Math 266, Practice Midterm 2

This is a 1-hour exam. No calculators or notes are allowed. Please show your work (except on multiple choice questions). Each multiple choice question has a single correct answer. If you finish early, you may bring your exam up to the front and leave the room.

Name: ____

Section: MWF 3:30-4:30 MWF 4:30 - 5:30

Useful things to remember: 1 N = 1 Newton = $1\,\text{kg}\cdot\text{m/s}^2$.

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

$$e^{it} = \cos(t) + i\sin(t).$$

It's all right to leave answers that would require complicated arithmetic in unsimplified form.

1. *Consider a differential equation of the form*

$$y'' + p(t)y' + q(t)y = g(t). \tag{1}$$

Suppose that $\{y_1, y_2\}$ *are an fundamental set of complex-valued solutions to the associated homogeneous equation*

$$y'' + p(t)y' + q(t)y = 0, \tag{2}$$

with $y_2 = \overline{y_1}$. *Also suppose that* $y = t^2$ *is a solution to* (1). *What is the general real-valued solution to* (1)*?*

   (a) $y = t^2 + C_1 y_1 + C_2 y_2$
   (b) $y = t^2 + C_1 \mathrm{Re}(y_1) + C_2 \mathrm{Re}(y_2)$
   (c) $y = t^2 + C_1 \mathrm{Re}(y_1) + C_2 \mathrm{Im}(y_1)$
   (d) $y = t^2 + C_1 \mathrm{Re}(y_1) + C_2 \mathrm{Im}(y_1) + C_3 \mathrm{Re}(y_2) + C_4 \mathrm{Im}(y_2)$

   Since $y_2 = \overline{y_1}$, we have $\mathrm{Re}(y_1) = \mathrm{Re}(y_2)$ and $\mathrm{Im}(y_1) = -\mathrm{Im}(y_2)$. In particular, $\mathrm{Re}(y_1)$ and $\mathrm{Re}(y_2)$ are linearly dependent, so they can't both be part of a fundamental set of solutions to (2). This rules out (b) and (d). Since (a) is complex-valued, **the correct answer must be (c)**.

   (Note that $\mathrm{Re}(y_1)$ and $\mathrm{Im}(y_1)$ can't be linearly dependent. If they were, say $\mathrm{Im}(y_1) = b\mathrm{Re}(y_1)$, then $y_1 = (1 + ib)\mathrm{Re}(y_1)$, and then $y_2 = (1 - ib)\mathrm{Re}(y_1)$, which would make y_1 and y_2 linearly dependent.)

2. *Consider the first-order initial value problem*

$$y' = t - y^2, \quad y(0) = y_0.$$

What is the approximate value for $y(2)$ *computed by Euler's method with step size 1?*

   (a) $y(2) \approx y_0 - y_0^2 + 1 - (y_0 - y_0^2)^2$
   (b) $y(2) \approx 2 - y_0^2$
   (c) $y(2) \approx 1 - (y_0 - y_0^2)^2$
   (d) $y(2) \approx y_0 + 2 - y_0^2 - (y_0 + 1 - y_0^2)^2$

   We calculate

$$\begin{aligned} y'(0) &= 0 - y(0)^2 = -y_0^2, \\ y(1) &\approx y(0) + 1 \cdot y'(0) = y_0 - y_0^2, \\ y'(1) &= 1 - y(1)^2 \approx 1 - (y_0 - y_0^2)^2, \\ y(2) &\approx y(1) + 1 \cdot y'(1) \approx y_0 - y_0^2 + 1 - (y_0 - y_0^2)^2. \end{aligned}$$

   So **the correct answer is (a)**.

3. *An RLC circuit has a resistor with a variable resistance, R. The current I is described by the formula*

$$4I'' + RI' + I/4 = 0.$$

For what values of R will the current decrease over time and oscillate as it does so?

(a) $0 < R < 1$
(b) $0 < R < 2$
(c) $0 \le R \le 2$
(d) $0 < R$
(e) $R \ge 2$

The characteristic polynomial is

$$4r^2 + Rr + 1/4 = 0,$$

which has roots

$$r = \frac{-R \pm \sqrt{R^2 - 4}}{8}.$$

For the current to oscillate, the general solution must include sine and cosine terms. This requires r to be non-real, meaning that $R^2 - 4 < 0$ or $|R| < 2$. For the current to decrease over time, the real part of r must be negative, which means that $R > 0$. **The correct answer is (b)**.

4. *Consider a differential equation of the form*

$$y'' + \alpha y' + 4y = e^{2t}.$$

For what value(s) of α will the equation have a solution of the form $y = Ate^{2t}$?

(a) $\alpha = -4$
(b) $\alpha = 4$
(c) $\alpha = \pm 4$
(d) *There is no such value of α.*

The right thing to use for y in the method of undetermined coefficients is $y = Ae^{2t}$. This will work unless e^{2t} is a solution of the associated homogeneous equation,

$$y'' + \alpha y' + 4y = 0. \tag{3}$$

The characteristic polynomial of (3) is

$$r^2 + \alpha r + 4 = 0.$$

If this were to factor as $(r-2)(r-c)$ for some number c, we see by checking the constant terms that $c = 2$, making $\alpha = -4$. But in this case, the characteristic polynomial has a repeated root, which means that the general solution to (3) is

$$y = C_1 e^{2t} + C_2 t e^{2t}.$$

In particular, Ate^{2t} is a solution to (3) and thus *not* a solution to the inhomogeneous equation given. So **the right answer is (d)**.

5. *The tides at Cardiff oscillate according to the formula*

$$y(t) = (5\,\mathrm{in})\cos(t/(12\,\mathrm{hr})) + (1\,\mathrm{ft})\cos(t/(12\,\mathrm{hr})).$$

(a) *What are the amplitude and period of the motion?*
We want to write

$$y(t) = R\cos(\omega t - \delta) = R\cos(\delta)\cos(\omega t) + R\sin(\delta)\sin(\omega t).$$

Comparing this with the given equation, we have $\omega = 1/(12\,\mathrm{hr})$. This is the frequency, and **the period is** $2\pi/12$ **hours**. Moreover, **the amplitude is**

$$R = \sqrt{5^2 + 12^2}\,\mathrm{in} = 13\,\mathrm{in}.$$

We also have

$$\delta = \tan^{-1}(12/5).$$

(Since both the coefficients are positive, δ is an angle in the first quadrant, meaning that we don't need to add π.)

(b) *What is the first time after* $t = 0$ *at which the tide is at its maximum?*
The above gave us

$$y = 13 \cdot \cos(t/12 - \tan^{-1}(12/5)).$$

The function $\cos(x)$ hits its maximum when x is a multiple of 2π. Clearly, the first of these multiples we will encounter here is when $x = 0$, or when

$$t = 12 \cdot \tan^{-1}(12/5)\,\mathrm{hr}.$$

6. *Find the general solution to the equation*

$$t^2y'' + t(t-3)y' - (t-3)y = 0, \quad t > 0.$$

(Hint: one solution is $y = t$.)

We use the method of reduction of order. Suppose that a solution has the form $y = tv$. Then $y' = tv' + v$ and $y'' = tv'' + 2v'$. Substituting into the equation, we get

$$t^2(tv'' + 2v') + t(t-3)(tv' + v) - (t-3)(tv) = 0.$$

The terms involving v cancel, leaving

$$t^3v'' + 2t^2v' + t^2(t-3)v' = 0.$$

Dividing by t^2 (which is harmless because $t > 0$) gives

$$tv'' = (1-t)v'.$$

Let $w = v'$; then this is a first-order equation for w, which separates to

$$\frac{1}{w}\,dw = \frac{1-t}{t}\,dt = \left(\frac{1}{t} - 1\right)dt.$$

Integrating gives

$$\begin{aligned} \ln|w| &= -t + \ln(t) + C \quad \text{(note that } t > 0\text{)} \\ |w| &= Ate^{-t} \quad (A > 0) \\ v' = w &= Ate^{-t} \quad (A \text{ arbitrary}) \end{aligned}$$

We can integrate this by parts to get

$$v = -Ate^{-t} + \int Ae^{-t}\,dt = -A(t+1)e^{-t} + B.$$

So, relabelling $-A$ as A, **the general solution is**

$$y = vt = A(t+1)te^{-t} + Bt.$$

7. *Find any solution to the equation*

$$y'' + y = 1 + \tan(x), \quad -\pi/2 < x < \pi/2.$$

The associated homogeneous equation is

$$y'' + y = 0,$$

which has general solution

$$y = C_1 \sin(x) + C_2 \cos(x).$$

For the inhomogeneous equation, we use variation of parameters, meaning we assume the solution takes the form

$$y = u_1 \sin(x) + u_2 \cos(x).$$

We additionally assume

$$u_1' \sin(x) + u_2' \cos(x) = 0, \tag{4}$$

which means that

$$y' = u_1 \cos(x) - u_2 \sin(x).$$

Differentiating again gives

$$y'' = u_1' \cos(x) - u_2' \sin(x) - u_1 \sin(x) - u_2 \cos(x).$$

Substituting into the original equation, we get

$$u_1' \cos(x) - u_2' \sin(x) - u_1 \sin(x) - u_2 \cos(x) + u_1 \sin(x) + u_2 \cos(x) = 1 + \tan(x)$$

or

$$u_1' \cos(x) - u_2' \sin(x) = 1 + \tan(x). \tag{5}$$

Now, (4) implies that $u_2' = -u_1' \tan(x)$. Plugging this into (5), we see that

$$\begin{aligned} u_1' \cos(x) + u_1' \sin(x) \tan(x) &= 1 + \tan(x) \\ u_1' \cos^2(x) + u_1' \sin^2(x) &= \cos(x) + \sin(x) \\ u_1' &= \sin(x) + \cos(x) \\ u_1 &= -\cos(x) + \sin(x) + C \end{aligned}$$

Likewise, $u_2' = -u_1' \tan(x) = -\sin^2(x)/\cos(x) + \sin(x)$. We can take the integral of $\sin^2(x)/\cos(x)$ by doing the substitution $u = \sin(x)$ and $du = \cos(x)\,dx$. So

$$\begin{aligned} \int \frac{\sin^2(x)}{\cos(x)}\,dx &= \int \frac{\sin^2(x)\cos(x)}{\cos^2(x)}\,dx \\ &= \int \frac{u^2}{1-u^2}\,du \\ &= -\frac{1}{2}\int \left(\frac{u}{u-1} + \frac{u}{u+1}\right) du \\ &= -\frac{1}{2}\int \left(1 + \frac{1}{u-1} + 1 - \frac{1}{u+1}\right) du \\ &= -u - \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \\ &= -\sin(x) - \frac{1}{2}\ln\left(\frac{1-\sin(x)}{1+\sin(x)}\right) + C. \end{aligned}$$

Finally,

$$u_2 = -\cos(x) + \sin(x) + \frac{1}{2}\ln\left(\frac{1-\sin(x)}{1+\sin(x)}\right) + C.$$

Taking both constants of integration to be zero gives one solution:

$$y = \sin^2(x) - \cos^2(x) + \frac{\cos(x)}{2}\ln\left(\frac{1-\sin(x)}{1+\sin(x)}\right).$$

8. *A 1 kg mass stretches a spring 0.4 m. The mass-spring system starts at equilibrium and is acted on by a variable force* $F_{\text{ext}}(t) = \cos(5t)$*. Write the equation describing the displacement of the mass as a function of time, and describe in words what happens to the spring. You may assume that the spring is undamped and* $g = 10\,\text{m/s}^2$.

First we calculate the spring constant using $mg = kL$, or

$$k = mg/L = (1\,\text{kg})(10\,\text{m/s}^2)/(0.4\,\text{m}) = 25\,\text{kg/s}^2.$$

The differential equation for the displacement of the mass is

$$u'' + 25u = \cos(5t).$$

We first solve the associated homogeneous equation,

$$u'' + 25u = 0.$$

The characteristic polynomial is $r^2 + 25 = 0$, with imaginary roots $r = \pm 5i$. So the general solution to the homogeneous equation is

$$u = C_1 \cos(5t) + C_2 \sin(5t).$$

The inhomogeneous equation can now be solved in a number of ways – I'll do variation of parameters. We would like to try $u = A\cos(5t) + B\sin(5t)$, but this overlaps with the solutions to the associated homogeneous equation. So instead we will use

$$u = At\cos(5t) + Bt\sin(5t).$$

Then

$$u' = A\cos(5t) + B\sin(5t) - 5At\sin(5t) + 5Bt\cos(5t),$$
$$u'' = -10A\sin(5t) + 10B\cos(5t) - 25At\cos(5t) - 25Bt\sin(5t).$$

Substituting into the equation gives

$$u'' + 25u = -10A\sin(5t) + 10B\cos(5t) = \cos(5t).$$

Thus $A = 0$ and $B = 1/10$, and we get the particular solution

$$u = \frac{1}{10}t\sin(5t).$$

The general solution is then

$$u = \frac{1}{10}t\sin(5t) + C_1\cos(5t) + C_2\sin(5t).$$

Since the spring starts from equilibrium, we have $u(0) = 0$ and $u'(0) = 0$, which translates to $C_1 = C_2 = 0$. So **the equation of the spring's motion is**

$$u = \frac{1}{10}t\sin(5t).$$

The spring oscillates with ever-increasing amplitudes (because the external force is acting at the resonant frequency, and there is no damping to resist the growth of the amplitude).

(Scratch paper)

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