# Lecture 12: Simplicial stuff

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We've been approximating $H^*\mathrm{map}(BV, X)$ with $T_V H^* X$. We've done a lot of algebra so far – time to put in the homotopy theory. This is a thing called the **(co)homology spectral sequence of a cosimplicial space**. It's easy to define, but convergence is harder to come by. There's also the **Bousfield-Kan spectral sequence**, which is an unstable version of the Adams spectral sequence.

## (Co)simplicial objects

As usual, $\mathbf{\Delta}$ is the category whose objects are the ordered finite sets $[n] = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}$ and whose morphisms are the order-preserving functions. Every order-preserving map can be written as a composition of standard order-preserving maps

$$d^i : [n] \hookrightarrow [n+1] \qquad \text{(skip } i\text{)}$$
$$s^i : [n] \twoheadrightarrow [n-1] \qquad \text{(double } i\text{)}$$

For example, the map $[1] \to [2]$ sending both 0 and 1 to 0 is $d^2 d^1 s^0$. (There are relations between the $d$'s and $s$'s, which you can look up.)

$$\{0\} \;\begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix}\; \{0,1\} \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix}\; \{0,1,2\} \qquad \cdots$$

A **cosimplicial object** of a category $\mathcal{C}$ is a functor $X : \mathbf{\Delta} \to \mathcal{C}$, and a **simplicial object** is $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

*Example* 1. There's a cosimplicial topological space

$$\Delta^\bullet : [n] \mapsto \Delta^n$$

where $\Delta^n$ is the standard $n$-simplex

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum x_i = 1 \right\}.$$

The $d^i$ include various faces, and the $s^i$ project onto various faces.

*Example* 2. If $X$ is a space, $S_\bullet X = \mathrm{Hom}_{\mathsf{Top}}(\Delta^\bullet, X)$ is a simplicial object.

*Example* 3. Let $\mathcal{K}$ be the category of unstable algebras, and $G : \mathcal{K} \to \mathcal{K}$ the functor $UF$, the free algebra on the forgetful functor to graded vector spaces. There is a map in $\mathcal{K}$, $\epsilon_K : G(K) \to K$ and a map in graded vector spaces $s_{-1} : K \to G(K)$.

We get a simplicial diagram

$$\cdots \qquad G^3(K) \;\begin{smallmatrix}\xrightarrow{G^2\epsilon_K}\\ \longrightarrow\\ \xrightarrow[\epsilon_{G^2K}]{}\end{smallmatrix}\; G^2(K) \;\begin{smallmatrix}\xrightarrow{G\epsilon_K}\\ \longleftarrow\\ \xrightarrow[\epsilon_{GK}]{Gs_{-1}}\end{smallmatrix}\; G(K) \xrightarrow[\epsilon_K]{} K.$$

This is an augmented simplicial object $\epsilon : G^\bullet K \to K$.

*Example* 4. The category of functors $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Sets}$ is the category $\mathsf{sSets}$ of simplicial sets. There are simplicial sets

$$\Delta^n = \mathrm{map}_{\mathbf{\Delta}}(\cdot, [n]).$$

We have

$$\mathrm{Hom}_{\mathsf{sSets}}(\Delta^n, X) \cong X_n := X([n]).$$

**Definition 5.** Let $A \in \mathsf{sAb}$ (the category of simplicial abelian groups). We define the **normalization** to be

$$NA_n = \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1})$$

Then $d_0 : NA_n \to NA_{n-1}$ and $d_0^2 = 0$ since $d_0^2 = d_0 d_1$ (one of the simplicial relations left unmentioned earlier). Since $NA_n$ is an abelian group, $(NA, d_0)$ is a chain complex. Let $\pi_n A = H_n(NA_\bullet, d_0)$.

**Theorem 6** (Dold-Kan Theorem)**.** *Let $A \in \mathsf{sAb}$. Then*

$$A_n \cong \bigoplus_{\phi:[n]\twoheadrightarrow[m]} \phi^* NA_m,$$

*and $N : \mathsf{sAb} \to \mathsf{Ch}_*(\mathsf{Ab})$ is an equivalence of categories.*

This is a way of encoding the observation that we can rewrite the simplicial object

$$A_0 \rightleftarrows A_1 \overset{\rightrightarrows}{\leftleftarrows} A_2 \qquad \cdots$$

as

$$A_0 = NA_0 \rightleftarrows NA_1 \oplus s_0 NA_0 \overset{\rightrightarrows}{\leftleftarrows} NA_2 \oplus s_0 NA_1 \oplus s_0 NA_1 \oplus s_0^2 NA_0 \qquad \cdots,$$

from which the simplicial structure maps are all determined: $d_0$ is the differential on $NA_i$, the other $d$'s vanish on $NA_i$, and the simplicial relations (like $d_0 s_0 = d_0 s_1 = id$) determine the maps on the other pieces.

Let

$$L_n A = \operatorname*{colim}_{\substack{\phi:[n]\twoheadrightarrow[m] \\ \phi \neq id}} \phi^* NA_m.$$

This is the $n$th **latching object**. Then there is a split short exact sequence

$$0 \to L_n A \to A_n \to NA_n \to 0,$$

and so $NA_n \cong A_n/(s_0 A_{n-1} + \cdots + s_{n-1} A_{n-1})$.

Define

$$N^{(k)} A_n = \bigcap_{i \geq k} \ker(d_i : A_n \to A_{n-1}).$$

So $N^{(1)} A_n = NA_n$, and $N^{(k)} A_n = A_n$ for $k > n$. Define

$$\partial = \sum (-1)^i d_i.$$

On $NA$, this is just $d_0$. There is a nested sequence of chain complexes

$$NA \subseteq N^{(2)} A_\bullet \subseteq \cdots \subseteq A_\bullet.$$

**Proposition 7.** *The inclusion $N^{(k-1)} A_\bullet \hookrightarrow N^{(k)} A_\bullet$ is a chain equivalence with retraction $r(x) = x - s_{k-1} d_k(x)$.*

*Proof.* You can check that $s_k$ (with some sign) is a chain equivalence from $id$ to $r : N^{(k)} A \to N^{(k)} A$. □

In particular, $\pi_* A_\bullet = H_*\left(A, \sum (-1)^i d_i\right)$. If $\mathbb{Z}[\cdot]$ is the free abelian group functor on sets, then $\pi_* \mathbb{Z}[S_\bullet(X)] = H_*(X)$, for $X$ a space.

## The skeletal filtration

Let $\mathcal{C}$ be a category with colimits, and define $\mathsf{s}\mathcal{C} = \{X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. Letting $\mathbf{\Delta}_{\leq n} \subseteq \mathbf{\Delta}$ be the full subcategory on objects $[m]$, $m \leq n$, we also define $\mathsf{s}_n\mathcal{C} = \{X_\bullet : \mathbf{\Delta}^{\mathrm{op}}_{\leq n} \to \mathcal{C}\}$. We have a restriction functor

$$(i_n)^* : \mathsf{s}\mathcal{C} \to \mathsf{s}_n\mathcal{C}$$

and this has a left adjoint

$$(i_n)_! : \mathsf{s}_n\mathcal{C} \to \mathsf{s}\mathcal{C},$$

which we can define by

$$((i_n)_! X)_m = \operatorname*{colim}_{\substack{\phi:[m]\to[k] \\ k\leq n}} X_k = \operatorname*{colim}_{\substack{\phi:[m]\twoheadrightarrow[k] \\ k\leq n}} X_k$$

(the second isomorphism since the surjections are cofinal in all maps to $[k]$).

For example, if you just have an object

$$X_0 \rightrightarrows X_1,$$

then the level 2 piece of $(i_1)_! X_\bullet$ has to be defined as

$$(i_1)_! X_2 = s_0 X_1 \coprod_{s_0 s_0 X_0} s_1 X_1.$$

**Definition 8.** If $X \in s\mathcal{C}$, then

$$sk_n(X) = (i_n)_!(i_n)^* X.$$

This maps to $X$ by the counit of the above adjunction. Again using the above example, if $X$ is a simplicial object, we have an obvious map

$$s_0 X_1 \coprod_{s_0 s_0 X_0} s_1 X_1 \to X_2.$$

There is a sequence

$$sk_0 X \subseteq sk_1 X \subseteq \cdots \subseteq X_\bullet.$$

The union of the skeleta is $X_\bullet$, and $sk_n X$ agrees with $X_\bullet$ in degrees up to $n$.

*Remark* 9.

$$(sk_{n-1} X)_n = \operatorname*{colim}_{\substack{\phi:[n]\twoheadrightarrow[m] \\ m<n}} X_n = L_n X,$$

the $n$th latching object.

**Construction 10.** Let $X \in \mathsf{s}\mathcal{C}$, $K \in \mathsf{sSet}$. Then

$$(K \otimes X)_n = \coprod_{K_n} X_n = K_n \otimes X_n,$$

with face and degeneracy maps coming from $K$ and $X$.

For example, if $X \in \mathcal{C}$, and thus a constant object in $\mathsf{s}\mathcal{C}$, then

$$\mathrm{Hom}_{\mathsf{s}\mathcal{C}}(\Delta^n \otimes X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y_n).$$

**Theorem 11.** *There is a pushout diagram in* $\mathsf{s}\mathcal{C}$

$$\begin{array}{ccc} \partial\Delta^n \otimes X_n \coprod_{\partial\Delta^n \otimes L_n X} \Delta^n \otimes L_n X & \longrightarrow & sk_{n-1} X \\ \downarrow & & \downarrow \\ \Delta^n \otimes X_n & \longrightarrow & sk_n X. \end{array}$$

This is a really categorical way of saying that you get the $n$-skeleton from the $(n-1)$-skeleton by adding in the non-degenerate $n$-simplices. This diagram is what you should get out of today.