TAF and the $K(n)$-local sphere

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1 Introduction and motivation

These notes are an attempt to understand some of the interesting ideas that take up the later part of Behrens and Lawson's monograph on topological automorphic forms [3]. In chapters 12 and 13, they use TAF and the theory of buildings to construct a cosimplicial resolution of a spectrum conjectured to be related to the $K(n)$-local sphere. In chapter 14, they calculate the $K(n)$-localization of this spectrum and of TAF itself as a homotopy fixed point spectrum of a product of Morava E-theories.

As always, the best way to motivate this is to look at the more familiar height 1 and 2 cases. Letting $\mathcal{E}$ be the sheaf of spectra on $\mathcal{M}_{ell}$ whose global sections are TMF, the $K(2)$-localization of TMF is then given by the sections of $\mathcal{E}$ over the formal neighborhood of the supersingular locus – that is,

$$L_{K(2)}TMF \simeq \mathcal{E}((\mathcal{M}_{ell}^{ss} \otimes \mathbb{F}_p)^\wedge) \simeq \mathcal{E}((\mathcal{M}_{ell}^{ss} \otimes \overline{F_p})^\wedge)^{\mathrm{hGal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}.$$

The supersingular locus, however, is very simple – there are a finite number of supersingular curves over $\overline{\mathbb{F}}_p$ (in fact, they're all defined over $\mathbb{F}_{p^2}$), each with a finite automorphism group that we can explicitly write down. In other words,

$$\mathcal{M}_{ell}^{ss} \otimes \mathbb{F}_p \cong \coprod_{C \text{ supersingular}} B\mathrm{Aut}(C).$$

Moreover, deformations of a supersingular curve C over an Artinian local ring R are the same as deformations of its formal group $\widehat{C}$ over R by the Serre-Tate theorem, which are the same as continuous maps $E(\overline{\mathbb{F}}_p, \widehat{C}) \to R$ by the Lubin-Tate theorem, $E(\overline{\mathbb{F}}_p, \widehat{C})$ being the Lubin-Tate ring associated to $\widehat{C}$. So in fact,

$$L_{K(2)}TMF \simeq \left(\prod_{C \text{ supersingular}} E(\overline{\mathbb{F}}_p, \widehat{C})^{\mathrm{hAut}(C)} \right)^{\mathrm{hGal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)} \simeq \left(\prod_{C \text{ supersingular}} E_2^{\mathrm{hAut}(C)} \right)^{\mathrm{hGal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}$$

When $p = 2$ or 3, there is just one supersingular curve C, defined over $\mathbb{F}_p$, and its automorphism group, sometimes written G_2, is a maximal finite subgroup of $\mathrm{Aut}(\widehat{C}) = \mathbb{S}_2$, the Morava stabilizer group. So in these cases, we get

$$L_{K(2)}TMF \simeq E_2^{\mathrm{h}(G_2 \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p))} \simeq EO_2^{\mathrm{hGal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)},$$

where $EO_2 = E_2^{\mathrm{h}G_2}$ is the height 2 **higher real K-theory** of Hopkins and Miller.

Since $L_{K(2)}S = E_2^{\mathrm{h}\mathbb{G}_2}$, the homotopy fixed points for the continuous action of the full Morava stabilizer group, this higher real K-theory can be thought of as our best approximation to the $K(2)$-local sphere using only equivariant homotopy theory of finite groups. In particular, it's better at detecting v_2-periodicity than E_2 itself. What we've done is construct this EO_2 explicitly in terms of TMF, which we at least pretend to understand better.

At $n = 1$, things are even simpler: $\mathbb{S}_1 \cong \mathbb{Z}_p^\times$, whose maximal finite subgroup is $\mathbb{Z}/(p-1)$ for p odd and $(\mathbb{Z}/2)^2$ for $p = 2$. So for $p = 2$, at least, we get

$$L_{K(1)}KO \simeq EO_1^{\mathrm{hGal}}.$$

Hence the name ‘higher real K-theory’.

Something similar is expected at larger heights. Recall that the Shimura variety $Sh(K^p)$ represents abelian varieties of dimension n^2, with complex multiplication by some fixed division algebra B, a prime-to-p polarization, and a K^p-level structure, where K^p is some compact open subgroup of $GU(\mathbb{A}^{p,\infty})$. The structure of B is such that any of these abelian varieties has a canonical 1-dimensional, height n summand of its p-divisible group, typically written $\epsilon A(u)$ in [3]. (u is one of the prime factors of p in a quadratic subfield F of B, and ϵ is the idempotent splitting this summand.) The formal part of this p-divisible group is thus a 1-dimensional formal group of height at most n. Define $Sh(K^p)^{[n]}$ to be the closed subscheme of $Sh(K^p)^{[n]} \otimes \mathbb{F}_p$ at which this formal group has height exactly n, i. e. $\epsilon A(u)$ is a formal group. As it turns out, this is a zero-dimensional scheme, and with a little work, we can produce a similar formula, of the form

$$L_{K(n)}TAF = \left(\prod E_n^{\mathrm{h}\Gamma}\right)^{\mathrm{hGal}},$$

where the product ranges over the points of $Sh(K^p)^{[n]}(\overline{\mathbb{F}}_p)$, and Γ is the automorphism group of each point. When we take sufficiently high level structure that the Shimura variety is a scheme, this reduces to

$$L_{K(n)}TAF = \left(\prod E_n\right)^{\mathrm{hGal}}.$$

One application is an attempt at resolving the $K(n)$-local sphere by simpler objects, as with the GHMR resolution [4]. In [1], Behrens defines a ‘half the sphere’ spectrum $Q(2)$ at the prime 3, fitting into a cofiber sequence

$$DQ(2) \to L_{K(2)}S \to Q(2),$$

and admitting an explicit cosimplicial resolution in terms of $K(2)$-local TMF with level structure that in some sense reinterprets that of [4]. Likewise, [3] defines an analogous spectrum Q_U at larger heights, proves a homotopy fixed-point formula similar to the one for TAF, and uses this to resolve Q_U in terms of TAF. Unfortunately, it is still unknown if Q_U is half the $K(n)$-local sphere in the above sense. (It is only known that $Q(\ell)$ is half the $K(2)$-local sphere when $\ell = 2$ and $p = 3$ or 5.)

As another application, one could try to construct EO_n using the homotopy fixed-point formula. It seems, though, that we got lucky early on: at $n = 2$ and $p = 2$ or 3, there is a single supersingular elliptic curve on which the Galois group acts trivially, and there’s a unique maximal finite subgroup of $\mathbb{S}_n$. All these things tend to fail in larger dimensions. In fact, Behrens and Hopkins prove [2] that EO_n is a summand of $L_{K(n)}TAF$ if and only if $p = 2, 3, 5$, or 7, $n = (p-1)p^{r-1}$, and p^r divides the order of G_n (some uncertainty remains at $p = 2$).

Note added August 2017: I’d intended to have these notes include the definition of the building complex and computations of the Galois cohomology of the similitude groups, but never got around to writing this up. This means that any appearances of these cohomology computations in the proofs will have to be taken on faith, or referred to careful study of [3] and [5]. I hope that the rest might still be of some use to those trying to understand TAF.

2 Notation, the Shimura variety, and the topological automorphic forms spectrum

We begin by fixing a whole lot of notation.

Definition 1. Let S be a set of primes, possibly including ∞. The **adèles at** S are the restricted product

$$\mathbb{A}_S = \prod_{p\in S}{}' \mathbb{Q}_p = \left\{ (x_p) \in \prod_{p\in S} \mathbb{Q}_p : x_p \in \mathbb{Z}_p \text{ for almost all } p \right\},$$

where $\mathbb{Q}_\infty$ is understood to be $\mathbb{R}$. The **adèles away from** S, $\mathbb{A}^S$ are the adèles at the complement of S.

Notation 2.
- p is a fixed prime of $\mathbb{Z}$.

- F is a quadratic imaginary extension of $\mathbb{Q}$, with $(\cdot)^c$ its nontrivial $\mathbb{Q}$-automorphism, such that p splits as uu^c in F.
- B is a central simple algebra over F of dimension n^2, which splits over u and u^c (that is, $B_u \cong M_n(F_u)$ and likewise with u^c).
- $(\cdot)^*$ is a positive involution on B extending $(\cdot)^c$. (Positive means that $\mathrm{Tr}_{B/\mathbb{Q}}(xx^*) > 0$ for all $0 \neq x \in B$.)
- $\mathscr{O}_F$ is the ring of integers of F, and $\mathscr{O}_B$ is a maximal order in B such that $\mathscr{O}_{B,(p)}$ is preserved by $(\cdot)^*$.
- Since B is split over u, we have $\mathscr{O}_{B,u} \cong M_n(\mathscr{O}_{F,u}) \cong M_n(\mathbb{Z}_p)$; fixing such an isomorphism, let $\epsilon \in \mathscr{O}_{B,u}$ be the matrix with a 1 in its (1,1) entry and 0 everywhere else.
- As before, V is B viewed as a free left B-module, together with a $\mathbb{Q}$-valued alternating $*$-hermitian form $\langle \cdot, \cdot \rangle$; L is an $\mathscr{O}_B$-lattice in V such that $\langle L, L \rangle \subseteq \mathbb{Z}$; and $(\cdot)^\dagger$ is the associated involution on $\mathrm{End}_B(V)$.
- We further impose the conditions that $L_{(p)}$ is self-dual for $\langle \cdot, \cdot \rangle$, and that the signature of $\langle \cdot, \cdot \rangle$ after tensoring with $\mathbb{R}$ (for one of the embeddings $F \hookrightarrow C$) is $(1, n-1)$. This means that $U(\mathbb{R}) \cong U(1, n-1)$, where U is the algebraic group defined in section 2.
- $K_0^p \subseteq GU(\mathbb{A}^{p,\infty})$ is the compact open subgroup of automorphisms preserving $L^p = L \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty}$, and K^p is a compact open subgroup of K_0^p.

Definition 3. The **height** n **PEL Shimura variety** $Sh(K^p)$ associated to the above data is the stack which sends a (locally noetherian, connected, locally killed by a power of p) scheme S to the groupoid of objects $(A, i, \lambda, [\eta])$, where

- A is an abelian scheme over S of dimension n^2,
- $\lambda \in \mathrm{Hom}(A, A^\vee)_{(p)}$ is a polarization,
- $i : \mathscr{O}_{B,(p)} \hookrightarrow \mathrm{End}(A)_{(p)}$ is a ring homomorphism that sends the involution $(\cdot)^*$ to the λ-Rosati involution, and such that the height-n summand $\epsilon A(u)$ of the p-divisible group of A is 1-dimensional,
- and $[\eta]$ is the K^p-orbit of an $\mathscr{O}_B$-linear isomorphism $L^p \cong T^p(A_s)$, for some geometric point $s \in S$, that is $\pi_1(S, s)$-invariant and sends the pairing $\langle \cdot, \cdot \rangle$ to a scalar multiple of the λ-Weil pairing.

The morphisms of the groupoid are isomorphisms of abelian schemes over S preserving all the data. Note that this is, in fact, a formal stack over $\mathrm{Spf}(\mathbb{Z}_p)$.

The data above is such that points of the Shimura variety are 'B-linear' abelian varieties of dimension n^2, whose p-divisible groups have canonical 1-dimensional height n summands $\epsilon A(u)$. The formal component of $\epsilon A(u)$, sometimes written $\epsilon A(u)^0$, is then a 1-dimensional formal group of height at most n associated to A. Using the Serre-Tate theorem and a theorem of Lurie that generalizes the Goerss-Hopkins-Miller theorem, Behrens and Lawson prove the following.

Theorem 4. *There is a sheaf of E_∞-ring spectra $\mathcal{E}(K^p)$ over $Sh(K^p)^\wedge_p$ such that if $U = \mathrm{Spf}\, R$ is a formal affine étale open of $Sh(K^p)^\wedge_p$, classifying an abelian scheme $(A, i, \lambda, [\eta])$ over R, then $\mathcal{E}(K^p)(U)$ is complex-oriented with formal group canonically isomorphic to $\epsilon A(u)^0$.*

Definition 5. The topological automorphic forms spectrum associated to $Sh(K^p)$ is the global sections

$$TAF(K^p) = \mathcal{E}(K^p)(Sh(K^p)^\wedge_p).$$

3 The isometry and similitude groups

Given an abelian variety A over $\mathbb{Z}_{(p)}$ of dimension n^2, we can collect all its ℓ-adic Tate modules for $\ell \neq p$ and form

$$V^p(A) = \left(\prod_{\ell \neq p} T_\ell(A) \right) \otimes \mathbb{Q},$$

a free $2n^2$-dimensional module over $\mathbb{A}^{p,\infty}$, with a canonical lattice

$$L^p(A) = \prod_{\ell \neq p} T_\ell(A).$$

A polarization λ on A induces a Weil pairing

$$\langle \cdot, \cdot \rangle : V^p(A) \otimes V^p(A) \to \mathbb{A}^{p,\infty}$$

which is integral on $L^p(A)$. The polarization also induces a Rosati involution on $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$, given by

$$f^\dagger : A \xrightarrow{\lambda} A^\vee \xrightarrow{f^\vee} A^\vee \xrightarrow{\lambda^{-1}} A.$$

This is related to the Weil pairing in that

$$\langle f(x), y \rangle = \langle x, f^\dagger(y) \rangle.$$

Definition 6. An isogeny $f : A \to A$ is an **isometry** if $f^\dagger f = 1$, which is to say that $f^*\lambda = \lambda$. It is a **similitude** if $f^\dagger f \in \mathbb{Q}^\times \subseteq \mathrm{End}^0(A)$, which is to say that f preserves λ up to a scalar.

These basic definitions can be abstracted in several ways. First, we could eliminate the abelian variety from the data, and just consider a $\mathbb{Q}$-vector space V together with a nondegenerate alternating form $\langle \cdot, \cdot \rangle$, with $(\cdot)^\dagger$ the involution induced on $\mathrm{End}(V)$. Isometries and similitudes are now certain endomorphisms of this vector space, and they more generally form algebraic groups, the **isometry group**

$$U : R \mapsto \{g \in (\mathrm{End}(V) \otimes_{\mathbb{Q}} R)^\times : g^\dagger g = 1\}$$

and the **similitude group**

$$GU : R \mapsto \{g \in (\mathrm{End}(V) \otimes_{\mathbb{Q}} R)^\times : g^\dagger g \in R^\times\}.$$

There's a short exact sequence

$$1 \to U \to GU \to \mathbb{G}_m \to 1.$$

Second, for the purposes of TAF, we often have complex multiplication by an algebra B on all our abelian varieties, and thus on V, and all this works in the B-linear setting [6], where B satisfies the hypotheses above.

The groups U and GU are then defined as above, with $\mathrm{End}(V)$ replaced by $\mathrm{End}_B(V) =\cong B^{op}$. When it won't cause confusion, we'll write U and GU for $U(\mathbb{A}^{p,\infty})$ and $GU(\mathbb{A}^{p,\infty})$ respectively. TAF can be formalized as a GU-equivariant spectrum.

What sort of groups are these? They're not profinite, but they are locally compact and totally disconnected because the ad'eles are (away from ∞). As a result, one has to jump through a few hoops in order to talk about GU-equivariant spectra – this is chapter 10 of [3]. The minimum I can get away with saying about this is that, if G-spectra for a finite group G are thought of as built out of equivariant cells $G/H \times D^n$, then G-spectra for our groups G should be built out of cells $G/H \times D^n$ where H is only allowed to be a *compact open* subgroup of G. These are called **smooth G-spectra**. One defines the spectrum of **smooth G-maps** between two smooth G-spectra as

$$\mathrm{Maps}_{G,sm}(X, Y) = \operatorname{colim} \mathrm{Maps}_H(X, Y),$$

where the colimit ranges over compact open subgroups $H \leq G$.

Finally, in the case where $R = \mathbb{Q}_\ell$, we additionally define

$$GU^1(\mathbb{Q}_\ell) = \{g \in \mathrm{End}(V_\ell)^\times : g^\dagger g \in \mathbb{Z}_\ell^\times\}.$$

This sits in between $U(\mathbb{Q}_\ell)$ and $GU(\mathbb{Q}_\ell)$, but does not extend to an algebraic group.

4 The building complex resolution

Bruhat-Tits buildings are simplicial complexes used to study algebraic groups. The ideas surrounding them were a big hang-up for me when I was first trying to learn this stuff, so with your permission, I'll say what we need from our buildings, do the homotopy theory involved in the resolution, and then take a look at the buildings for TMF and TAF.

The building complex $\mathcal{B}_\bullet$ is a simplicial set satisfying the following properties.

- $\mathcal{B}_\bullet$ is obtained from a finite-dimensional simplicial complex by freely adding degeneracies.
- A locally compact, totally disconnected group G acts on $\mathcal{B}_\bullet$, with compact open stabilizers.
- $\mathcal{B}_\bullet$ is contractible, and more generally satisfies the following key lemma, whose proof will be given in the next section.

Lemma 7. *For X a smooth G-spectrum, the map*

$$X \to \left|\mathrm{Maps}_{G,sm}(\mathcal{B}_\bullet, X)\right|$$

induced by $\mathcal{B}_\bullet \to *$ is an equivalence of smooth G-spectra. (G acts on the target by conjugation.)

In the case at hand, G is a non-compact open subgroup of $GU(\mathbb{A}^{p,\infty})$ of the form

$$G = GU(\mathbb{Q}_\ell) \times_{m \neq p,\ell} K_m$$

where the K_m are all compact open subgroups of $GU(\mathbb{Q}_m)$. (These latter factors are supposed to come from the level structure $K^p = \prod_{m\neq p} K_m$ already imposed on the TAF spectrum we're working with; they act trivially on the building complex.) We also define

$$G^1 = GU^1(\mathbb{Q}_\ell) \times_{m \neq p,\ell} K_m.$$

Modelling $TAF(K^p)$ as a smooth GU-spectrum, we can then define

$$Q_{GU}(K^{p,\ell}) = TAF(K^p)^{\mathrm{h}G} \quad \text{and} \quad Q_U(K^{p,\ell}) = TAF(K^p)^{\mathrm{h}G^1}.$$

These spectra, especially the latter, are meant to generalize Behrens's $Q(\ell)$.

Remark 8. Even more generally, we could replace the single prime ℓ above by a set S of primes. It eludes me why $GU^1(\mathbb{Q}_\ell)$ rather than $U(\mathbb{Q}_\ell)$ is used to define Q_U.

The building complex allows us to construct a resolution of $Q_{GU}(K^{p,\ell})$ in terms of finitely many spectra $TAF(K^p)^{\mathrm{h}H^p} = TAF(H^p)$, for H^p a compact open subgroup of $GU(\mathbb{A}^{p,\infty})$. To wit, write

$$TAF \simeq \left|\mathrm{Maps}_{GU,sm}(\mathcal{B}_\bullet, TAF)\right| \simeq \mathrm{colim}_H \operatorname{holim}_n \mathrm{Maps}_H(\mathcal{B}_n, TAF).$$

Here H ranges over compact open subgroups of GU. Since the colimit is filtered and the homotopy limit is finite (as $\mathcal{B}_\bullet$ is a finite-dimensional simplicial complex), they commute, so we get

$$TAF \simeq \operatorname{holim}_n \mathrm{Maps}_{GU,sm}(\mathcal{B}_n, TAF).$$

Now take homotopy fixed points with respect to the open subgroup G. This gives

$$TAF^{\mathrm{h}G} \simeq \left(\operatorname{holim}_n \mathrm{Maps}_{GU,sm}(\mathcal{B}_n, TAF)\right)^{\mathrm{h}G} \simeq \operatorname{holim}_n \mathrm{Maps}_G(\mathcal{B}_n, TAF).$$

But $\mathcal{B}_n$ is just a GU-set, so this splits up as

$$TAF^{\mathrm{h}G} \simeq \operatorname{holim}_n \prod_{[\sigma]\in\mathcal{B}_n/G} TAF^{\mathrm{h}K(\sigma)} \simeq \operatorname{holim}_n \prod_{[\sigma]\in\mathcal{B}_n/G} TAF(K(\sigma)).$$

Here $[\sigma]$ ranges over the G-orbits of $\mathcal{B}_n$, and $K(\sigma)$ is the stabilizer of σ, which is compact and open, allowing us to make the final equivalence.

Thus, we've written $Q_{GU} = TAF^{\mathrm{h}G}$ as a finite homotopy limit of spectra of topological automorphic forms with level structure. It's noted in [3] that this is actually a diagram of E_∞-ring spectra.

5 The resolution for TMF

To illustrate how this works, and delay the inevitable discussion of buildings, let's take a look at Behrens's resolution of $Q(\ell)$ using level-2 structures on TMF. Since elliptic curves carry canonical principal polarizations, we can ignore anything to do with the pairing, so the relevant algebraic groups are GL and SL rather than GU and U. Let $V_\ell \cong \mathbb{Q}_\ell^2$. The building for $GL(V_\ell)$ is defined as

$$\mathcal{B}_n = \{L_0 < \cdots < L_n \leq \ell^{-1} L_0\},$$

where the L_i are $\mathbb{Z}_\ell$-lattices in $\mathbb{Q}_\ell$. The face maps are given by forgetting lattices in the chain. This is a 2-dimensional simplicial complex with a $GL(V_\ell)$-action.

Extend V_ℓ to a 2-dimensional adèlic vector space $V^p = \prod_{m \neq p} V_m$, and for $m \neq \ell, p$, fix a lattice $L_0 \subseteq V_m$. (You should think of this lattice as being the m-adic Tate module of an elliptic curve, and V^p the rationalization of all the Tate modules.) The trivial level structure on TMF is that corresponding to the group $K_0^p \subseteq GL(V^p) = GL(\mathbb{A}^{p,\infty})$ of automorphisms preserving the given lattices. The group G from the previous section, then, is $GL(V_\ell) \times K_0^{p,\ell}$. This acts on the building complex if $GL(V_\ell)$ is given the obvious action and the other factors act trivially.

The G-orbits of the simplices of the building complex are easily computed.

0. $\mathcal{B}_0$ is just the set of lattices in $\mathbb{Q}_\ell$, and GL acts transitively on these, with stabilizer $K_{\ell,0}$. So the stabilizer for the full G-action is just K_0^p.

1. The 1-simplices are of the form $L_0 < L_1 \leq \ell^{-1} L_0$, and either $L_1 = \ell^{-1} L_0$, or it contains L_0 with index ℓ. These give two GL-orbits. In the first case, the stabilizer is again K_0^p. In the second case, the stabilizer is the group of automorphisms that preserve a lattice together with an index-ℓ subgroup, which is to say a $\Gamma_0(\ell)$-structure in the notation of Katz-Mazur.

2. The 2-simplices are of the form $L_0 < L_1 < L_2 = \ell^{-1} L_0$, just as in the second case above. They form a single orbit whose stabilizer corresponds to a $\Gamma_0(\ell)$-structure.

The building complex is thus of the form

$$\begin{tikzcd} Q(\ell) \arrow[r] & TMF \arrow[r, shift left] \arrow[r, shift right] & TMF \times TMF_0(\ell) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & TMF_0(\ell). \end{tikzcd}$$

With a little more thought, we can describe the arrows here too, which we'll do via the associated map of stacks

$$\begin{tikzcd} \mathcal{M}_{ell} & \mathcal{M}_{ell} \sqcup \mathcal{M}_{ell,0}(\ell) \arrow[l, shift left] \arrow[l, shift right] & \mathcal{M}_{ell,0}(\ell). \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

The first two arrows are given by forgetting one of the lattices in a chain $L_0 < L_1 \leq \ell^{-1} L_0$, which is given by either an elliptic curve E (when $L_1 = \ell^{-1} L_0$) or an elliptic curve with level-ℓ structure $A = L_1/L_0$ (when $L_1 < \ell^{-1} L_0$). In the first case, forgetting L_1 gives the same elliptic curve and forgetting L_0 gives that elliptic curve mod $[\ell]$. In the second case, forgetting L_1 forgets the level-ℓ structure, and forgetting L_0 gives the elliptic curve E/A.

The 2-simplices are full chains $L_0 < L_1 < L_2 = \ell^{-1} L_0$, which come from elliptic curves E with level-ℓ structure $A = L_1/L_0$. Forgetting L_1 puts us in the first case above – we've just forgotten the level structure on the curve. Forgetting L_2 gives the identity map $\mathcal{M}_{ell,0} \to \mathcal{M}_{ell,0}$. Forgetting L_0 sends (E, A) to $(E/A, E[\ell]/A)$.

6 The height n locus

We now turn to the study of $K(n)$-local TAF, which is mostly the study of the height n locus on the Shimura variety.

Definition 9. For a height n Shimura variety $Sh(K^p)$ as above, its **height n locus** $Sh(K^p)^{[n]}$ is the closed substack of $Sh(K^p) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ consisting of points where $\epsilon A(u)$ is a formal group.

The importance of this is as follows: the summand $\epsilon A(u)$ defines a map from $Sh(K^p)$ to the moduli stack $\mathcal{M}_p(n)$ of 1-dimensional, height n p-divisible groups, which is formally étale, basically by the Serre-Tate theorem. Composing this map with the map $\mathcal{M}_p(n) \to \mathcal{M}_{fg}^{\leq n}$ sending a p-divisible group to its formal part, and invoking a theorem of Lurie, we are able to construct a sheaf of E_∞-ring spectra on $Sh(K^p)$, whose global sections are $TAF(K^p)$. The $K(n)$-localization of TAF is the sections of this sheaf of spectra over the formal neighborhood of $\mathcal{M}_{fg}^{=n}$ (a closed point of $\mathcal{M}_{fg}^{\leq n}$), which is to say its sections over a formal neighborhood of $Sh(K^p)_{[n]}$.

Proposition 10. *If $Sh(K^p)$ is a scheme, then $Sh(K^p)^{[n]}$ is étale over* $\operatorname{Spec} \mathbb{F}_p$*, and in particular, 0-dimensional if it's not empty.*

Proof. Let A be a height n point of $Sh(K^p)$, defined over a finite field k. The Serre-Tate theorem gives an isomorphism

$$(Sh(K^p) \otimes \mathbb{F}_p)^\wedge_A \cong \mathrm{Def}_\epsilon A(u) \otimes \mathbb{F}_p \cong \operatorname{Spf} k[[u_1, \ldots, u_{h-1}]]$$

where the right-hand side is the Lubin-Tate space of deformations of the height n formal group $\epsilon A(u)$. The intersection of $(Sh(K^p) \otimes \mathbb{F}_p)^\wedge_A$ with $Sh(K^p)^{[n]}$ consists of those deformations of A whose one-dimensional formal group is still height n, meaning that $u_1, \ldots, u_{h-1}$ must vanish. Thus, $(Sh(K^p)^{[n]})^\wedge_A \cong \operatorname{Spec} k$ is étale over $\operatorname{Spec} \mathbb{F}_p$, and so $Sh(K^p)^{[n]}$ is formally étale over $\operatorname{Spec} \mathbb{F}_p$. If $Sh(K^p)$ is a scheme, it's finite type (even quasiprojective, see [6]), so $Sh(K^p)^{[n]}$ is étale over $\operatorname{Spec} \mathbb{F}_p$. □

Proposition 11. *$Sh(K^p)^{[n]}$ has an $\overline{\mathbb{F}}_p$-point.*

Proof. The B-linear structure of A induces a splitting

$$A(p) \cong (\epsilon A(u))^n \oplus (\epsilon^* A(u^c))^n \cong (\epsilon A(u) \oplus (\epsilon A(u))^\vee)^n.$$

If $\epsilon A(u)$ is dimension 1 and height n, then $\epsilon^* A(u^c)$ must have dimension $n-1$ and height n, and $\epsilon A(u)$ being all formal means that both are simple. So A is a point of $Sh(K^p)^{[n]}(\overline{\mathbb{F}}_p)$ iff the Newton polygon of $A(p)$ consists of a line of horizontal length n^2 and slope $1/n$ and a line of horizontal length n^2 and slope $(n-1)/n$. (In other words, if there were an étale component in $\epsilon A(u)$, this would show up as a line of slope 0 in the Newton polygon.) By the Honda-Tate classification, an abelian variety A with complex multiplication by $\mathscr{O}_{B,(p)}$ and the right slope decomposition exists over $\overline{\mathbb{F}}_p$.

The key point is that the formal height n condition is a condition on the Newton polygon of $A(p)$, which is a complete invariant of p-divisible groups up to isogeny; moreover, an isogeny of p-divisible groups $A(p) \to \mathbb{G}$ extends to an isogeny of abelian varieties from A to another abelian variety, since its kernel is just a subgroup of A. For example, we've constructed A to have the right slope decomposition, which means that $A(p)$ is *isogenous* to a p-divisible group that splits as required; but in order to make sure that $A(p)$ actually splits as required, we can just replace A by an isogenous abelian variety.

At this point, we have the abelian variety A and the B-linear structure i. Now we construct the $\mathbb{Z}_{(p)}$-polarization λ, whose Rosati involution is required to extend the involution on B. The existence of such things is established in Lemma 9.2 of [6]. Basically, after tensoring everything with $\mathbb{R}$, the compatibility condition is just a condition on the signature of the Weil pairing, which is satisfied by a nonempty convex cone in $\mathrm{Hom}_{\mathrm{sym}}(A, A^\vee) \otimes \mathbb{R}$; but $\mathrm{Hom}_{\mathrm{sym}}(A, A^\vee) \otimes \mathbb{Z}_{(p)}$ has a nonempty intersection with this.

Finally, we need the λ-Weil pairing to be similar to the given pairing on V^p. Similitude classes of such pairings are parametrized by the Galois cohomology

$$\prod_{\ell \neq p} H^1(\mathbb{Q}_\ell, GU).$$

When ℓ splits in F, or n is odd, this group is 0. When n is even and ℓ does not split, significantly more work is required. □

Next we need to determine the automorphisms of points of $Sh(K^p)^{[n]}$. Let $(A, i, \lambda, [\eta])$ be such a point. Since $A(p)$ is simple over B, $\mathrm{End}^0_B(A)$ is a central simple algebra over $Z(B) = F$ of dimension n^2. Such things are classified by the Brauer group $Br(F)$, which fits into the **fundamental exact sequence of class field theory**

$$0 \longrightarrow Br(F) \xrightarrow{\sum_v inv_v} \bigoplus_v Br(F_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

In other words, the structure of a CSA over F is given by a $\mathbb{Q}/\mathbb{Z}$-valued **local invariant** of each of its completions at places of F; the local invariants sum to $0 \in \mathbb{Q}/\mathbb{Z}$ and are almost all 0. The definition of the local invariants depends on the following proposition.

Proposition 12. *Let D be a central division algebra over a nonarchimedean local field F, of dimension n^2. Then D contains a maximal subfield L which is unramified over F. (Necessarily, $[L : F] = n$ and L splits D.)*

There's only one unramified degree n field extension L of a local field F of degree n, and $\mathrm{Gal}(L/F)$ is cyclic of order n, generated by an element σ. By the Noether-Skolem theorem, there's an $\alpha \in D^\times$, uniquely determined up to multiplication by $L^\times$, such that

$$\sigma(x) = \alpha x \alpha^{-1} \quad \text{for } x \in L.$$

Now, there's a unique extension of the valuation v on F to D: for $x \in D$, define $v(x)$ to be the valuation of x in any subfield of D containing x, and check that these agree. In general, this is valued in $\frac{1}{n}\mathbb{Z}$, assuming we want $v(F) = \mathbb{Z}$. Since L is unramified, we moreover have $v(L) = \mathbb{Z}$. So if $\sigma(x) = \alpha x \alpha^{-1}$ as above, the quantity $v(\alpha) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ doesn't depend on our choice of α. We define

$$inv_v(D/F) = v(\alpha) \in \mathbb{Q}/\mathbb{Z}.$$

To deal with archimedean places, recall that $Br(\mathbb{R}) \cong \mathbb{Z}/2$, generated by the quaternions $\mathbb{H}$; we define $inv_\infty(\mathbb{H}) = \frac{1}{2}$.

The Honda-Tate classification, or rather Kottwitz's B-linear extension [6], tells us the complete structure of the endomorphism algebra of a B-linear abelian variety A in the height n locus.

Theorem 13. *If A is a simple B-linear abelian variety over $\overline{F}_p$ such that $\mathrm{End}^0_B(A)$ and B are both central simple over F, then the local invariants of $\mathrm{End}^0_B(A)$ are given by*

- *$inv_v(\mathrm{End}^0_B(A)) = \frac{1}{2} - inv_v(B)$ for v archimedean,*
- *$inv_v(\mathrm{End}^0_B(A)) = -inv_v(B)$ for v not dividing p,*
- *and $inv_v(\mathrm{End}^0_B(A)) = s_v - inv_v(B)$ is the (unique) slope of the summand of $A(p)$ corresponding to v, if v divides p.*

In our case, B was assumed to split at p and ∞, so this says that

$$inv_u(\mathrm{End}^0_B(A)) = \frac{1}{n}, \qquad inv_{u^c}(\mathrm{End}^0_B(A)) = \frac{n-1}{n},$$

$\mathrm{End}^0_B(A)$ is ramified at the infinite places, and for other places v, $inv_v(\mathrm{End}^0_B(A)) = -inv_v(B)$.

Remark 14. As always, you should think about a supersingular elliptic curve, whose endomorphism $\mathbb{Q}$-algebra is the unique quaternion algebra over $\mathbb{Q}$ which is ramified at p and ∞.

What about the algebra $\mathrm{End}_B(A)$ itself? This is a maximal order in $\mathrm{End}^0_B(A)$, and

$$\mathrm{End}_B(A)_{(p)} = \mathrm{End}_B(A(p))_{(p)}$$

is also a maximal order (the unique one) in the division algebra $\mathrm{End}^0_B(A)_{(p)}$, since $A(p)$ is simple over B. The polarization λ is a $\mathbb{Z}_{(p)}$-quasi-isogeny, meaning that it becomes an honest isogeny after multiplication by some integer prime to p. As a result, the Rosati involution does not change p-adic valuation, meaning that it preserves the order $\mathrm{End}_B(A)_{(p)}$.

The upshot is that we can define 'integral' versions of our similitude groups:

$$GU_A : R \mapsto \{g \in (\mathrm{End}_B(A)_{(p)} \otimes_{\mathbb{Z}_{(p)}} R)^\times : g^\dagger g \in R^\times\}$$
$$U_A : R \mapsto \{g \in (\mathrm{End}_B(A)_{(p)} \otimes_{\mathbb{Z}_{(p)}} R)^\times : g^\dagger g = 1\}$$

and an exact sequence

$$1 \to U_A \to GU_A \to \mathbb{G}_m \to 1.$$

(Note these depend on i and λ as well as just A!)

Proposition 15. *There is an isomorphism*

$$U_A(\mathbb{Z}_p) \xrightarrow{\cong} \mathbb{S}_n,$$

where $\mathbb{S}_n$ is the nth Morava stabilizer group, via the action of the isometry group on $\epsilon A(u) \subseteq A(p)$.

Proof. I'm having trouble understanding this. By a theorem of Tate, $\mathrm{End}_B(A)_p \cong \mathrm{End}_B(A(p))$, so it seems like we should have

$$\mathbb{S}_n = \mathrm{End}(\epsilon A(u))^\times = \mathrm{End}_B(A(p))^\times = \mathrm{End}_B(A)_p^\times = GU_A(\mathbb{Z}_p).$$

□

Define

$$\Gamma = U_A(\mathbb{Z}_{(p)}),$$

an infinite group which is dense in $U_A(\mathbb{Z}_p)$, and maps into $GU(\mathbb{A}^{p,\infty})$ via its action on $V^p(A) \cong V^p$.

Proposition 16. *The automorphisms of the point $(A, i, \lambda, [\eta]) \in Sh(K^p)(\overline{F}_p)$ are given by $\Gamma \cap K^p$.*

Proof. We defined the Shimura variety so that the automorphisms of (A, i, λ, η) are automorphisms of (A, i) that are similitudes for λ (so in $GU_A(\mathbb{Z}_{(p)})$) and preserve the level structure (so in K^p). We just need to show that the norm of such an automorphism – the scalar by which it scales the Weil pairing – is 1. Well, this is automorphism is an honest isogeny, so its norm is in $\mathbb{Z}^\times = \{\pm 1\}$. The isogeny also scales the Rosati involution by its norm, but the Rosati involution of an abelian variety is positive definite, so the norm must be 1, as desired. □

Proposition 17. *If $(A, i, \lambda, [\eta])$ and $(A', i', \lambda', \eta')$ are two points of $Sh(K^p)^{[n]}(\overline{\mathbb{F}}_p)$, then there is a prime-to-p isogeny $(A, i, \lambda) \to (A, i', \lambda')$ (possibly not preserving the level structures.)*

Proof. $A(p)$ and $A'(p)$ have the same slope decompositions, so they are isogenous and thus A and A' have a prime-to-p isogeny. By the results of [6], these statements are also true B-linearly if we remember the B-action on the p-divisible groups. As in the proof of Proposition 11, we can make this quasi-isogeny a quasi-similitude of λ and λ'. We want to bump this up to a prime-to-p quasi-isometry, meaning that it's an isometry for λ and λ' and in $\mathrm{End}(A, A')_{(p)}$. The trick is to multiply it by some $R \in GU_A(\mathbb{Q})$ so that it's a quasi-isometry, and then multiply it by some $T \in U_A(\mathbb{Q})$ so that its denominator is prime to p. We can obviously do both these things in $GU_A(\mathbb{Q}_p)$ and $U_A(\mathbb{Q}_p) \cong \mathrm{End}^0_B(A)^\times_u$ respectively. So to finish the proof, we have to prove that $GU_A(\mathbb{Q})$ is dense in $GU_A(\mathbb{Q}_p)$, and $U_A(\mathbb{Q})$ is dense in $U_A(\mathbb{Q}_p)$. This is the content of the lemmas in [3][Section 14.2], building on work of Naumann [7]. □

Corollary 18. *If $Sh(K^p)$ is a scheme, then the set $Sh(K^p)^{[n]}(\overline{\mathbb{F}}_p)$ may be described by the double coset formulas*

$$Sh(K^p)^{[n]}(\overline{\mathbb{F}}_p) \cong GU_A(\mathbb{Z}_{(p)})\backslash GU(\mathbb{A}^{p,\infty})/K^p \cong GU_A(\mathbb{Q})\backslash GU_A(\mathbb{A}^\infty)/K^p GU_A(\mathbb{Z}_p) \cong \Gamma\backslash GU^1(\mathbb{A}^{p,\infty})/K^p.$$

Proof. We have just seen that any two of these points are in the same GU-orbit, and with level structure attached, they are in the same GU/K^p-orbit. The stabilizer of a point A is $GU_A(\mathbb{Z}_{(p)})$ almost by definition. This establishes the first isomorphism. The second one follows from the fact that $GU_A(\mathbb{Q})$ is dense in $GU_A(\mathbb{Q}_p)$. For the third, we have to check that the images of $GU_A(\mathbb{A}^\infty)$ and $GU_A(\mathbb{Q})$ under the compositions

$$GU_A(\mathbb{Q}) \to \mathbb{Q}^\times \xrightarrow{\text{valuations}} \bigoplus_{\ell \text{ prime}} \mathbb{Z}$$

and

$$GU_A(\mathbb{A}^\infty) \to (\mathbb{A}^\infty)^\times \xrightarrow{\text{valuations}} \bigoplus_{\ell \text{ prime}} \mathbb{Z}$$

agree.

Using the short exact sequence $1 \to U_A \to GU_A \to \mathbb{G}_m \to 1$, these images are the same as the kernels of the maps $\mathbb{Q}^\times \to H^1(\mathbb{Q}, U_A)$ and $(\mathbb{A}^\infty)^\times \to H^1(\mathbb{A}^\infty, U_A) = \bigoplus_\ell H^1(\mathbb{Q}_\ell, U_A)$. Using Galois cohomology computations, this is done. □

7 $K(n)$-local TAF

$TAF(K^p)$ is defined as the global sections of the sheaf of spectra $\mathcal{E}(K^p)$ over $Sh(K^p)^\wedge_p$. To calculate its $K(n)$-localization, it's helpful to introduce the auxiliary spectrum

$$TAF(K^p)_{\overline{\mathbb{F}}_p} = \mathcal{E}(K^p)((Sh(K^p) \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_p))^\wedge_p) = \mathrm{colim}_k\, \mathcal{E}(K^p)((Sh(K^p) \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_{p^k}))^\wedge_p).$$

From this, we can recover TAF by taking the Galois fixed points:

$$TAF(K^p) = (TAF(K^p)_{\overline{\mathbb{F}}_p})^{\mathrm{hGal}} = \mathrm{hofib}(TAF(K^p)_{\overline{\mathbb{F}}_p} \overset{\mathrm{Fr}-1}{T} AF(K^p)_{\overline{\mathbb{F}}_p}).$$

Proposition 19. *$TAF(K^p)$, $TAF(K^p)_{\overline{\mathbb{F}}_p}$, and $\mathcal{E}(K^p)(U)$ for any étale open $U \to Sh(K^p)^\wedge_p$ are all E_n-local.*

Proof. Essentially, this is because the formal group associated to any section in $Sh(K^p)(U)$ is height $\leq n$. This immediately proves that $\mathcal{E}(K^p)(U)$ is E_n-local if U is an affine formal scheme, and the general case, including the first two statements, follows by homotopy descent and the fact that E_n-localization is smashing. ☐

As a result, we can $K(n)$-localize any of these spectra via the completion formula

$$L_{K(n)}X \simeq \mathrm{holim}_{(j_0,\dots,j_{n-1})}\, X \wedge BP/(p^{j_0}, \cdots, v_{n-1}^{j_{n-1}}), \tag{1}$$

where $BP/(p^{j_0}, \cdots, v_{n-1}^{j_{n-1}})$ is a generalized Moore spectrum, defined by the Periodicity Theorem for a cofinal collection of multi-indices $(j_0, \dots, j_{n-1})$.

Now suppose that $Sh(K^p)$ is a scheme, so that the height n locus is a 0-dimensional subscheme. For $x \in Sh(K^p)^{[n]}$, the formal neighborhood of x in $Sh(K^p)$ is just Lubin-Tate space, and x is cut out from $Sh(K^p)$ precisely by the ideal $(p, \dots, v_{n-1})$. Thus,

$$(Sh(K^p) \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^k}))^\wedge_{(p,\dots,v_{n-1})} \cong \coprod \mathrm{Spf}\, W(\mathbb{F}_{p^k})[[v_0, \dots, v_{n-1}]], \tag{2}$$

a coproduct of Lubin-Tate spaces indexed by the points of the height n locus defined over $\mathbb{F}_{p^k}$. But there are only finitely many points of the height n locus, so they're all defined over some $\mathbb{F}_{p^k}$. With a little care I'll brush under the rug, one can show that the $K(n)$-localization of $TAF(K^p)_{\mathbb{F}_{p^k}}$, computed via (1), is the same as the sections of $\mathcal{E}$ over (2); taking the colimit over all k, the same goes for $L_{K(n)}TAF(K^p)_{\overline{\mathbb{F}}_p}$. But the height n dimensional formal group law obtained from any point of $Sh(K^p)^{[n]}$, defined over $\mathbb{F}_{p^k}$, is isomorphic to the Honda formal group law over $\overline{\mathbb{F}}_p$, and thus over some further extension $\mathbb{F}_{p^{k'}}$. Thus we get

$$L_{K(n)}TAF(K^p)_{\overline{\mathbb{F}}_p} \simeq \prod E_n,$$

the product still indexed over the geometric points of $Sh(K^p)^{[n]}$. If we'd like, we can express these points by the double coset formula

$$GU_A(\mathbb{Z}_{(p)})\backslash GU(\mathbb{A}^{p,\infty})/K^p \cong \Gamma\backslash GU^1(\mathbb{A}^{p,\infty})/K^p.$$

We thus obtain

Theorem 20. *If $Sh(K^p)$ is a scheme, then the $K(n)$-localization of $TAF(K^p)$ is*

$$L_{K(n)}TAF(K^p) \simeq \left(\prod_{\Gamma\backslash GU^1(\mathbb{A}^{p,\infty})/K^p} E_n \right)^{\mathrm{hGal}}.$$

In general, $Sh(K^p)$ is not a scheme but is a coequalizer of them, and we can make the analogous calculation by taking the colimit over covers $Sh(H^p)$ which are schemes. We quickly conclude the following.

Theorem 21. *In general,*

$$L_{K(n)}TAF(K^p) \simeq \left(\prod_{[g]\in\Gamma\backslash GU^1(\mathbb{A}^{p,\infty})/K^p} E_n^{\mathrm{h}(\Gamma\cap gK^pg^{-1})} \right)^{\mathrm{hGal}}.$$

The groups appearing in the homotopy fixed points are actually finite subgroups of GU, since they are the intersection of a discrete group with a compact group. By the same arguments, the map $\Gamma \to GU^1(\mathbb{A}^{p,\ell,\infty})$, and the definition

$$Q_U(K^{p,\ell}) = TAF^{\mathrm{h}GU^1(\mathbb{A}^{p,\ell,\infty})},$$

it follows that

$$L_{K(n)}Q_U(K^{p,\ell}) = \left(\prod_{[g]\in\Gamma\backslash GU^1(\mathbb{A}^{p,\ell,\infty})/K^{p,\ell}} E_n^{\mathrm{h}(\Gamma\cap gK^{p,\ell}g^{-1})} \right)^{\mathrm{hGal}}.$$

The groups appearing here are now infinite and discrete.

Again, it's helpful to think about TMF and the spectrum $Q(\ell)$ in the case $p = 3$. In this case, $\Gamma \cong (\mathrm{End}(C)_{(p)})^\times$ for the supersingular elliptic curve C, and so

$$\Gamma \cap K^{p,\ell} = (\mathrm{End}(C)[1/\ell])^\times, \qquad \text{and} \qquad \Gamma \cap K^p = \mathrm{Aut}(C) \cong G_2.$$

We have

$$L_{K(2)}Q(\ell) \cong E_2^{\mathrm{h}(\Gamma\rtimes\mathrm{Gal})}, \quad L_{K(2)}TMF \cong E_2^{\mathrm{h}(G_2\rtimes\mathrm{Gal})}, L_{K(2)}S \cong E_2^{\mathrm{h}(\mathbb{S}_2\rtimes\mathrm{Gal})}.$$

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