p-DIVISIBLE GROUPS

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1. p-divisible groups and finite group schemes

Here are two related problems with the algebraic geometry of formal groups.

Problem 1. The height of a formal group is not well-behaved under base change. For instance, the Lubin-Tate formal group over the ring $R = \mathbb{Z}_p[[u_1, \ldots, u_{h-1}]]$, given in p-typical form by the p-series

$$[p]_F(x) = px +_F u_1 x^p +_F \cdots +_F u_{h-1} x^{p^{h-1}} +_F x^{p^h},$$

has height h, but inverting $u_t, \ldots, u_{h-1}$ lowers the height to t. From a computational point of view, this makes it hard to use formal groups to study phenomena that involve a change in chromatic level, such as chromatic fracture squares.

Problem 2. To define tmf, we obtained one-dimensional, height 2 formal groups from supersingular elliptic curves, by taking the p-power torsion

$$E(p) = \varinjlim E[p^i].$$

Most elliptic curves, though, are ordinary and only give formal groups of height 1; the above construction (over an algebraic closure of the base field, say) will then give us something of the form

$$E(p) = F \times \underline{\mathbb{Q}_p/\mathbb{Z}_p},$$

where F is a formal group of height 1 and $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is a constant group scheme. To make matters worse, if k has characteristic different from p, then we'll just have

$$E(p) = \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^2},$$

with no formal data at all. There's a strong argument, particularly when we enter the world of higher-dimensional abelian varieties, to consider all of these cases as instances of a single phenomenon.

This leads to the following definition.

Definition 1. A **p-divisible group of height** h over a scheme S is an ind-group scheme $\mathbb{G}$ of the form

$$0 = \mathbb{G}_0 \to \mathbb{G}_1 \to \mathbb{G}_2 \to \cdots$$

with each $\mathbb{G}_i$ a finite, flat, commutative group scheme over S of constant rank p^{ih}, the maps are closed immersions, and $\mathbb{G}_i$ is the kernel of the multiplication-by-p^i map $[p^i] : \mathbb{G}_{i+1} \to \mathbb{G}_{i+1}$.

Most of the things we can say about p-divisible groups are easy generalizations of things we can say about finite flat commutative group schemes. These objects are so important that from now on, I'll just call them **finite group schemes**. Here are some basic facts about these objects:

- Over a small affine $\operatorname{Spec} R$ of the base, a finite group scheme is of the form $\operatorname{Spec} A$, where A is a cocommutative Hopf algebra that is finite and free as an R-module.
- Over a field, the category of finite group schemes is abelian. The monomorphisms are the closed immersions and the epimorphisms are the faithfully flat maps.

- Still over a field, we can talk about the order of a finite group scheme – it's just its algebra's rank as a k-module. If $0 \to G' \to G \to G'' \to 0$ is exact, then $|G| = |G'| \cdot |G''|$.

  It is easily deduced that in the situation of a p-divisible group, each $\mathbb{G}_i$ is not only the p^i-torsion of $\mathbb{G}_{i+1}$, but also of all higher $\mathbb{G}_{i+j}$. There are exact sequences
$$0 \to \mathbb{G}_i \to \mathbb{G}_{i+j} \overset{[p^i]}{\to} \mathbb{G}_j \to 0.$$
  It thus makes sense to write $\mathbb{G}[p^i]$ instead of $\mathbb{G}_i$. As a final note, the diagram defining a p-divisible group is canonically defined given its colimit, so it makes no difference whether we talk about ind-schemes or these diagrams.
- Finally, base change to a field extension is an exact (and faithful) functor of these abelian categories.

2. Examples

Here are some examples of p-divisible groups, all cribbed from the first chapter of [2].

Example 1. The **constant group of height** h**,** $\underline{(\mathbb{Q}_p/\mathbb{Z}_p)^h}$**, with**
$$\mathbb{G}[p^i] = \coprod_{(\mathbb{Z}/p^i)^h} S,$$
the group structure given by permuting the factors.

Example 2. Any formal group $G \cong \operatorname{Spf} R[[x]]$, with formal group law F, automatically gives rise to a p-divisible group with
$$\mathbb{G}[p^i] = \operatorname{Spec} R[[x]]/[p^i]_F(x).$$
If R is p-adically complete, then you can use p-adic approximation to reconstruct the formal group law from the p-divisible group. This gives an inclusion of formal groups into p-divisible groups.

As an argument that our definitions so far are good, this inclusion is realized topologically. After completing at p, we have
$$\mathbb{C}P^\infty = BS^1 \simeq \operatorname{holim}(B\mathbb{Z}/p^i)^\wedge_p.$$
So if E is complex oriented and p-complete, we have a natural isomorphism
$$E^0[[x]] = E^0\mathbb{C}P^\infty \cong \varinjlim E^0 B\mathbb{Z}/p^i = \varinjlim E^0[[x]]/[p^i]_{F_E}(x).$$
This is precisely the above construction.

Example 3. If A is an abelian variety of dimension d, then $A(p) = \varinjlim A[p^i]$, the group of p-power torsion points of A, is a p-divisible group of height $2d$.

3. Duality

We now examine the wealth of natural structure that arises on finite group schemes and p-divisible groups. First is a duality functor, called **Cartier duality** or **Serre duality**. In brief, this is defined for a finite group scheme G by
$$G^\vee = \operatorname{Hom}_{\mathbf{GrpSch}/S}(G, \mathbb{G}_m).$$
In the highbrow way of looking at this, we're using the internal Hom on the category of abelian fppf sheaves over S, which manages in this case to land back in finite group schemes. That is, $G^\vee$ is the functor of points $T \mapsto \operatorname{Hom}_{\mathbf{GrpSch}}(G \times_S T, \mathbb{G}_m \times_S T)$.

In the lowbrow way of looking at this, we take $G = \operatorname{Spec}(A)$ and $S = \operatorname{Spec}(R)$, with A a Hopf algebra over R. Then
$$\operatorname{Hom}_{\mathbf{GrpSch}}(G, \mathbb{G}_m)(S) = \operatorname{Hom}_{\mathbf{Hopf}}(R[t^{\pm 1}], A) \subseteq A^\times.$$
Specifically, a map of Hopf algebras $R[T^{\pm 1}] \to A$ is equivalently an element $u \in A$ with $\Delta(u) = u \otimes u$ and $\epsilon(u) = 1$, also called a **grouplike** element of A. Since we want to represent the Cartier dual as an affine group scheme, we'd like this to be represented by maps to R from some Hopf algebra. An

obvious choice is the **dual Hopf algebra** $A^\vee = \mathrm{Hom}_R(A, R)$, which inherits a multiplication from the *comultiplication* of A, and a comultiplication from its *multiplication*. Of course, if A is free and finite over R, a $u \in A$ is equivalently a map of R-modules $\mathrm{Hom}_R(A, R) \to R$. If $f, g \in \mathrm{Hom}_R(A, R)$, then

$$(fg)(u) = \mu(f \otimes g)(u) = (f \otimes g)(\Delta(u)) = f(u)g(u)$$

if and only if $\Delta(u) = u \otimes u$. Likewise, the multiplicative unit of $A^\vee$ is ϵ, so the maps of unital algebras $A^\vee \to R$ precisely correspond to the $u \in A$ satisfying the conditions $\Delta(u) = u \otimes u$ and $\epsilon(u) = 1$. Finally, it's clear that all of this is preserved by any base change whatsoever, giving us

$$\mathrm{Spec}(A)^\vee = \mathrm{Spec}(A^\vee) :$$

the Cartier dual of a finite group scheme is Spec of its dual Hopf algebra. One should note that this functor is exact, contravariant, and preserves orders.

For p-divisible groups, the dual of the epimorphism $[p] : \mathbb{G}[p^{i+1}] \to \mathbb{G}[p^i]$ is a monomorphism $\mathbb{G}[p^i]^\vee \to \mathbb{G}[p^{i+1}]^\vee$. The p^i-torsion of $\mathbb{G}[p^{i+1}]^\vee$ is the subscheme of maps to $\mathbb{G}_m$ that factor through $\mathbb{G}_m[p^i]$, or equivalently, through $\mathbb{G}[p^i]$ along $[p]$ – but this is precisely $\mathbb{G}[p^i]^\vee$. Thus, the diagram

$$0 \to \mathbb{G}[p]^\vee \to \mathbb{G}[p^2]^\vee \to \cdots ,$$

where the arrows are $[p]^\vee$, defines a p-divisible group, called the **Cartier dual** or **Serre dual** of $\mathbb{G}$.

4. The fundamental exact sequence and its dual

The second interesting piece of structure is a natural exact sequence splitting a group into an 'étale part' and a 'connected part'. By 'connected,' I mean that its fibers are all local, or in the p-divisble case, complete local – this corresponds to a formal group. Since we're working over fields most of the time, one way to think about étale groups is that they become constant after base changing to an algebraic closure. The following proof is from [5].

Proposition 1. *Let G be a finite group scheme over $S = \mathrm{Spec}\, R$, with R complete, noetherian, and local. There is a unique, natural exact sequence*

$$0 \to G^0 \to G \to G^{et} \to 0$$

where G^0 is connected and G^{et} is étale. If $S = \mathrm{Spec}\, k$ with k a perfect field, then this sequence splits.

Proof. G^0 is obviously the connected component of the identity. This is a closed subscheme, and since R is local, $G^0 \times_S G^0$ is still connected, so the restriction of the multiplication map to this subscheme factors through G^0. Thus, G^0 is a closed subgroup. In Hopf algebra language, A is a finite product of local extensions of R, and $\epsilon : A \to R$ factors through the projection to one of them, which will then be A^0.

G^{et} corresponds to the maximal étale subalgebra of A. To get at this, base change to the residue field k of R, making A a product of finite local k-algebras A_i, each of whose residue fields will be a finite extension of k. The separable closure of k in $A_i/\mathfrak{m}A_i$ is of the form $k[\theta]/(P(\theta))$, by the primitive element theorem; using Hensel's lemma, one can lift this θ to an element in A_i, giving an embedding of a (maximal) finite separable extension of k into each A_i. By the uniqueness part of Hensel's lemma, this subalgebra A^{et} is unique, and it's clearly étale over k. Standard stuff about étale morphisms tells us that $A^{et} \cong k[x]_g/(f)$ for some polynomials f and g such that f' is a unit in the localization. Now by Hensel's lemma again, we can (uniquely) lift these polynomials to polynomials over R satisfying the same condition, and get a subalgebra $A^{et} \cong R[x]_g/(f)$ that is étale over R and maximal among étale subalgebras. (If there were an étale subalgebra containing this, its reduction to k would have to be the same, and it would have to be the same as A^{et} by Nakayama's lemma.)

Any map from a connected group to an étale group is trivial, so the composition $G^0 \to G \to G^{et}$ is zero. On the other hand, reducing to k and base changing to the algebraic closure, G^{et} becomes the union of the closed geometric points, one of which is in each connected component, so the sequence becomes exact. Since these base changes are exact and *faithful* for finite flat group schemes, $0 \to G^0 \to G \to G^{et} \to 0$ is the desired exact sequence.

For naturality, observe that if $G \to H$ is a map of finite group schemes, then G^0 must map to H^0 by topology.

Now suppose that the base scheme is a perfect field k, and let G^{red} be the reduction of G (the spectrum of A mod its nilradical). Since k is perfect, $A^{red} \otimes_k A^{red}$ is again a reduced ring, and so $G^{red} \times G^{red}$ is a reduced scheme – thus, the multiplication map sends this into G^{red}, making G^{red} a closed subgroup. Finally, $G^{red} \to G^{et}$ is a map of group schemes that becomes an isomorphism after base changing to $\overline{k}$ (where both group schemes are constant), and thus it's an isomorphism over k. The inclusion $G^{red} \to G$ splits the exact sequence, and is evidently unique. □

By naturality, the above goes through for p-divisible groups, giving a natural exact sequence

$$0 \to \mathbb{G}^0 \to \mathbb{G} \to \mathbb{G}^{et} \to 0.$$

The **dimension** of $\mathbb{G}$ is the dimension of the formal group $\mathbb{G}^0$.

Finally, there's a second, less intuitive decomposition – we can apply the formal-étale decomposition to $\mathbb{G}^\vee$ and dualize the resulting exact sequence. This gives a natural exact sequence

$$0 \to \mathbb{G}^{mult} \to \mathbb{G} \to \mathbb{G}^{un} \to 0.$$

Here $\mathbb{G}^{mult}$ is **multiplicative** (the prime example is $\widehat{\mathbb{G}_m}$) and $\mathbb{G}^{un}$ is **unipotent** (the prime example is $\widehat{\mathbb{G}_a}$). There are some good descriptions of these in Demazure's book, but for now I just want to point out that they exist.

5. Frobenius and Verschiebung

For S a scheme of characteristic p, there's a natural Frobenius map $\sigma_S : S \to S$. We define $G^{(p)} = G \times_S^{\sigma_S} S$. There's then a diagram

$$\begin{tikzcd}
G \arrow[dr, "F_G"] \arrow[drr, bend left, "\sigma_G"] \arrow[ddr, bend right, "\eta"'] & & \\
& G^{(p)} \arrow[r] \arrow[d] & G \arrow[d, "\eta"] \\
& S \arrow[r, "\sigma_S"'] & S.
\end{tikzcd}$$

Note that this only depends on the scheme structure of G, and not its group structure. If $G = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, then $A^{(p)}$ is given by 'adjoining pth roots of R' to A, and $F : A^{(p)} \to A$ is $x \mapsto x^p$, which is then R-linear.

Now using the group structure of G, we can take the Frobenius of $G^\vee$ and dualize it to get a map $V : G^{(p)} \to G$ called the **Verschiebung**.[1] This uses the fact, which you should check, that

$$(G^\vee)^{(p)} = (G^{(p)})^\vee.$$

In Hopf algebra terms, V is the composition

$$A \to (A^{\otimes} p)^{\Sigma_p} \to A^{(p)}.$$

The first map is comultiplying p times, and the second is the unique linear map sending $(a \otimes \cdots \otimes a) \mapsto a$.

You should check that F and V give a factorization

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "F"] \arrow[rr, bend right, "{[p]}"'] & \mathbb{G}^{(p)} \arrow[r, "V"] & \mathbb{G}.
\end{tikzcd}$$

[1]This is the German word for 'shift.' If G is the Witt vectors – which are something like the universal example of a p-divisible group – the Verschiebung map is given by shifting all the Witt components to the right.

We can use this, and understanding of orders, to prove our first proposition that's purely about p-divisible groups.

Proposition 2. *The height of* $\mathbb{G}$ *is the sum of the dimension of* $\mathbb{G}$ *and the dimension of* $\mathbb{G}^\vee$*.*

Proof. Since $[p] = VF$, there's an exact sequence of finite group schemes

$$0 \to \ker F \to \ker[p] \to \ker V \to 0.$$

Of course, $\ker[p]$ is just $\mathbb{G}_1$, which is a finite group scheme of order p^h. F raises each formal coordinate to the pth power and acts as an isomorphism on the étale part, so $\ker F$ is just the connected part of $\mathbb{G}_1$, and thus of order p^d. Thus $\ker V$ is order p^{n-d}. But $\ker V$ is the dual of the cokernel of $F^\vee : \mathbb{G}^\vee \to (\mathbb{G}^\vee)^{(p)}$; it's also the cokernel of $F^\vee : \mathbb{G}_1^\vee \to (\mathbb{G}_1^\vee)^{(p)}$. This is a map of finite group schemes of the same order, so its cokernel has the same order as its kernel, which is $p^{d'}$ where $d' = \dim \mathbb{G}^\vee$. Thus $d' = n - d$. $\square$

6. Deformations

I now make a brief leap to section 7 of [2] to prove a generalization of the Lubin-Tate theorem. Suppose we have a p-divisible group $\mathbb{G}$ of height h over a field k of characteristic p whose formal part is height n and dimension 1. The étale part is then simply $\underline{(\mathbb{Q}_p/\mathbb{Z}_p)^{h-n}}$. Let $\mathrm{Def}_{\mathbb{G}}$ be the functor sending a complete local ring R with residue field k to the groupoid of deformations of $\mathbb{G}$ to R. By Lubin-Tate, $\mathrm{Def}_{\mathbb{G}^0}$ is discrete ('valued in setoids') and represented by the universal deformation ring

$$W(k)[[u_1, \dots, u_{n-1}]].$$

Furthermore, $\mathrm{Def}_{\mathbb{G}^{et}}$ is trivial – étale schemes have no deformations. We finally need to consider the various extensions of $\mathbb{G}^{et}$ by $\mathbb{G}^0$. These are classified by

$$\mathrm{Ext}^1(\mathbb{G}^{et}, \mathbb{G}^0) = \mathrm{Ext}^1(\underline{(\mathbb{Q}_p/\mathbb{Z}_p)^{h-n}}, \mathbb{G}^0).$$

Using the projective resolution

$$0 \to \underline{(\mathbb{Z}_p)^{h-n}} \to \underline{(\mathbb{Q}_p)^{h-n}} \to \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^{h-n}},$$

we obtain an isomorphism

$$\mathrm{Ext}^1(\mathbb{G}^{et}, \mathbb{G}^0) \cong \mathrm{Hom}(\underline{(\mathbb{Z}_p)^{h-n}}, \mathbb{G}^0).$$

This is just given by $h - n$ more freely adjoined formal parameters. Thus the deformation functor for p-divisible groups is again discrete and represented by

$$W(k)[[u_1, \dots, u_{n-1}]][[u_n, \dots, u_{h-1}]] = W(k)[[u_1, \dots, u_{h-1}]].$$

So the generalization of Lubin-Tate is the 'obvious' one, but it happens in sort of a funny way – we deform the formal part as expected, but rather than deforming the étale part, the new parameters are deforming the extension!

7. Classification and Dieudonné modules

In proving that

$$\mathrm{height}\, \mathbb{G} = \dim \mathbb{G} + \dim \mathbb{G}^\vee,$$

we observed that F is an isomorphism on the étale part of $\mathbb{G}$ and does something murderous to the formal part. This suggests that F and V tell us a lot about the shape of the p-divisible group. To be precise, we have the following.

- G is connected iff F is nilpotent.
- G is étale iff F is an isomorphism.
- G is unipotent iff V is topologically nilpotent.
- G is multiplicative iff V is an isomorphism.

From now on, let k be a perfect field of characteristic p, and $\mathbb{G}$ a formal p-divisible group over k. We define the **curves functor** by

$$C(\mathbb{G}) = \mathrm{Hom}_{\mathrm{formalschemes}}(\widehat{\mathbb{A}^1}, \mathbb{G}).$$

The addition in $\mathbb{G}$ gives this set of curves an abelian group structure. But it actually has even more structure, which is perhaps best expressed by saying that there's a free formal group on the formal scheme $\mathbb{A}^1$ – namely, the formal completion of the big Witt vector scheme over k, $\widehat{\mathbb{W}}$.[2] So we actually have

$$C(\mathbb{G}) = \mathrm{Hom}_{\mathrm{formalgroups}}(\widehat{\mathbb{W}}, \mathbb{G}).$$

In particular, this is not just an abelian group, but a $W(k)$-module. Moreover, the Frobenius and Verschiebung maps on define operations F and V on this module $C(G)$.

Most of us are probably more familiar with p-typical Witt vectors than big Witt vectors, and we can likewise look at

$$D(\mathbb{G}) = \mathrm{Hom}_{\mathrm{formalgroups}}(\widehat{\mathbb{W}_p}, \mathbb{G}).$$

This corresponds to 'p-typical curves' in $\mathbb{G}$, in a sense I won't make precise. It's naturally a module over the noncommutative **Dieudonné ring**

$$D(k) = W(k)[F, V]/(FV - p, VF - p, Fx - \sigma(x)F, V\sigma(x) - xV),$$

where σ is the Frobenius map on $W(k)$. We now have the following theorem.

Theorem 1. *The functor D is an equivalence between the category of formal p-divisible groups and the category of finitely generated $D(k)$-modules which are free over $W(k)$ and satisfying the reducedness condition*

$$\bigcap_i V^i M = 0$$

and the uniformity condition that

$$V : V^i M/V^{i+1}M \to V^{i+1}M/V^{i+2}M$$

is an isomorphism.

This doesn't look pretty because I've whizzed through it, but what it does is reduce a lot of statements about p-divisible groups to statements about linear algebra (or maybe 'Frobenius-linear algebra'). For example, the rank $D(\mathbb{G})$ as a $W(k)$-module is the height of $\mathbb{G}$; the dimension of $\mathbb{G}$ can be recovered as the rank of $D(\mathbb{G})/VD(\mathbb{G})$ as a k-module.

This extends to all p-divisible groups, but I've only seen awkward ways of doing it. Define the dual of a a Dieudonné module M by

$$M^\vee = \mathrm{Hom}_{W(k)}(M, W(k)),$$

with

$$F^\vee = \sigma \circ V^*, \qquad V^\vee = \sigma^{-1} \circ F^*.$$

Since p-divisible groups are never both étale and multiplicative, any étale p-divisible group must have a connected *dual*, so we can apply the Dieudonné correspondence to $\mathbb{G}^\vee$ and then dualize the resulting module. Since the two natural exact sequences split over a perfect field, we can define the Dieudonné correspondence for general $\mathbb{G}$ by doing one of the above two things to each factor, and checking that they agree in the connected, unipotent case where they're both defined. We end up with the following theorem

[2]One way to think about this is to write

$$\mathbb{W}(R) = W(R) = \varprojlim_n W(R)/V^n(R),$$

while

$$\widehat{\mathbb{W}}(R) = \varinjlim_n W(R)/V^n(R),$$

the set of Witt vectors with only finitely many nonzero Witt coordinates.

Theorem 2. *Over a perfect field k, the functor D is an equivalence between the category of p-divisible groups and the category of* **DieuMod** *of finitely generated $D(k)$-modules which are free over $W(k)$.*

As a final remark, things aren't completely lost over a non-perfect field. One possibility is to replace our modules with **crystals**, which are quasi-coherent sheaves on a certain site. Applying the above functor D to this entire site gives us the **Dieudonné crystal**, and there's a corresponding correspondence theorem. If you're curious, [4] is a good place to start.

8. The Newton polygon

I know decidedly less about this part of the theory, and am only mentioning it because Behrens and Lawson do. One way to make the classification problem easier is to weaken our notion of equivalence. An **isogeny** of p-divisible groups (or of abelian varieties) is a surjection with finite kernel. This isn't a terrible weakening: isogenous p-divisible groups have the same height and dimension. We also have the following correspondence theorem:

Theorem 3. *Over a perfect field k, the category of p-divisible groups and isogenies is equivalent to the category $\mathbf{DieuMod} \otimes_{\mathbf{W(k)}} \operatorname{Frac} \mathbf{W(k)}$.*

Intuitively, the picture is this: an isogeny $\mathbb{G} \to \mathbb{H}$ induces an isomorphism $D(\mathbb{G}) \to \frac{1}{p^i} D(\mathbb{H})$ for some i, where we are now regarding $\mathbb{H}$ as a sub-$W(k)$-module of $\mathbb{H} \otimes_{W(k)} \operatorname{Frac} W(k)$. So we can think of these isogeny classes as $\operatorname{Frac} W(k)$-vector spaces with actions of F and V, but not every such 'Dieudonné space' appears in this category: only those that are 'effective' in the sense of being generated by a sub-$D(k)$-module.

To an isogeny class of p-divisible groups we can associate the invariant $\frac{d}{h}$, called the **slope**. In [3], it is shown that over an algebraically closed field, the simple Dieudonné spaces are precisely those for which d and h are relatively prime. It is easy to exhibit a Dieudonné space with given h and d: one can take $M \otimes \operatorname{Frac} W(k)$, where

$$M = W(k)[T]/(T^h - p^d).$$

Moreover, this is the *only* such space. Finally, the category of Dieudonné spaces is semisimple, in the sense that every space is a product of simple ones. In conclusion, we have the following.

Theorem 4. *Over an algebraically closed field k, an isogeny class of p-divisible groups is uniquely specified by a set of slopes, which are fractions $\frac{d_i}{h_i}$ between 0 and 1, with d_i and h_i relatively prime for each i.*

Given an isogeny class, order its slopes $\frac{d_1}{h_1}, \ldots, \frac{d_n}{h_n}$ in increasing order. The **Newton polygon** associated to this isogeny class is the polygon with vertices

$$(0,0),\ (h_1, d_1),\ (h_1 + h_2, d_1 + d_2), \ldots, \left(\sum_{i=1}^n h_i, \sum_{i=1}^n d_i\right),\ \left(\sum_{i=1}^n h_i, 0\right).$$

So we're visualizing the basic information of the p-divisible group as a concave polygon with integer endpoints. I don't know how useful this is, but I'll conclude by stating one cool theorem, from [1].

Theorem 5 (Grothendieck)**.** *Let $\mathbb{G}$ be a p-divisible group over a scheme S, and x and y two points of S such that $x \in \overline{\{y\}}$ (that is, x is a specialization of y). Then the Newton polygon of $\mathbb{G} \times_S k(x)$ contains the Newton polygon of $\mathbb{G} \times_S k(y)$.*

References

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