① # Intro

Def Category for k field $\mathrm{AbVar}_k^\circ = \begin{cases} \text{obj} = \text{AV over } k \\ \text{mor} = \mathrm{Hom}(A,B) \otimes \mathbb{Q} =: \mathrm{Hom}^\circ(A,B) \end{cases}$

AV = alg var over k
- also a group and mult & inv are regular maps
- connected
- complete $X \times Y \to Y$ closed for all var Y. (equiv projective)

$\overset{?}{\longleftrightarrow}$ group scheme $A \to \mathrm{Spec}(\mathbb{F}_q)$ smooth, proper & geom irr fibres
If we weaken $\mathrm{Spec}(\mathbb{F}_q) \rightsquigarrow$ scheme $=:$ abel schem

quasi-isogeny = iso in AbVar_k° ie invertible $f \in \mathrm{Hom}^\circ(A,B)$

isogeny = quasi-iso $\in \mathrm{Hom}(A,B)$
ie $f \in \mathrm{Hom}(A,B)$ with $g \in \mathrm{Hom}(B,A)$ st $fg = gf = \hat{n}$ $n \in \mathbb{Z}$.

AbVar_k° is
- semisimple ie all obj are semisimple
- $\mathbb{Q}$-linear ie additive and Hom-sets are $\mathbb{Q}$-vector spaces

Hence it is sufficient to describe isomorphism classes of simple objects and their endo algebras.

Def An algebraic int π is called a q-Weil-int if $\forall$ $\mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$ the image has abs value $\sqrt{q}$. Set of those denoted by $W(q)$.

$\pi \sim \pi'$ conjugate if one of TFE:
1) min pol over $\mathbb{Q}$ are same
2) $\exists$ iso $\mathbb{Q}[\pi] \to \mathbb{Q}[\pi']$ w/ $\pi \mapsto \pi'$
3) π & π' in same orbit of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

② $AV/\mathbb{F}_{p^n}$

Thm $\{$ ~~quasi-iso classes~~ simple $AV/\mathbb{F}_q\}/_{\text{quasi-iso}} \longrightarrow W(q)/\text{conjugacy}$

defined by sending A to it's Frobenius $\pi_A \in End^{\circ}(A)$ is a bijection.

Pf Sketch
1) this is well defined (Weil)
2) this is injective (Tate)
3) this is surjective (Honda)

1) π_A commutes with endo's so $\pi_A \in$ centre of $End^{\circ}(A)$. Since A simple $End^{\circ}(A)$ is a div alg and $\mathbb{Q}[\pi]$ is a field. An quasi-isogeny carries $\pi_A \mapsto \pi_B$ in $End^{\circ}(A) \xrightarrow{\simeq} End^{\circ}(B)$.
So π_A is a Weil-int by the Riemann-h

2) Similar to what we have seen from Paul Tate proved that

$$Hom(A,B)_{\ell} \to Hom_{Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)}(V_\ell A, V_\ell B) \ \cancel{(A(\ell), B(\ell))}$$

is an iso.

The gal-action is as π_A and π_B and these actions are ss. ie $\exists$ basis of eigenvectors for some ext of $\mathbb{Q}_\ell$

Hence this second set is iso to

$$hom = \# \{(i,j) \mid a_i = b_j \quad \begin{matrix} a_i \text{ root of } P_{\pi_A} \\ b_j \text{ root of } P_{\pi_B} \end{matrix}\}$$

Alternatively, A simp, B simple
$f_{\pi_A} = \underset{\text{in } \mathbb{Q}[x]}{h^n}$ characteristic pol. Since

min poly $P_{\pi_A} = P_{\pi_B}$ and $f_{\pi_B} = h^m \Rightarrow f_{\pi_B} \mid f_{\pi_A}$ or $f_{\pi_A} \mid f_{\pi_B}$.

$x \mapsto x^p$ induces
$fr: Spec(\mathbb{F}_q) \to Spec(\mathbb{F}_q)$

$$\begin{tikzcd}
A \arrow[r, "fr_A"] \arrow[rr, bend left] \arrow[dr, "fr^*"'] & A^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Spec\, \mathbb{F}_q \arrow[d, "fr"] \\
& A \arrow[r] & Spec\, \mathbb{F}_q
\end{tikzcd}$$

$\pi_A : A \to A^{(p)} \to A^{(p)^2} \cdots \to A^{(p)^n} = A$

③

3) We call $\pi \in W(q)$ effective if it is conj to ~~an~~ the q-Frob of a simple AV over k.

Lemma If π^{n} is eff, then so is π.

with some knowledge of $Br(\mathbb{Q}[\pi])$ we can reconstruct endoring by setting the right local inv.

$\exists$ fin ext $\mathbb{F}_{q^n}$ & AV B over $\mathbb{F}_{q^n}$ st $\pi_B = \pi^n$

To construct a simple AV A, ~~corresponding~~ mapping to a $\pi \in W(q)$ we will first ~~[illegible]~~ construct it's endo ring. ~~[illegible]~~
$End^0(A)$ is div alg with $\mathbb{Q}[\pi]$ in it's center
So look at $Br(\mathbb{Q}[\pi]) = \left\{ \begin{array}{l} CSA\text{'s} / M_n(A) \simeq M_n(B) \\ [A]\cdot[B] = [A \otimes_M B] \end{array} \right.$
$\exists$ injective map $inv_k : Br(k) \hookrightarrow \mathbb{Q}/\mathbb{Z}$
Then a CSA$/M$, E, is determined by
$$\{ inv_{M_x}(E \otimes M_x) \in \mathbb{Q}/\mathbb{Z} \mid x \text{ primes of } M \}$$
Pick ~~the one~~ E "$inv_x E$" ~~with~~ over $\mathbb{Q}[\pi]$ with
$$inv_x E = 1/2 \quad \text{if } x \text{ real}$$
$$inv_x E = 0 \quad \text{if } x \nmid p$$
$$inv_x E = \frac{x(\pi)}{x(q)} \left[\mathbb{Q}[\pi]_x : \mathbb{Q}_p\right].$$

Lemma $\exists$ CM field $L \supset \mathbb{Q}[\pi]$ st L splits E, ie $E \otimes_{\mathbb{Q}[\pi]} L \simeq M_n L$, and
$$[L : \mathbb{Q}[\pi]] = \sqrt{[E : \mathbb{Q}[\pi]]}$$

Sketch Pf: If $\mathbb{Q}[\pi] = \mathbb{Q} \vee \mathbb{Q}(\sqrt{p})$, pick $L = \mathbb{Q}[\pi](\sqrt{-p})$
If $\mathbb{Q}[\pi]$ is CM with real subfield $\mathbb{Q}[\pi]^+$, take $L = \mathbb{Q}[\pi] \otimes_{\mathbb{Q}[\pi]^+} L^+$ with $[L^+ : \mathbb{Q}[\pi]^+]$ as desired and 1) $\forall$ $v_0 \in \mathbb{Q}[\pi]^+$ over p $\exists!$ w_0 over v_0
2) w_0 unram over v_0 if $\mathbb{Q}[\pi]/\mathbb{Q}[\pi]^+$ is ram over v_0.
Found by weak approx.

Lemma $\exists$ AV A over a fin ext K over $\mathbb{Q}_p$ st $\exists$ ring map $L \to End^0_K(A)$ st $[L:\mathbb{Q}] = 2\dim A$ and st $\pi_A^n = \pi^{mn}$ for some n, m.

(4)

Now for $\mathrm{End}^\circ(A)$

One observation we can make, which will help to understand these def later (I think) is the (proven) Tate ~~conj~~ isogeny conj

$$\mathrm{Hom}_{\mathbb{F}_q}(A,B)_\ell \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}(A(\ell),B(\ell)).$$

Pf Enough to prove $\mathrm{End}(A)_\ell \simeq \mathrm{End}_{gal}(A(\ell))$ (take $A = B \times C$ later)

Replace $\mathrm{End}_{gal}\, A(\ell)$ by $\mathrm{End}_{gal}\, V_\ell A$

$V_\ell A = \varprojlim A[\ell^i] \otimes \mathbb{Q}$

$\mathrm{End}(A)_\ell \hookrightarrow \mathrm{End}(V_\ell A)$ double centralizer thm

$\mathrm{End}(A)_\ell = Z_{\mathrm{End}\, V_\ell A}\, Z_{\mathrm{End}\, V_\ell A}\, \mathrm{End}(A)_\ell$

Gal commutes with $\mathrm{End}(A)_\ell$.

Pick $\alpha \in \mathrm{End}_{gal} V_\ell A$ and define $W = \{(x, \alpha x) \in (V_\ell A)^2\}$

Show for any $c \in Z_{\mathrm{End}\, V_\ell A}\, \mathrm{End}(A)_\ell$ that

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} W \subseteq W$$

Hence $\forall x \in V_\ell A$ $(cx, c\alpha x) \in W$ so $c\alpha x = \alpha c x$

Thus $\alpha \in Z_{..} Z_{..} \mathrm{End}(A)_\ell = \mathrm{End}(A)_\ell$.

(5)

~~Now for $\mathrm{End}^\circ(A)$~~

Thm (Tate)

1) $Z(\mathrm{End}^\circ(A)) = \mathbb{Q}[\pi_A]$

2) $2\dim A = [\mathbb{Q}[\pi_A]:\mathbb{Q}]\sqrt{[\mathrm{End}^\circ(A):\mathbb{Q}[\pi]]}$

Hence $\mathrm{End}^\circ(A)$ contains a CM-subfield (namely $\mathbb{Q}[\pi_A]^+$)

3) CSA $\mathrm{End}^\circ(A)/\mathbb{Q}[\pi_A]$

- does not split above reals $\quad \mathrm{inv}_x \mathrm{End}^\circ A = 1/2 \quad x$ rea
- does split not-above p $\quad \mathrm{inv}_x \mathrm{End}^\circ(A) = 0 \quad x \nmid p$
- $\mathrm{inv}_x \mathrm{End}^\circ(A) = \dfrac{x(\pi_A)}{x(q)}[\mathbb{Q}[\pi_A]_x : \mathbb{Q}_p]$

"= slope of $A(x)$"

And this data recovers $\mathrm{End}^\circ(A)$

Pf 1) Tate shows that, as we just saw,

$$\mathrm{Hom}(A,B)_\ell \xrightarrow{\simeq} \mathrm{Hom}_{gal}(A(\ell), B(\ell))$$

Since $\mathrm{Gal}(\ )$ is top gen by π_A, so this shows

$$Z_{\mathrm{End}(A(\ell))}\mathbb{Q}[\pi]_\ell = \mathrm{End}_{gal}(A(\ell)) = \mathrm{End}(A)_\ell$$

By the double centralizer thm

$$\begin{aligned}\mathbb{Q}[\pi]_\ell &= Z_{\mathrm{End}(A(\ell))} Z_{\mathrm{End}(A(\ell))}\mathbb{Q}[\pi_A]_\ell \\ &= Z_{\mathrm{End}(A(\ell))}\mathrm{End}(A)_\ell \\ &\supset Z_{\mathrm{End}(A)_\ell}\mathrm{End}(A)_\ell = C(\mathrm{End}(A))_\ell\end{aligned}$$

2) Say $f_A = \mathrm{char}(\pi_A)$, $h_A = \mathrm{irr}(\pi_A)$ $\quad \deg h_A = [\mathbb{Q}[\pi_A]:\mathbb{Q}]$

A simple, so $f_A = h_A^e$. Hence

$2\dim(A) = \deg f_A = e\cdot[\mathbb{Q}[\pi_A]:\mathbb{Q}]$. And

$\mathrm{End}(A)_\ell \simeq \mathrm{End}_{gal}(A(\ell))$ uses that

$[\mathrm{End}^\circ(A):\mathbb{Q}] = e^2\cdot[\mathbb{Q}[\pi]:\mathbb{Q}]$.

3) involved.

⑥ $\overline{\mathbb{F}_p}$

$(\overline{\mathbb{F}}_p = \cup \mathbb{F}_{p^n} = \varinjlim \mathbb{F}_{p^n})$

Now we'll try a similar thing ~~one~~ for a AV $/\overline{\mathbb{F}_p}$

A simple AV

M center of $End^0(A)$; Div alg, contains CM field (now we don't have a π)

Suppose $p = x_1^{e_{x_1}} \cdots c(x_1)^{e_{x_1}} \cdots x_1'^{e_{x_1'}} \cdots$ over M.

$c(x_i') = x_i'$

Def $d_x = [M_x : \mathbb{Q}_p]$ and f_x residue degree $[M_x/xM_x : \mathbb{F}_p]$

So $d_x = e_x f_x$.

The decomposition $M_p = \prod_{x|p} M_x$ induces

$$End^0(A)_x \simeq End(A(x))$$

Since $[End^0(A)_x : M_x] = m^2$, $A(x)$ has height m as p-div M_x-mod., so height $d_x m$ a p-div group.

Since $End^0(A)_x$ is simple $A(x)$ has pure slope

$$s_x = \frac{\dim A(x)}{d_x m} = \eta_x f_x \quad \text{for some } \eta_x \in \mathbb{Q}$$

($\lambda : A(x_i) \to A(c(x_i))^\vee$ leads to $s_{x_i} = 1 - s_{c(x_i)}$ so the Newton polygon is symmetric. (Manin conj. every symm Newton polygon comes from an AV.))

Note that in the $\mathbb{F}_q$ case $M = \mathbb{Q}[\pi]$ determined an AV of height $h = [\mathbb{Q}[\pi]:\mathbb{Q}] \cdot m$ and $\dim = \text{slope} \cdot h = \frac{1}{2}h$ and m is l.c. denominator of inv_x and $End^0(A)$ constructed by the local inv. $inv_x End^0(A)$ x prime in M

In the case $\overline{\mathbb{F}_p}$, M determines heights $h_x = d_x m$ but not $\dim(A(x)) = h_x s_x = \eta_x h_x f_x$, so we also have to remember η_x (or s_x). The $A(x)$ piece together A, so $(M, \{\eta_x\})$ call p-adic type

Thm $\{\text{simple AV}/\overline{\mathbb{F}_p}\}/_{\text{quasi-iso}} \to \{\text{p-adic types}\}/_{\text{conjugacy}} \simeq \{\text{min p-adic type}\}$ is a bij.

The structure of $End^\circ(A)$ for A of p-adic type $(M, \{\eta_x\})$ is given by

$$\begin{array}{lll} inv_x End^\circ(A) & = 1/2 & x \text{ real} \\ \quad '' & = 0 & x \nmid p \\ \quad '' & = \eta_x f_x & x \mid p \end{array}$$

<u>Def</u> For B simple alg st $Z(B)$ is a CM field we define cat B-lin AV ~~as~~, $AbVar^\circ_{\overline{\mathbb{F}}_p}(B)$ with objects (A, i) where $i: B \hookrightarrow End^\circ(A)$ and morphism B-lin quasi-homo's.

<u>Lemma</u> (Kott) Every simple B-lin AV is isogeneous (in $AbVar^\circ_{\overline{\mathbb{F}}_p}$) to A_0^j for some j and simple AV A_0.

<u>Sketch</u> $AbVar^\circ_{\overline{\mathbb{F}}_p} \simeq \bigoplus_{X \in S} AbVar^\circ_X$ $\quad$ S set of rep of simples

$AbVar^\circ_X$ full subcat gen X^j.

$AbVar^\circ_{\overline{\mathbb{F}}_p}(B) \simeq \bigoplus_{X \in S} AbVar^\circ_X(B)$ $\square$

Observe that $Z(End^\circ_B(A)) \longleftrightarrow Z(End^\circ_B(A_0))$ $\quad$ ~~End~~°

$\longrightarrow Z(End^\circ(A))$

$i: Z(B) \hookrightarrow Z(End^\circ_B A)$ ~~$Z(End^\circ_B A)$~~

$Z(End^\circ(A_0)) \otimes_{\mathbb{Q}} Z(B) \longrightarrow Z(End^\circ_B A)$

Kottwitz observes that this makes Z a factor and that

$$\{\text{simple B-lin AV } (A, i),\ A \text{ isotypical to fixed } A_0\}$$
$$\downarrow$$
$$\{\text{fields } L \text{ factor of } Z(End^\circ A_0) \otimes_{\mathbb{Q}} Z(B)\}$$

is bij.

$Z(End^\circ A_0)$ is p-adic type of A_0

If $(M, \{\varepsilon_y\})$, then to A we may associate the p-adic type $(L, \{\eta_x\})$ over $Z(B)$, ie p-adic type st L extension of $Z(B)$, where

$$\eta_x = e_{x/y}\,\varepsilon_y \quad \text{if } y \mid x$$

So L takes over the function of center in p-adic type.

We conclude

Thm 1) $\{\text{simple } B\text{-lin AV} / \overline{\mathbb{F}}_p\}/_{\text{quasi-iso}} \xrightarrow{\simeq} \{\min p\text{-adic types} / Z(B)\}$

2) For a simpl B-lin AV A of p-adic type $(L, \{\eta_x\})$ the central div alg $D = \mathrm{End}^0_B A$ over L may be recovered from

$$\begin{aligned} \mathrm{inv}_x D &= 1/2 - \mathrm{inv}_x (B \otimes_{Z(B)} L) \\ &= 0 - \mathrm{inv}_x (B \otimes_{Z(B)} L) \\ &= \eta_x f_x - \mathrm{inv}_x (B \otimes_{Z_B} L) \end{aligned}$$

For stack Sh we fix

~~Moreover pick~~ F = quad imaginary ext of $\mathbb{Q}$ st $p = v v^c$

B = CSA / F, $[B:F] = n^2$ split over v & v^c

Then this class shows $\exists$ B-lin AV's A with $A(v)$ of dim n and slope $\frac{1}{n}$. And $A(v) \simeq (\varepsilon A(v))^n$ for an idempotent $\varepsilon \in B_v = M_n(F_v)$. $\varepsilon A(v)$ is a 1-dim p-div group of ~~slope~~ height n.

By the splitting $0 \to G^{for} \to G \to G^{et} \to 0$ we now get a height n formal group. This last bit can be formulated as a map

$$\underset{\text{p-div groups of height } n}{\mathcal{M}_p(n)} \longrightarrow \mathcal{M}_{FG}.$$

if we start doing this over schemes

And this can be formulated as

$$Sh_{p,F,B,n,\varepsilon,\ldots} : \{\text{loc noeth formal schemes} / \mathrm{Spf}(\mathbb{Z}_p)\} \to \text{Groupoids}$$

$$S \longmapsto \langle (A, i, \lambda) : \begin{array}{l} A \text{ AV over } S \text{ dim } n^2 \\ \lambda \text{ pol} \\ i : \mathcal{O}_{B,(p)} \hookrightarrow \mathrm{End}(A)_{(p)} \end{array}$$

+ level str

$$Sh_{\ldots} \to \mathcal{M}_p(A)$$

Now we need some info about def. to make a GHM kind Thm work (says Lurie)