4.5. Again, let $\xi_1, \xi_2, \xi_3, \dots$ be i.i.d. random variables, but now assume that $P(\xi_1 > t) \sim bt^{-2}$ as $t \to \infty$. Note that this tail decay implies that $\mathrm{Var}(\xi_1) = \infty$ so that the central limit theorem does not apply. Nevertheless, if we take a scaling that is slightly larger than $\sqrt{n}$ we can still obtain a Gaussian limiting distribution. That is, there exists an $a > 0$ such that

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^n \xi_i - nE[\xi_1]}{a\sqrt{n \log n}} \leq x \right) = \Phi(x).$$

4.1.2 Stable Distributions and Poisson Point Processes

Next, we briefly recall the relationship between Poisson point process and α-stable distributions when $\alpha < 2$. Recall that a point proccess $N = \sum_{i\geq 1} \delta_{x_i}$ is a measure valued random variable. The x_i are called the atoms of the point process N (note that the ordering of the atoms does not matter), and for any (Borel-measurable) $A \subset \mathbb{R}$, $N(A)$ is the number of atoms contained in A.

Definition 4.1. N is a non-homogeneous Poisson point process with intensity $\lambda(x)$ if

(i) $N \sim \text{Poisson}\left(\int_A \lambda(x)\, dx\right)$ for all $A \subset \mathbb{R}$.

(ii) $\{N(A_1), N(A_2), \dots, N(A_k)\}$ are independent if the sets $A_1, A_2, \dots, A_k$ are disjoint.

Example 4.6. Let $M = \sum_{i\geq 1} \delta_{t_i}$ be a homogeneous rate 1 Poisson point process on $(0,\infty)$ (that is the intensity $\mathbf{1}_{x>0}$). Fix a constant $\lambda > 0$ and $\alpha > 0$ and let $N_{\lambda,\alpha}$ be the transformed point process

$$N_{\lambda,\alpha} = \sum_{i\geq 1} \delta_{\lambda^{1/\alpha} x_i^{-1/\alpha}}.$$

Then $N_{\lambda,\alpha}$ is a Poisson point process with intensity $\lambda\alpha x^{-\alpha-1}$. This is a standard exercise in transformed Poisson point process, but as a review we will note that since $\lambda^{1/\alpha}x^{-1/\alpha} \in [a,b]$ if and only if $x \in [\lambda b^{-\alpha}, \lambda a^{-\alpha}]$ it follows that

$$N_{\lambda,\alpha}([a,b]) = M([\lambda b^{-\alpha}, \lambda a^{-\alpha}]) \sim \text{Poisson}\left(\lambda(a^{-\alpha} - b^{-\alpha})\right) = \text{Poisson}\left(\int_a^b \lambda\alpha x^{-\alpha-1}\, dx \right).$$

We now show how the Poisson point processes from Example 4.6 are related to the totally skewed to the right α-stable distributions from Example 4.3.

Example 4.7. Let $N_{\lambda,\alpha}$ be a Poisson point process with intensity $\lambda\alpha x^{-\alpha-1}$. If $\alpha \in (0,1]$ the random variable

$$Z = \int x\, N_{\lambda,\alpha}(dx).$$

is almost surely well defined and has distribution $L_{\alpha,\lambda}$ as defined in Example 4.3.

To see that Z is well defined let $N_{\lambda,\alpha} = \sum_{i\geq 1} \delta_{z_i}$ so that $Z = \sum_{i\geq 1} z_i$. Recall from Example 4.6 that we can represent the atoms z_i of $N_{\lambda,\alpha}$ by $z_i = \lambda^{1/\alpha} x_i^{-1/\alpha}$, where the x_i are the atoms of a homogeneous Poisson process with rate 1. Since we know that $x_i \sim i$ as $i \to \infty$ it follows that $z_i \sim \lambda^{1/\alpha} i^{-1/\alpha}$ as $i \to \infty$. Since $\sum_{i\geq 1} i^{-1/\alpha} < \infty$ when $\alpha < 1$ this shows that Z is almost surely well defined.

The fact that Z has distribution $L_{\alpha,\lambda}$ can be verified by directly computing the characteristic function. This is a somewhat involved computation, but more simply one can easily check that Z has a stable distribution. Let $N_1, N_2, \ldots, N_n$ be n independent point processes, all with the same distribution as $N_{\lambda,\alpha}$. Then if $Z_i = \int x\, N_i(dx)$, the random variables $Z_1, \ldots, Z_n$ are i.i.d. and all with the same distribution as Z. It follows from the superposition of Poisson point processes that

$$\sum_{i=1}^{n} Z_i = \int x\, N(dx),$$

where N is a Poisson point process with intensity $n\lambda\alpha x^{-\alpha-1}$. Also, if $N_{\lambda,\alpha} = \sum_{i\geq 1} \delta_{x_i}$ then $\sum_{i\geq 1} \delta_{n^{1/\alpha} x_i}$ is a Poisson point process with intensity $n\lambda\alpha x^{-\alpha-1}$ as well. From this it is clear that

$$\frac{Z_1 + Z_2 + \cdots Z_n}{n^{1/\alpha}} \stackrel{\text{Law}}{=} Z.$$

Example 4.8. Let $N_{\lambda,\alpha}$ be a Poisson point process with intensity $\lambda\alpha x^{-\alpha-1}$. If $\alpha \in (1, 2)$ then the random variable

$$Z = \lim_{\delta\to 0} \left(\int_\delta^\infty x\, N_{\lambda,\alpha}(dx) - \frac{\lambda\alpha\delta^{1-\alpha}}{\alpha - 1} \right), \tag{27}$$

is almost surely well defined and has distribution $L_{\alpha,\lambda}$ as defined in Example 4.3.

For convenience of notation, let $Z_\delta = \int_\delta^\infty x\, N_{\lambda,\alpha}(dx) - \frac{\lambda\alpha\delta^{1-\alpha}}{\alpha-1}$. To see that the limit $\lim_{\delta\to 0} Z_\delta$ exists, first note that

$$E\left[\int_\delta^\infty x\, N_{\lambda,\alpha}(dx) \right] = \int_\delta^\infty \lambda\alpha x^{-\alpha}\, dx = \frac{\lambda\alpha\delta^{1-\alpha}}{\alpha - 1},$$

so that $E[Z_\delta] = 0$ for all $\delta > 0$. Secondly, note that for any $0 < \varepsilon < \delta$

$$\mathrm{Var}(Z_\delta - Z_\varepsilon) = \mathrm{Var}\left(\int_\varepsilon^\delta x\, N_{\lambda,\alpha}(dx) \right) = \int_\varepsilon^\delta \lambda\alpha x^{1-\alpha}\, dx = \frac{\lambda\alpha}{2-\alpha}\left(\delta^{2-\alpha} - \varepsilon^{2-\alpha}\right) \leq \frac{\lambda\alpha\delta^{2-\alpha}}{2-\alpha}.$$

Since $\alpha < 2$ this vanishes as $\delta \to 0$. This shows that Z_δ is Cauchy in probability and thus converges in probability. In fact, it can be shown that the limit converges almost surely, but we will content ourselves with the above argument for now.

As in the previous example it can be checked using the superposition of Poisson processes that $n^{-1/\alpha}(Z_1 + \cdots Z_n)$ has the same distribution as Z for any n. Finally, it can be checked by direct computation that $\lim_{\delta\to 0} E[e^{iuZ_\delta}] = \hat{L}_{\alpha,\lambda}(u)$ so that Z does have distribution $L_{\alpha,\lambda}$.

4.2 Averaged Limiting Distributions - $\kappa \leq 2$

Having reviewed the necessary information on stable distributions, we are now ready to begin the study of the limiting distributions for RWRE when $\kappa \in (0, 2)$. In contrast to the previous section we will discuss the averaged limiting distributions first since they are much easier. However, as in the previous section we fill first study the limiting distributions for hitting times and then deduce the corresponding limiting distributions for X_n.

Theorem 4.9. *Let Assumptions 1, 2, and 3 hold. If κ is defined as in* (24) *then* *(i)* *If $\kappa \in (0,1)$, then there exists a $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/\kappa}} \leq t \right) = L_{\kappa,b}(t), \quad \forall t.$$

(ii) *If $\kappa = 1$, then there exists a constant $b > 0$ and a sequence $D(n) \sim b \log n$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - nD(n)}{n} \leq t \right) = L_{1,b}(t), \quad \forall t.$$

(iii) *If $\kappa \in (1,2)$, then there exists a $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{n^{1/\kappa}} \leq t \right) = L_{\kappa,b}(t), \quad \forall t.$$

(iv) *If $\kappa = 2$, then there exists a $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{b\sqrt{n \log n}} \leq t \right) = \Phi(t), \quad \forall t.$$

Note the similarity in the above limiting distributions to those for sums of i.i.d. non-negative heavy tailed random variables in Examples 4.4 and 4.5. On the one hand this is not surprising since $T_n = \sum_{k=1}^n \tau_k$, but under the averaged measure $\mathbb{P}$ the random variables τ_k are neither independent nor identically distributed. However, as we will see below the main idea of the proof is that if we group certain of the τ_k together the sums of the τ_k within the groups become essentially independent and identically distributed with heavy tails.

Recall the definition of the potential of the environment V in (2). For a given environment ω we will define a sequence of "ladder locations" of the environment as follows.

$$\nu_0 = 0, \quad \text{and} \quad \nu_k = \inf\{i > \nu_{k-1} : V(i) < V(\nu_{k-1})\} \quad \text{for } k \geq 1. \tag{28}$$

The idea of the proof of Theorem 4.9 is to show that the times to cross between ladder locations

$$U_k := T_{\nu_k} - T_{\nu_{k-1}} \tag{29}$$

have slowly varying tails and are approximately i.i.d.

First of all, we note that the crossing times U_k are "almost" a stationary sequence. The reason for this is that the distribution on the environment near $\nu_0 = 0$ is different from the distribution of the environment near ν_k for $k \geq 1$. In particular, while the definition of the ladder locations implies that $V(j) > V(\nu_k)$ for all $j \in [0, \nu_k)$ it is possible that $V(j) \leq V(0) = 0$ for some $j \leq 0$. To rectify this problem we define a new measure Q on environments by

$$Q(\cdot) = P(\cdot \,|\, V(j) > 0, \, \forall j \leq -1). \tag{30}$$

Note that the measure Q is well defined since $E_P[\log \rho_0] < 0$ implies that $P(V(j) > 0, \, \forall j \leq -1) > 0$. The environment is no longer i.i.d. under the measure Q, but it does have the following useful properties.

Lemma 4.10. *If the measure Q is defined as in* (30) *then* *(i)* *Under the measure Q on environments the environment ω is stationary under shifts of the ladder locations in the sense that $\{\theta^{\nu_k}\omega\}_{k\geq 0}$ is a stationary sequence.*

(ii) *P and Q can be coupled so that there exists a distribution on pairs of environments (ω, ω') such that $\omega \sim P$, $\omega' \sim Q$, and $\omega_x = \omega'_x$ for all $x \geq 0$.*

(iii) *The "blocks" of the environment between ladder locations are i.i.d. That is, if*

$$B_k = (\omega_{\nu_{k-1}}, \omega_{\nu_{k-1}+1}, \ldots, \omega_{\nu_k - 1}),$$

then the sequence $\{B_k\}_{k\geq 1}$ is i.i.d.

Proof. Stationarity under shifts of the ladder locations follows easily from the definition of Q. The coupling of P and Q is easy to construct since the conditioning event in the definition of Q only depends on the environment to the left of the origin. It is easy to see that the blocks between ladder locations B_k are i.i.d. under the measure P since the environment is i.i.d. under P and $\nu_k - \nu_{k-1}$ only depends on the environment to the right of ν_{k-1}. Finally, since the B_k only depend on the environment to the right of the origin they have the same distribution under Q as under P. □

We will use the notation $\mathbb{Q}$ to denote the averaged distribution of the RWRE when the environment has distribution Q. That is $\mathbb{Q}(\cdot) = E_Q[P_\omega(\cdot)]$. Part (1) of Lemma 4.10 can easily be seen to imply the following Corollary.

Corollary 4.11. *Under the measure $\mathbb{Q}$, the crossing times of ladder locations $U_k = T_{\nu_k} - T_{\nu_{k-1}}$ are a stationary sequence.*

Remark 4.12. The proof of Corollary 4.11 is essentially the same as that of Lemma 2.8 and is therefore ommitted. Also, it can be shown in fact that under Q the environment is ergodic under the shifts of the ladder locations and thus that U_k is ergodic under $\mathbb{Q}$.

We still need to show that the sequence U_k has (nicely behaved) heavy tails and is "fast-mixing" so that it is almost i.i.d. To accomplish this, it will be helpful to seperate the randomness in U_k due to the environment and the random walk, respectively. Define for any $k \geq 1$,

$$\beta_k = \beta_k(\omega) = E_\omega[U_k] = E_\omega^{\nu_{k-1}}[T_{\nu_k}]. \tag{31}$$

Also, by possibly expanding the probability space P_ω, let $\{\eta_k\}_{k\geq 1}$ be a sequence of i.i.d. Exp(1) random variables. The main idea of the proof is to show that $\sum_k U_k$ has approximately the same distribution as $\sum_k \beta_k \eta_k$.

Lemma 4.13. *Under the measure Q, the sequence $\{\beta_k\}_{k\geq 1}$ is stationary. Moreover, there exists a constant $C_0 > 0$ such that*

$$Q(\beta_1 > t) \sim C_0 t^{-\kappa}, \qquad \text{as } t \to \infty.$$

Proof. The stationarity of the β_k under Q follows easily from Theorem 4.10 part (1). The proof of the tail asymptotics of β_1 is quite technical and therefore ommitted (the details can be found in [PZ09]). However, note that the nice polynomial tail decay is not surprising in light of the similar tail decay of $E_\omega[\tau_1]$ as stated in Lemma 4.1. □

Corollary 4.14. *If C_0 is the constant from Lemma 4.13, then*

$$\mathbb{Q}(\beta_1\eta_1 > t) \sim \Gamma(\kappa+1)C_0 t^{-\kappa}, \qquad \text{as } t \to \infty.$$

Proof. We need to show that $\lim_{t\to\infty} t^\kappa \mathbb{Q}(\beta_1\eta_1 > t) = \Gamma(\kappa+1)C_0$. By conditioning on η_1 we obtain

$$t^\kappa \mathbb{Q}(\beta_1\eta_1 > t) = \int_0^\infty t^\kappa Q(\beta_1 > t/y)e^{-y}\, dy.$$

The tail asymptotics of β_1 from Lemma 4.13 imply that there is a constant $K < \infty$ such that $t^\kappa Q(\beta_1 > t) \leq K$ for all $t \geq 0$. Thus, $t^\kappa Q(\beta_1 > t/y) = y^\kappa (t/y)^\kappa Q(\beta_1 > t/y) \leq y^\kappa K$ and so we may apply the dominated convergence theorem and Lemma 4.13 to obtain

$$\lim_{t\to\infty} \mathbb{Q}(\beta_1\eta_1 > t) = \int_0^\infty \lim_{t\to\infty} t^\kappa Q(\beta_1 > t/y)e^{-y}\, dy = \int_0^\infty C_0 y^\kappa e^{-y}\, dy = C_0\Gamma(\kappa+1).$$

□

As noted above, the β_k are stationary but not independent. However, the following Lemma shows that they the dependence is rather weak.

Lemma 4.15. *For each $n \geq 1$, there exists a stationary sequence $\{\beta_k^{(n)}\}_{k\geq 1}$ such that*

(i) If $I \subset \mathbb{N}$ is such that $|k-j| > \sqrt{n}$ for all $k, j \in I$ with $k \neq j$, then $\{\beta_k\}_{k\in I}$ is an independent family of random variables.

(ii) There exist constants $C, C' > 0$ such that

$$Q\left(|\beta_k - \beta_k^{(n)}| > e^{-n^{1/4}}\right) \leq Ce^{-C\sqrt{n}}.$$

Sketch of proof. For any $k, n \geq 1$, let $\omega^{(k,n)}$ be then environment ω modified by

$$\omega_x^{(k,n)} = \begin{cases} 1 & \text{if } x = \nu_{k-1} - \lfloor\sqrt{n}\rfloor \\ \omega_x & \text{if } x \neq \nu_{k-1} - \lfloor\sqrt{n}\rfloor. \end{cases}$$

That is, we modify the environment by putting a reflecting barrier to the right at a distance $\sqrt{n}$ to the left of the ladder location ν_{k-1}. We then define $\beta_k^{(n)} = E_{\omega^{(n,k)}}^{\nu_{k-1}}[T_{\nu_k}]$. The claimed independence properties of the sequence $\{\beta_k^{(n)}\}_{k\geq 1}$ is then obvious.

Since backtracking of the random walk is exponentially unlikely (recall Lemma 4.13), it seems reasonable that modifying the environment a distance $\sqrt{n}$ to the left of the starting location won't change the expected crossing time by much. In fact, by using the exact formulas for quenched expectations of hitting times in (5) it can be shown that

$$\beta_k - \beta_k^{(n)} = 2\left(\sum_{j=\nu_{k-1}}^{\nu_k - 1} \prod_{i=\nu_{k-1}}^{j} \rho_i\right)\left(\sum_{i\leq \nu_{k-1}-\lfloor\sqrt{n}\rfloor} \prod_{j=i}^{\nu_{k-1}-1} \rho_j\right).$$

Note that in the second term in parenthesis on the right, each product in the sum has at least $\sqrt{n}$ terms, and as was shown in the proof of Lemma 4.13 with high probability these products are exponentially small in the number of terms in the product. This can be used to show the second claimed property of the sequence $\{\beta_k^{(n)}\}$. □

We are now ready to give a (sketch) of the proof of Theorem 4.9

Sketch of proof of Theorem 4.9. First of all, since our assumptions imply that the random walk is transient, the walk spends only finitely many steps to the left of the origin. Since the measures P and Q can be coupled so that they only differ to the left of the origin it can be shown that if the limiting distributions hold under $\mathbb{Q}$ then they also hold under $\mathbb{P}$.

Secondly, note that the gaps between ladder locations $\nu_k - \nu_{k-1}$ are i.i.d. and thus

$$\lim_{n\to\infty} \frac{\nu_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \nu_k - \nu_{k-1} = E_P[\nu_1]$$

In fact, it is easy to see that ν_1 has exponential tails so that the deviations of ν_n/n are exponentially unlikely. From this, it can be shown that if $\bar{\alpha} = 1/E_P[\nu_1]$ then $n^{-1/\kappa}(T_{\nu_{\bar{\alpha}n}} - T_n) \to 0$ in $\mathbb{Q}$-probability.

We have thus reduced the problem to proving limiting distributions for $T_{\nu_n} = \sum_{k=1}^{n} U_k$ under the measure $\mathbb{Q}$. As mentioned above, the key will be to be able to approximate the crossing times between ladder locations $U_k = T_{\nu_k} - T_{\nu_{k-1}}$ by $\beta_k \eta_k$. To this end, we will create a coupling of the random variables U_k and $\beta_k \eta_k$. For simplicity we will describe this coupling when $k = 1$ only. The crossing time $U_1 = T_{\nu_1}$ can be thought of as a series of excursions away from the origin. There will be a random number G of excursions that return to the origin before reaching ν_1 (we will call these excursions "failures") followed by an excursions that goes from 0 to ν_1 without first returning to 0 (we will call this a "success" excursion). That is, if we time of the i-th failure excursion by F_i and the time of the success excursion by S then we can represent

$$T_{\nu_1} = \sum_{i=1}^{G} F_i + S.$$

It is easy to see that the number of failure excursions in this decomposition is geometric with distribution

$$P_\omega(G = k) = (1 - p_\omega)^k p_\omega, \qquad \text{where } p_\omega = \omega_0 P_\omega^1(T_{\nu_1} < T_0).$$

We will couple T_{ν_1} with $\beta_1 \eta_1$ by coupling the exponential random variable η_1 with the geometric random variable G. This is accomplished by letting

$$G = \lfloor c_\omega \eta_1 \rfloor, \qquad \text{where } c_\omega = \frac{-1}{\log(1 - p_\omega)}.$$

It can be shown that this coupling is good enough enough so that

$$\lim_{n\to\infty} n^{-2/\kappa} Var_\omega \left(T_{\nu_n} - \sum_{k=1}^{n} \beta_k \eta_k \right) = 0, \qquad \text{in } Q\text{-probability.} \tag{32}$$

Now, for any $\varepsilon, \delta > 0$

$$\begin{aligned}\mathbb{Q}\left(\left|T_{\nu_n} - \sum_{k=1}^n \beta_k \eta_k\right| > \delta n^{1/\kappa}\right) &= E_Q\left[P_\omega\left(\left|T_{\nu_n} - \sum_{k=1}^n \beta_k \eta_k\right| > \delta n^{1/\kappa}\right)\right] \\ &\leq \varepsilon + Q\left(P_\omega\left(\left|T_{\nu_n} - \sum_{k=1}^n \beta_k \eta_k\right| > \delta n^{1/\kappa}\right) > \varepsilon\right) \\ &\leq \varepsilon + Q\left(n^{-2/\kappa}\mathrm{Var}_\omega\left(T_{\nu_n} - \sum_{k=1}^n \beta_k \eta_k\right) \geq \varepsilon\delta\right).\end{aligned}$$

Since (32) implies that this last probability vanishes as $n \to \infty$ and since $\varepsilon > 0$ was arbitrary we can conclude that

$$\lim_{n\to\infty} n^{-1/\kappa}\left(T_{\nu_n} - \sum_{k=1}^n \beta_k \eta_k\right) = 0, \qquad \text{in } \mathbb{Q}\text{-probability.}$$

Finally, we are down to proving a limiting distribution for $\sum_{k=1}^n \beta_k \eta_k$ under the measure $\mathbb{Q}$. However, Lemma 4.14 shows that the random variables $\beta_k \eta_k$ have well behaved polynomial tails, and Lemma 4.15 shows that they are close enough to i.i.d. to have limiting distributions of the same form as in Examples 4.4 and 4.5. □

We now state the corresponding averaged limiting distributions for X_n when $\kappa \in (0, 2]$.

Theorem 4.16. *Let Assumptions 1, 2, and 3 hold. If κ is defined as in* (24) *then*

(i) If $\kappa \in (0,1)$, then there exists a $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/\kappa}} \leq t\right) = 1 - L_{\kappa,b}(t^{-1/\kappa}), \quad \forall t.$$

(ii) If $\kappa = 1$, then there exists a constant $b > 0$ and a sequence $\delta(n) \sim n/(b \log n)$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - \delta(n)}{n/(\log n)^2} \leq t\right) = 1 - L_{1,b}(-b^2 t), \quad \forall t.$$

(iii) If $\kappa \in (1,2)$, then there exists a $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{n^{1/\kappa}} \leq t\right) = 1 - L_{\kappa,b}(t\mathrm{v}_P^{-1-1/\kappa}), \quad \forall t.$$

(iv) If $\kappa = 2$, then there exists a $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{\mathrm{v}_P^{3/2} b\sqrt{n \log n}} \leq t\right) = \Phi(t), \quad \forall t.$$

Remark 4.17. Note that we have stated Theorem 4.16 so that the constants b in each case are the same scaling parameters appearing in the limiting distributions of the hitting times in Theorem 4.9. Note that when $\kappa \in [1,2)$ one can simplify the limits on the right hand side by using the fact that $L_{\kappa,b}(ct) = L_{\kappa,bc^{-\kappa}}(t)$ for any $c > 0$. In this way the right hand side can be written as

$$1 - L_{\kappa,\bar{b}}(-t), \quad \text{where} \quad \bar{b} = \begin{cases} \frac{1}{b} & \kappa = 1 \\ b\mathrm{v}_P^{1+\kappa} & \kappa \in (1,2). \end{cases}$$

From this it is clear that when $\kappa \in [1,2)$ the limiting distribution is a totally skewed to the left κ-stable distribution. In contrast, when $\kappa \in (0,1)$ the limiting distribution is $1 - L_{\kappa,b}(t^{-1/\kappa})$ which is not a κ-stable distribution but is instead a transformation of a κ-stable distribution. This particular transformation of κ-stable distributions is sometimes referred to as a Mittag-Leffler distribution.

Proof. The proof of Theorem 4.16 follows from Theorem 4.9 in essentially the same way that Theorem 3.8 followed from Theorem 3.5. For example, when $\kappa \in (0,1)$ we have that

$$\mathbb{P}\left(\frac{X_n^*}{n^\kappa} < t\right) = \mathbb{P}(X_n^* < tn^\kappa) = \mathbb{P}(T_{\lceil tn^\kappa \rceil} > n) = \mathbb{P}\left(\frac{T_{\lceil tn^\kappa \rceil}}{\lceil tn^\kappa \rceil^{1/\kappa}} > \frac{n}{\lceil tn^\kappa \rceil^{1/\kappa}}\right).$$

Since $\frac{n}{\lceil tn^\kappa \rceil^{1/\kappa}} \to t^{-1/\kappa}$ as $n \to \infty$ it follows from Theorem 4.9 that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n^*}{n^\kappa} < t\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{T_{\lceil tn^\kappa \rceil}}{\lceil tn^\kappa \rceil^{1/\kappa}} > t^{-1/\kappa}\right) = 1 - L_{\kappa,b}(t^{-1/\kappa}).$$

The proofs of the cases when $\kappa \in (1,2)$ or $\kappa = 2$ are similar and therefore ommitted. The proof of the case when $\kappa = 1$ is slightly more difficult due to the somewhat strange centering term $nD(n)$ in the limiting distribution for T_n. While Theorem 4.9 states that $D(n) \sim b \log n$, one actually better control of the function $D(n)$ to prove the limiting distribution for X_n in this case. In fact, it turns out that the proof of Theorem 4.9 gives $D(n) = (1/\bar{\nu})E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \leq n\}}]$ (note the similarity to the centering term in Example 4.4 when $\alpha = 1$). From this explicit form for $D(n)$ and the tail asymptotics of β_1 in Lemma 4.13, it follows that there exists a function $\delta(x)$ such that

$$\delta(x) D(\delta(x)) = x + o(1), \qquad \text{as } x \to \infty.$$

If the centering term for X_n is chosen in this way, then one can prove the claimed limiting distribution for X_n in the case $\kappa = 1$. (The details of this argument in the case when $\kappa = 1$ can be found in [KKS75, pp. 167-8].) $\square$

4.3 Weak Quenched Limiting Distributions - $\kappa < 2$

We now turn our attention the study of the asymptotics of the quenched distribution of hitting times when $\kappa < 2$. Much of the work that we did in the proof of the averaged limiting distributions in Theorem 4.9 was done with this in mind. Recall that in the case $\kappa > 2$ we proved a quenched central limiting distribution. We will refer to this as a *strong* quenched limiting distribution since the convergence holds for P-a.e. environment ω. The main result in this subsection shows that there is no such strong quenched limiting distribution for the hitting times when $\kappa < 2$. Instead, we will prove a what we will call a *weak* quenched limiting distribution.

Let $\mathcal{M}_1(\mathbb{R})$ denote the space of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field. Recall that prohorov metric ρ on $\mathcal{M}_1(\mathbb{R})$ is defined by

$$\rho(\mu, \pi) = \inf\{\varepsilon : \mu(A) \leq \pi(A^{(\varepsilon)}) + \varepsilon, \, \forall A \in \mathcal{B}(\mathbb{R})\},$$

where $A^{(\varepsilon)} = \{x : \operatorname{dist}(x, A) < \varepsilon\}$. The Prohorov metric ρ induces the topology of weak convergence (i.e., convergence in distribution) on the space $\mathcal{M}_1(\mathbb{R})$, and the metric space $(\mathcal{M}_1(\mathbb{R}), \rho)$ is a Polish space.

We will be interested in studying random probability measures - that is $\mathcal{M}_1(\mathbb{R})$-valued random variables. If π_n is a sequence of $\mathcal{M}_1(\mathbb{R})$-valued random variables and π is another $\mathcal{M}_1(\mathbb{R})$ valued random variable we will use the notation $\mu_n \Longrightarrow \mu$ to denote convergence in distribution of $\mathcal{M}_1(\mathbb{R})$-valued random variables.

Remark 4.18. Note that the notation $\Longrightarrow$ for convergence in distribution of random probability measures should not be confused with the standard convergence of measures in the space $\mathcal{M}_1(\mathbb{R})$. The notation $\mu_n \Longrightarrow \mu$ means that

$$\lim_{n\to\infty} \mathbf{E}[\phi(\mu_n)] = \mathbf{E}[\phi(\mu)], \quad \text{for all bounded continuous } \phi : \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}.$$

On the other hand, pointwise convergence in the space $\mathcal{M}_1(\mathbb{R})$, which we would denote $\mu_n \to \mu$, is equivalent to

$$\lim_{n\to\infty} \int \phi(x) \, d\mu_n = \int \phi(x) \, d\mu, \quad \text{for all bounded continuous } \phi : \mathbb{R} \to \mathbb{R}.$$

Now, for any environment $\omega \in \Omega$ and any $n \geq 1$ define $\mu_{n,\omega,\kappa} \in \mathcal{M}_1(\mathbb{R})$ by

$$\mu_{n,\omega,\kappa}(\cdot) = P_\omega\left(\frac{T_n - E_\omega[T_n]}{n^{1/\kappa}} \in \cdot \right).$$

Since the environment ω is itself random, then we can view $\mu_{n,\omega,\kappa}$ as a $\mathcal{M}_1(\mathbb{R})$-valued random variable (or a *random probability measure*). In order to define the limiting random probability measures that will arise we need to introduce some notation. Let $\mathcal{M}_p$ denote the space of Radon point processes on $(0, \infty]$ - i.e., point processes with finitely many points on $[x, \infty]$ for any $x > 0$. We will equip $\mathcal{M}_p$ with the standard topology of vague convergence (see [Res08] for more information on point processes and the definition of vague convergence). Let $F \subset \mathcal{M}_p$ denote the subset of point processes $N = \sum_{i\geq 1} \delta_{x_i}$ such that

$$\int x^2 \, N(dx) = \sum_{i\geq 1} x_i^2 < \infty.$$

Then, define the function $H : \mathcal{M}_p \to \mathcal{M}_1(\mathbb{R})$ by

$$H(N) = \begin{cases} \mathbf{P}_\eta\left(\sum_{k\geq 1} x_k(\eta_k - 1) \in \cdot \right) & \text{if } N = \sum_{k\geq 1} \delta_{x_k} \in F, \\ \delta_0 & \text{if } N \notin F, \end{cases} \tag{33}$$

where $\{\eta_k\}_{k\geq 1}$ is an i.i.d. sequence of Exp(1) random variables with distribution $\mathbf{P}_\eta$.

Remark 4.19. Note that the condition $N \in F$ guarantees that the random sum $\sum_{k\geq 1} x_k \eta_k$ is finite $\mathbf{P}_\eta$-a.s. The definition of $H(N)$ when $N \notin F$ is arbitrary and will not matter since we will only be considering point processes that are almost surely in F.

Theorem 4.20. *If Assumptions 1, 2, and 3 hold and the parameter $\kappa < 2$, then there exists a $\lambda > 0$ such that $\mu_{n,\omega,\kappa} \Longrightarrow H(N_{\lambda,\kappa})$, where $N_{\lambda,\kappa}$ is a non-homogeneous Poisson point process with intensity $\lambda\kappa x^{-\kappa-1}$.*

Sketch of proof. As in the proof of the averaged stable limit laws for T_n, the proof of Theorem 4.20 is accomplished by the following reductions. First we show that $\mu_{n,\omega,\kappa}$ has approximately the same distribution on $\mathcal{M}_1(\mathbb{R})$ when $\omega \sim P$ and when $\omega \sim Q$. Secondly, we show that it is enough to prove a similar weak quenched limiting distribution for the quenched distribution of T_{ν_n} instead of T_n, and finally we show that we can couple T_{ν_n} with a sum of exponential random variables so that it is enough to study the quenched distribution of $\sum_{k=1}^n \beta_k \eta_k$, where $\beta_k = \beta_k(\omega)$ is as defined in (31) and the η_k are i.i.d. Exp(1) random variables that are independent of the β_k. That is, letting $\sigma_{n,\omega,\kappa} \in \mathcal{M}_1(\mathbb{R})$ be the random probability measure defined by

$$\sigma_{n,\omega,\kappa}(\cdot) = \mathbf{P}_\eta \left(\frac{1}{n^{1/\kappa}} \sum_{k=1}^n \beta_k(\eta_k - 1) \in \cdot \right), \tag{34}$$

it is enough to show that there exists a $\lambda' > 0$ such that $\sigma_{n,\omega,\kappa} \Longrightarrow H(N_{\lambda',\kappa})$ as $n \to \infty$ when ω has distribution Q.

Note that in the definition of $\sigma_{n,\omega,\kappa}$ in (34), the distribution is entirely determined by the coefficients β_k. Thus, the key to understanding the random probability distribution $\sigma_{n,\omega,\kappa}$ is understanding the joint distribution of the coefficients β_k. To this end, let $N_{n,\omega,\kappa}$ be the point process

$$N_{n,\omega,\kappa} = \sum_{k=1}^n \delta_{\beta_k / n^{1/\kappa}}. \tag{35}$$

Then, it can be shown that $N_{n,\omega,\kappa}$ converges in distribution under Q to a non-homogeneous Poisson point process $N_{\lambda',\kappa}$. Recall that in the proof of Theorem 4.9 we remarked that under the distribution Q the β_k have heavy tails and are fast-mixing enough to be close to i.i.d. If the β_k were i.i.d. then the convergence of $N_{n,\omega,\kappa}$ to the Poisson process $N_{\lambda',\kappa}$ would be standard, but since the β_k are not quite i.i.d. it takes a little extra work.

Recall the definition of the function $H : \mathcal{M}_p \to \mathcal{M}_1(\mathbb{R})$ from (33). Then, the definitions of the point process $N_{n,\omega,\kappa}$ and the random measure $\sigma_{n,\omega,\kappa}$ imply that $\sigma_{n,\omega,\kappa} = H(N_{n,\omega,\kappa})$. Since we know the point processes $N_{n,\omega,\kappa}$ converge in distribution, this suggests that $\sigma_{n,\omega,\kappa}$ should converge in distribution to $H(N_{\lambda',\kappa})$. Unfortunately the function H is not continuous, and so we need to do a modification. The details of this truncation and the rest of the full proof of Theorem 4.20 can be found in [PS10]. □

5 RWRE on $\mathbb{Z}^d$ - $d \geq 2$

We now turn to discussion of multi-dimesional RWRE. For nearest-neighbor RWRE on $\mathbb{Z}$ an environment can be encoded by a single number $\omega_x \in [0,1]$ at every site $x \in \mathbb{Z}$. However, for multi-dimensional RWRE we need a probability vector at every site. To simplify things we will only consider the case of nearest-neighbor RWRE, but obviously one can consider RWRE on $\mathbb{Z}^d$ with bounded jumps as well (although less is known in the more general bounded jumps case). In this case an environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}^d}$ where ω_x is a probability distribution on $\mathcal{E} = \{z \in \mathbb{Z}^d : |z| = 1\} = \{\pm e_i,\, i = 1, 2, \dots d\}$ in the sense that

$$\omega_x = (\omega_x(z))_{z\in\mathcal{E}} \in [0,1]^{\mathcal{E}} \quad \text{with } \sum_{z\in\mathcal{E}} \omega_x(z) = 1.$$

Given an environment $\omega = \{\omega_x\}_x = \{\omega_x(z)\}_{x,z}$, the quenched transition probabilities for the random walk are given by

$$P_\omega(X_{n+1} = x + z \,|\, X_n = x) = \begin{cases} \omega_x(z) & \text{if } z \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

As with our coverage of one-dimensional RWRE we will restrict ourselves to i.i.d. uniformly elliptic environments.

Assumption 4. *The environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}^d}$ is i.i.d. under the distribution P on environments.*

Assumption 5. *There exists a constant $c > 0$ such that*

$$P(\omega_x(z) \geq c, \forall x \in \mathbb{Z}^d, z \in \mathcal{E}) = 1.$$

5.1 Directional transience/recurrence

The first natural thing to study for multi-dimensional RWRE is the question of recurrence or transience. Unfortunately, as we will see below, in contrast to the one-dimensional case where there is a nice explicit criterion for recurrence/transience (see Theorem 2.1) even the question of directional transience/recurrence is not yet settled. Let $S^{d-1} = \{z \in \mathbb{R}^d : |z| = 1\}$ be the $d-1$-dimensional sphere. We will refer to a fixed $\ell \in S^{d-1}$ as a *direction* in $\mathbb{R}^d$. For any such fixed direction ℓ we will define the event of transience in direction ℓ by

$$A_\ell = \left\{ \lim_{n\to\infty} X_n \cdot \ell = +\infty \right\}. \tag{36}$$

The following lemma was proved by Kalikow in [Kal81].

Lemma 5.1 (Kalikow's 0-1 Law)**.** *If the distribution on environments P satisfies Assumptions 4 and 5, then*

$$\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0,1\}, \quad \forall \ell \in S^{d-1}.$$

Proof. First, we claim that $\mathbb{P}(A_\ell \cup A_{-\ell} \cup \mathcal{O}_\ell) = 1$, where $\mathcal{O}_\ell$ is the event

$$\mathcal{O}_\ell = \{X_n \cdot \ell \text{ changes sign infinitely many times}\}.$$

If none of the events A_ℓ, $A_{-\ell}$ or $\mathcal{O}_\ell$ are satisfied, then either

$$0 \le \liminf_{n\to\infty} X_n \cdot \ell < \infty \quad \text{or} \quad -\infty < \limsup_{n\to\infty} X_n \cdot \ell \le 0. \tag{37}$$

Indeed, in the first case in (37) there must be some $x > 0$ such that $|X_n \cdot \ell - x| \le 1$ infinitely many times. However, by uniform ellipticity the probability of the random walk visiting $\{|z \cdot \ell - x| \le 1\}$ infinitely many times without ever reaching the half-space $\{z \cdot \ell < 0\}$ is zero.

Next, for $\ell \in S^{d-1}$ let $D_\ell := \inf\{n \ge 0 : X_n \cdot \ell < X_0 \cdot \ell\}$ be the first time the random walk "backtracks" in direction ℓ from its initial location (note that D_ℓ is a stopping time and that we have stated the definition to account for starting locations other than $X_0 = \mathbf{0}$). If $\mathbb{P}(D_\ell = \infty) = 0$ then $P_\omega^x(D_\ell < \infty) = 1$ for all $x \in \mathbb{Z}^d$ and P-a.e. environment ω. By the strong Markov property this implies that $\mathbb{P}(\liminf_{n\to\infty} X_n \cdot \ell < 0) = 1$ and so $\mathbb{P}(A_\ell) = 0$. Taking the contrapositive of this we obtain that

$$\mathbb{P}(A_\ell) > 0 \quad \Longrightarrow \quad \mathbb{P}(D_\ell = \infty) > 0.$$

To complete the proof of the lemma, we may assume that either $\mathbb{P}(A_\ell) > 0$ or $\mathbb{P}(A_{-\ell}) > 0$ since otherwise the conclusion of the lemma is obvious. Without loss of generality we will assume that $\mathbb{P}(A_\ell) > 0$. Since we showed above that $\mathbb{P}(A_\ell \cup A_{-\ell} \cup \mathcal{O}_\ell) = 1$, it will be enough to show that $\mathbb{P}(\mathcal{O}_\ell) = 0$ whenever $\mathbb{P}(A_\ell) > 0$. To this end, first note that

$$\mathbb{P}(\mathcal{O}_\ell \cap \{\sup_{n\ge 0} X_n \cdot \ell < \infty\}) = 0,$$

for by uniform ellipticity every time $X_n \cdot \ell$ switches from negitive the probability that the random walk reaches the halfspace $\{z \cdot \ell > x\}$ before $\{z \cdot \ell < 0\}$ is uniformly bounded below. Next, we claim that $\mathbb{P}(\mathcal{O}_\ell \cap \{\sup_{n\ge 0} X_n \cdot \ell = \infty\}) = 0$ as well. To see this, we introduce a sequence of stopping times $B_1 \le F_1 \le B_2 \le F_2 \le \dots$ defined as follows.

$$B_1 = D_\ell, \quad F_k = \inf\{n > B_k : X_n \cdot \ell > \max_{i<n} X_i \cdot \ell\}, \quad \text{and} \quad B_{k+1} = \inf\{n > F_k : X_n \cdot \ell < 0\}.$$

(Note that if $B_k = \infty$ for some k then $F_j = B_j = \infty$ also for all $j \ge k$.) The times B_k are certain "backtracking" times where the random walk enters the halfspace to the left of the origin, and the F_k are the first "fresh times" where the random walk reaches a new portion of the environment farther to the right than it had previously reached. On the event $\mathcal{O}_\ell \cap \{\sup_{n\ge 0} X_n \cdot \ell < \infty\}$ it is clear that $B_k < \infty$ for all $k < \infty$. However, when $B_{k+1} < \infty$ by decomposing according to the location of the random walk at time F_k we obtain

$$\begin{aligned}
\mathbb{P}(B_{k+1} < \infty) &= \sum_z \mathbb{P}(F_k < \infty,\, X_{F_k} = z,\, B_{k+1} < \infty) \\
&= \sum_z E_P\left[P_\omega(F_k < \infty,\, X_{F_k} = z) P_\omega^z(\inf_{n\ge 0} X_n \cdot \ell < 0)\right] \\
&\le \sum_z E_P\left[P_\omega(F_k < \infty,\, X_{F_k} = z) P_\omega^z(D_\ell < \infty)\right]
\end{aligned}$$

Note that the quenched probabilities inside the last expectation are independent since $P_\omega(F_k < \infty,\, X_{F_k} = z)$ is $\sigma(\omega_x : x \cdot \ell < z \cdot \ell)$-measureable and $P_\omega^z(D_\ell < \infty)$ is $\sigma(\omega_x : x \cdot \ell \ge z \cdot \ell)$-measurable.

Thus, we obtain that

$$\begin{aligned}\mathbb{P}(B_{k+1} < \infty) &\le \sum_z \mathbb{P}(F_k < \infty,\, X_{F_k} = z)\mathbb{P}^z(D_\ell < \infty) \\ &= \mathbb{P}(F_k < \infty)\mathbb{P}(D_\ell < \infty) \\ &\le \mathbb{P}(B_k < \infty)\mathbb{P}(D_\ell < \infty).\end{aligned}$$

Since $\mathbb{P}(B_1 < \infty) = \mathbb{P}(D_\ell < \infty)$ we obtain by induction that $\mathbb{P}(B_k < \infty) \le \mathbb{P}(D_\ell < \infty)^k$ for any $k \ge 1$. Since $\mathbb{P}(D_\ell < \infty) < 1$ whenever $\mathbb{P}(A_\ell) > 0$ we have that

$$\mathbb{P}(\mathcal{O}_\ell \cap \{\sup_{n\ge 0} X_n \cdot \ell = \infty\}) \le \mathbb{P}(B_k < \infty,\, \forall k \ge 1) \le \lim_{k\to\infty} \mathbb{P}(D_\ell < \infty)^k = 0.$$

Thus we have shown that $\mathbb{P}(\mathcal{O}_\ell) = 0$ whenever $\mathbb{P}(A_\ell) > 0$ and so $\mathbb{P}(A_\ell \cup A_{-\ell}) = 1$ whenever $\mathbb{P}(A_\ell) > 0$. □

MORE YET TO BE ADDED....

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