Introduction to ∞-categories

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1 Introduction

Good evening. We've got a spectacular show for you tonight – full of scares, spooks, and maybe a few laughs too. The standard reference for most of this is [4] and/or the nLab, but [5] is a much more comfortable introduction and formed the basis for most of the material here.

2 Intuition: simplicial and topological categories

In homotopy theory, we're used to dealing with categories in which two objects have a space of maps between them, rather than just a set. A classic example is the category Top of compactly generated Hausdorff spaces. In this case, the set $\mathrm{Maps}(X, Y)$ of continuous maps from X to Y can be equipped with a topology, called the compact-open topology, such that composition

$$\mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z) \to \mathrm{Maps}(X, Z)$$

is continuous.

Likewise, Top is enriched over the category sSet of simplicial sets: the simplicial mapping spaces is that with

$$\mathrm{Fun}(X, Y)_n = \mathrm{Fun}(X \times \Delta^n, Y)$$

where Δ^n is the standard topological n-simplex. The identical definition shows that sSet is enriched over itself.

Another example: a finite group G can be represented by a category with one object, in which G is the set of automorphisms of that object. A *topological* group G should be represented by a category in which G is the *space* of automorphisms. Moreover, there should be a nerve functor from topological categories to spaces, such that the nerve of this category G is the space BG we talked about last week.

Since we're doing homotopy theory, we might want to go one step further, looking at categories enriched not just over spaces, but over homotopy types. We can think of a simplicial or topological category as a presentation of such a homotopically enriched category. However, an actual simplicial category is a very special kind of presentation: one in which two composable maps have a chosen composition. If we're really thinking homotopically, we should only be asking for a homotopy class of compositions. A homotopy between two choices of compositions will then appear as a higher simplex in the mapping space, which can then be composed with other maps, and higher homotopies show up in choosing these compositions, and so on.

The purpose of this talk is to work out a model which more adequately captures the intuition of a 'category enriched in homotopy types'. This model is a certain kind of simplicial set, called a **weak Kan complex** by Boardman & Vogt, a **quasicategory** by Joyal, and an **∞-category** by Lurie. By work of Bergner, ∞-categories are part of a model structure that's Quillen equivalent to a model category of simplicial categories. However, ∞-categories are in many respects more convenient for applications and constructions.

3 Intuition: horn filling in nerves and Kan complexes

Let's recall the construction of a nerve of a category $\mathcal{C}$. This is the simplicial set $N\mathcal{C}$ whose 0-simplices are the objects of $\mathcal{C}$, whose 1-simplices are the morphisms of $\mathcal{C}$, and whose n-simplices for $n \geq 1$ are the chains of n composable morphisms of $\mathcal{C}$.

In particular, a pair of composable morphisms $f : x \to y$, $g : y \to z$, gives us a 2-simplex

- whose vertices are, in order, x, y, and z;
- whose 01 edge is f, whose 12 edge is g, and whose 02 edge is gf;
- and whose body expresses the composition.

The two 1-simplices corresponding to f and g form a subcomplex of the 1-simplex called a **horn** Λ^2_1. More generally, Λ^n_k is the subcomplex of the n-simplex generated by all $(n-1)$-faces containing vertex k. The fact that composable morphisms in $\mathcal{C}$ can be uniquely composed means that there's a unique filler in every diagram of the form:

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N\mathcal{C} \\ \Delta^2 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Question for the audience. Does the same horn-filling property hold for higher-dimensional horns? What about Λ^2_2 or Λ^2_0? What does it tell us about $\mathcal{C}$ if it does?

We can summarize this by saying that in the nerve of a category, every *inner* horn admits a unique filler – every horn Λ^n_k with $0 < k < n$.

On the other hand, we also model spaces by simplicial sets. Given a space X, its associated simplicial set is the singular complex

$$\mathrm{Sing}(X)_n = \mathrm{Maps}(\Delta^n, X).$$

This simplicial set also satisfies a horn-filling condition: every (not necessarily inner) horn in $\mathrm{Sing}(X)$ admits a (not necessarily unique) filler. This is equivalent to the condition that every map of topological spaces, $\Lambda^n_k \to X$, extends to $\Delta^n \to X$. But Λ^n_k is a retract of Δ^n, so we can just compose with the retraction, $\Delta^n \to \Lambda^n_k \to X$.

Definition 3.1. A **Kan complex** is a simplicial set in which every horn admits a filler.

An ∞-category captures the best of both worlds. Like nerves of categories, we only attempt to fill inner horns. Like Kan complexes, we don't ask that these fillers are unique.

Definition 3.2. An **∞-category** is a simplicial set in which every inner horn admits a filler. That is, a dotted map exists completing every diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

for $0 < n < k$.

4 Examples and constructions

We've seen two sorts of examples of ∞-categories: Kan complexes and nerves of categories. A more general class of example is constructed as follows. There's a **homotopy coherent nerve functor**

$$\mathbf{N} : \mathsf{sSetCat} \to \mathsf{sSet}$$

that generalizes the nerve functor on ordinary categories. If $\mathcal{C}$ is enriched, not just in simplicial sets, but in Kan complexes, then $\mathbf{N}\mathcal{C}$ is actually an ∞-category. In particular, let $\mathcal{C}$ be a simplicial model category; then the subcategory of cofibrant-fibrant objects, $\mathcal{C}^{cf}$, is enriched in Kan complexes, and so we get an ∞-category $\mathbf{N}\mathcal{C}^{cf}$ that's 'presented' by the model category $\mathcal{C}$. Moreover, a simplicial Quillen equivalence of simplicial model categories translates to an equivalence of ∞-categories, in a sense to be described later.

Thus, taking the model category $\mathsf{sSet}_{\text{Quillen}}$ of simplicial sets with the Quillen model structure, in which every object is cofibrant and the fibrant objects are the Kan complexes – or the model category Top of (nice) topological spaces, in which every object is fibrant and the cofibrant objects are the cell complexes – we get an ∞-category of spaces,

$$\mathcal{S} = \mathbf{N}\mathsf{sSet}_{\text{Quillen}} \simeq \mathbf{N}\mathsf{Top}.$$

Taking the model category $\mathsf{sSet}_{\text{Joyal}}$ I'll describe later in this talk, in which every object is cofibrant and the fibrant objects are the ∞-categories, we get an ∞-category $\infty\mathsf{Cat}$ of ∞-categories.

The homotopy coherent nerve fits into an adjunction

$$|\cdot| : \mathsf{sSet} \leftrightarrows \mathsf{sSetCat} : N.$$

In fact, this adjunction is a Quillen equivalence when we give both sides the appropriate model structure.

I'll give a concrete example of a homotopy coherent nerve of a (2,1)-category. There's a (2,1)-category Cat whose objects are categories, whose morphisms are functors, and whose 2-morphisms are natural isomorphisms. The nerve of this is a simplicial set

- whose 0-simplices are categories,
- whose 1-simplices with vertices C_0 and C_1 are functors $F_{01} : C_0 \to C_1$,
- whose 2-simplices with edges $F_{01} : C_0 \to C_1$, $F_{12} : C_1 \to C_2$, and $F_{02} : C_0 \to C_2$ are natural isomorphisms $\alpha_{012} : F_{12} \circ F_{01} \Rightarrow F_{02}$,
- and, for $n \geq 3$, a unique n-simplex for each set of data $(C_i, F_{ij}, \alpha_{ijk})$, $0 \leq i < j < k \leq n$, such that for any $i < j < k < \ell$,

$$\alpha_{ij\ell} \circ (\alpha_{jk\ell} 1_{F_{ij}}) = \alpha_{ik\ell} \circ (1_{F_{k\ell}} \alpha_{ijk}) : F_{k\ell} F_{jk} F_{ij} \Rightarrow F_{i\ell}.$$

We can check the inner horn-filling condition on this explicitly. By definition, there's at most one n-simplex for $n \geq 3$ with given 2-faces, and there's exactly one n-simplex for $n \geq 4$ with given 3-faces. So

- all horns Λ^n_k for $n \geq 4$ fill trivially,
- all inner horns Λ^3_k fill uniquely, by finding the unique natural isomorphism that makes the diagram commute,
- the inner horns Λ^2_1 fills by defining F_{02} to be $F_{12}F_{01}$ and α_{012} to be the identity.

I'll now mention two constructions that, together with the above, account for probably most of the ∞-categories you'll encounter on a first reading of Lurie. First, let $\mathcal{C}$ be an ∞-category and $S \subseteq \mathcal{C}_0$. The **full subcategory spanned by** S is the subcomplex of $\mathcal{C}$ consisting of the simplices all of whose vertices are in S. It's not hard to see that this is also an ∞-category.

Second, let K and L be two simplicial sets. The **function space** L^K is the simplicial set with

$$(L^K)_n = \text{Hom}_{\mathsf{sSet}}(K \times \Delta^n, L).$$

If $\mathcal{C}$ is an ∞-category, then $\mathcal{C}^K$ is also an ∞-category, which we think of as a **functor category**. Time permitting, I'll prove this statement this talk, but not before I establish some machinery.

5 Homotopy in an ∞-category

I'm going to be following Rezk's notation for simplicial sets from now on. If s is an n-simplex of X and I is an ordered subset of $[n] = \{0, \dots, n\}$, then s_I is the $(|I|-1)$-dimensional face of s corresponding to the inclusion $I \hookrightarrow [n]$. If J is a list of elements of $[n]$, in order, with repetitions, corresponding to a surjection $[|J|] \twoheadrightarrow [n]$, then s_J is the $(|J|-1)$-dimensional degeneracy of s corresponding to this surjection.

If you're reading this, this is the part where I start describing diagrams in words that I didn't want to TeX. You should start drawing them.

We've defined ∞-categories as simplicial sets, but in the sequel, we'll often be speaking about them in the language of higher categories. The following definitions will get us started.

Definition 5.1. The **objects** of an ∞-category $\mathcal{C}$ are its 0-simplices. The **morphism set** between two objects x and y is the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of 1-simplices f such that $f_0 = x$, $f_1 = y$. (Later, we'll replace this set with a space.)

Thus, for example, if we have a Λ^2_1 horn with sides $f : x \to y$ and $g : y \to z$, the inner horn filling condition tells us that there's a filler. The third 1-simplex of this filler will be a composition of g and f. On the other hand, this filler, and thus the composition of the two maps, isn't uniquely determined. On the mutant third hand, let a and b be two possible fillers, giving compositions (gf) and $(gf)'$, and consider the Λ^3_1 horn whose 012-face is a, whose 013-face is b, and whose 123-face is g_{011}. This horn has a filler whose 023-face has 1-simplices gf, z_{00}, and $(gf)'$. So, even though the composition isn't well-defined, it is well-defined up to a homotopy or 2-morphism. This is the idea behind the next definition.

Definition 5.2. If f, $g \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, say that

- $f \sim_\ell g$ if there exists a 2-simplex a such that $a_{01} = x_{00}$, $a_{02} = f$, and $a_{12} = g$.
- $f \sim_r g$ if there exists a 2-simplex b such that $b_{01} = f$, $b_{12} = y_{00}$, and $b_{02} = g$.

Proposition 5.3. *If $\mathcal{C}$ is an ∞-category, then $\sim_\ell$ and $\sim_r$ are the same relation on $\mathrm{Hom}_{\mathcal{C}}(x, y)$, and an equivalence relation. Composition is well-defined on homotopy classes.*

Proof. First, to prove $\sim_\ell$ is reflexive, observe that the degenerate 2-simplex f_{001} has sides x_{00}, f, and f.

The remaining claims here are all consequences of the inner horn-filling condition. First, take a left homotopy $a : f \sim_\ell g$. We will show that $g \sim_r f$. Consider the Λ^3_1 horn whose 012-face is g_{001}, whose 013-face is a, and whose 123-face is g_{011}. This horn admits a filler whose 023-face is a right homotopy from g to f.

We will also show that $f \sim_r g$. Consider the Λ^3_1 horn whose 012-face is a, whose 013-face is g_{001}, and whose 123-face is g_{011}. This horn admits a filler whose 023-face is a right homotopy from f to g. Both these arguments can be reversed using Λ^3_2 horns, showing that $\sim_\ell$ and $\sim_r$ are the same relation, and are symmetric.

Next, take left homotopies $a : f \sim_\ell g$ and $b : g \sim_\ell h$. Consider the Λ^1_3 horn whose 012-face is x_{000}, whose 013-face is a, and whose 123-face is b. This horn admits a filler whose 023-face is a left homotopy from f to h. Thus, left homotopy is transitive.

The final step, to show that composition is well-defined on homotopy classes, is left as an exercise. $\square$

Definition 5.4. If $\mathcal{C}$ is an ∞-category, let the **homotopy category** $h\mathcal{C}$ be the category whose objects are the objects of $\mathcal{C}$, and with $h\mathcal{C}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)/\sim_\ell$.

The homotopy category is the initial category whose nerve receives a map from $\mathcal{C}$. That is, the homotopy category functor is left adjoint to the nerve:

$$\mathrm{Hom}_{\mathsf{sSet}}(\mathcal{C}, N\mathcal{D}) = \mathrm{Hom}_{\mathsf{Cat}}(h\mathcal{C}, \mathcal{D}).$$

This left adjoint can be defined generally on all simplicial sets. However, the definition is hard to use: to find the morphisms in hX from x to y, we have to take all paths in X from x to y and quotient by the equivalence relation generated by homotopy. We should think of ∞-categories as simplicial sets whose homotopy categories are effectively computable. The relation between simplicial sets and ∞-categories, then, is analogous to the relation between categories with weak equivalences (which have homotopy categories) and model categories (in which these homotopy categories can actually be computed).

Thinking of ∞-categories as higher categories again rather than homotopical categories, we note from the above proof that every 2-morphism (i. e., every left homotopy) is invertible. For similar reasons, every n-morphism for $n > 1$ is invertible, once these words have been suitably interpreted. This is why ∞-categories are sometimes called $(\infty, 1)$-categories: the 1 indicates that every n-morphism for $n > 1$ is invertible.

Definition 5.5. A morphism in an ∞-category $\mathcal{C}$ is an **isomorphism** if its image in $h\mathcal{C}$ is an isomorphism. An **∞-groupoid** is an ∞-category in which every morphism is an isomorphism.

Example 5.6. The nerve of an ordinary category $\mathcal{C}$ is an ∞-groupoid iff $\mathcal{C}$ is a groupoid.

Meanwhile, any Kan complex is an ∞-groupoid, for the following reason. Let $f : x \to y$ be a morphism, and consider the (outer) Λ^2_0 horn with 01-face f and 02-face x_{00}. This horn has a filler whose 12-face is a $g : y \to x$ with $gf = 1$ in the homotopy category. Filling a Λ^2_2 horn will likewise give f a right inverse h in the homotopy category. Now, in the homotopy category, $g = gfh = h$ so $g = h$ and f is invertible.

Finally, let $\mathcal{C}$ be an ∞-category. Define $\mathcal{C}^{\simeq}$ to be the full subcategory whose 0-simplices are all the 0-simplices of $\mathcal{C}$, and whose 1-simplices are the isomorphisms of $\mathcal{C}$. This is an ∞-groupoid (as you should check), the **maximal ∞-subgroupoid** of $\mathcal{C}$.

In fact, it's also true that any ∞-groupoid is a Kan complex. This is a little bit harder to prove than its converse. [5] proves it (26.1) as a consequence of the

Theorem 5.7 (Joyal extension theorem)**.** *If $\mathcal{C}$ is an ∞-category, and $\Lambda^n_0 \to \mathcal{C}$ is a horn sending the 01-edge to an isomorphism, then the horn admits a filler.*

We've already identified Kan complexes with spaces, and so you'll often hear things like '∞-groupoids are just spaces' in this neck of the woods.

6 Model structures: Quillen and Joyal

So far, many of the ideas we've been using about ∞-categories also make sense more generally for simplicial sets. It's worth it to think of ∞-categories as a particularly well-chosen subcategory of sSet, and I'll now explain this in detail. This will require me to talk for a little about model categories, and a convenient example of a model category is the Quillen model structure on simplicial sets.

Definition 6.1. A **model category** is a category $\mathcal{C}$ with all small limits and colimits, and with distinguished classes of **weak equivalences** $W = \{\bullet \xrightarrow{\sim} \bullet\}$, **cofibrations** $C = \{\bullet \hookrightarrow \bullet\}$, and **fibrations** $F = \{\bullet \twoheadrightarrow \bullet\}$. An **acyclic** cofibration or fibration is one which is also a weak equivalence. These classes of maps are required to satisfy the following properties.

- Every isomorphism is a weak equivalence, and if two of f, g, and fg are weak equivalences, so is the third.
- Every morphism can be factorized as an acyclic cofibration followed by a fibration, and as a cofibration followed by an acyclic fibration.
- Say that f has the **left lifting property** with respect to a class of morphisms T, $f \in \mathrm{LLP}(T)$, if a lift exists in any diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

for any $g \in T$. Say that g has the **right lifting property** with respect to a class of morphisms S, $g \in \mathrm{RLP}(S)$, if such a lift exists in any diagram for any $f \in S$.

Then: $W \cap C = \mathrm{LLP}(F)$ and $F = \mathrm{RLP}(W \cap C)$. Also: $C = \mathrm{LLP}(W \cap F)$ and $W \cap F = \mathrm{RLP}(C)$.

An object X is **cofibrant** if $\emptyset \to C$ is a cofibration, and **fibrant** if $C \to *$ is a fibration.

Let's remind ourselves of a few facts about these things; if this is news to you, see [1] or [2] for good introductions. First, any two of the distinguished classes of morphisms completely determine the third. For example, if we know C and W, then F has to be $\mathrm{RLP}(C \cap W)$. A little less obviously, if we know C and F, then the lifting axiom determines $W \cap F$ and $W \cap C$ – but by the 2-out-of-3 and factorization axioms, any weak equivalence factors as an acyclic cofibration followed by an acyclic fibration.

There's also a standard way of describing model categories in terms of generating sets of cofibrations and fibrations. Say that $I \subseteq C$ is a **generating set of cofibrations** if $W \cap F = \mathrm{RLP}(I)$. In this case, $C = \mathrm{LLP}(\mathrm{RLP}(I))$ has an explicit description in terms of I: it's the closure of I under pushouts (along any map), transfinite composition, and retracts. There's an analogous definition of a **generating set of acyclic cofibrations**.

Remark 6.2. By these two facts, the data of a model category can be entirely given by a category $\mathcal{C}$ with all small limits and colimits, and two generating sets I and J of cofibrations and acyclic cofibrations respectively. But not every such set of data determines a model category: the factorization and 2-out-of-3 axioms have to be proved.

Let's look at this in action for the Quillen model structure on simplicial sets. The generating sets of cofibrations and acyclic cofibrations are

$$I = \{\partial\Delta^n \to \Delta^n : n \geq 0\},$$

$$J = \{\Lambda^n_k \to \Delta^n : n \geq 1, 0 \leq k \leq n\}.$$

Proposition 6.3. *The cofibrations generated by I are precisely the monomorphisms.*

Proof. If $K \to L$ is a monomorphism, we can build L from K by attaching simplices, starting in lower dimensions and moving up to higher dimensions. Each n-simplex attachment is a pushout of a map $S \to \Delta^n$, where $S \subseteq \partial\Delta^n$; this is a retract of $\partial\Delta^n \to \Delta^n$. The map $K \to L$ is a transfinite composition of these cell attachments.

Conversely, transfinite composition, pushouts, and retracts all preserve monomorphisms, so the class generated by I is precisely the class of monomorphisms. □

The fibrations don't admit a description more convenient than the lifting property:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

However, this condition is important enough that it has its own name: **Kan fibration**. And note that the Kan complexes are precisely the fibrant objects, which are also the cofibrant-fibrant objects since every object is cofibrant.

Finally, if we were doing this for real, we'd define the homotopy groups of a simplicial set, and prove:

Proposition 6.4. *The weak equivalences are precisely the maps that induce isomorphisms on all homotopy groups.*

As an immediate corollary, we see that the *monomorphisms* that induce isomorphisms on all homotopy groups are precisely the class of maps generated by the horn inclusions – something that's not at all obvious!

When I originally wrote this talk, I wrote that the Joyal model structure on simplicial sets is defined by the generating cofibrations

$$I = \{\partial\Delta^n \to \Delta^n : n \geq 0\},$$

and acyclic cofibrations

$$J = \{\Lambda^n_k \to \Delta^n : n \geq 1, 0 < k < n\}.$$

I'm glad I didn't have time to say this, because as Piotr pointed out, it's completely wrong! The set I is a set of generating cofibrations. Thus, the Quillen and Joyal model structures have the same acyclic fibrations, and they have the same cofibrations, namely the monomorphisms. Fibrations – called **categorical fibrations** – are harder to define, and no nice generating set is known. A categorical fibration *whose target is an* ∞*-category* is a map that has the RLP with respect to inner horn fillings $\Lambda^n_k \to \Delta^n$, and which also has the RLP with respect to the map $\{0\} \to N(\mathrm{Iso})$, where Iso is the 1-category with two objects and an isomorphism between them.

Meanwhile, a map with the RLP with respect to the set of inner horn fillings is called a **inner fibration**, and a map with the LLP with respect to inner fibrations is an **inner anodyne map**. So, inner anodyne maps are the closure of the set of inner horn fillings under retracts, pushouts along arbitrary maps, and transfinite compositions. We have strict inclusions

$$\{\text{Kan fibrations}\} \subsetneq \{\text{categorical fibrations}\} \subsetneq \{\text{inner fibrations}\}.$$

In particular, ∞-categories are precisely objects that are inner fibrant over a point. By the above description of categorical fibrations to an ∞-category, any inner fibration to a point is also a categorical fibration, so ∞-categories are the fibrant objects in the Joyal model structure.

Likewise, weak equivalences in the Joyal model structure, called **categorical equivalences**, are hard to define directly. Again, there's a nicer description for maps between ∞-categories: a categorical equivalence $\mathcal{C} \to \mathcal{D}$ is an isomorphism in the ∞-category of ∞-categories.

In practice, inner fibrations are easier to work with than categorical fibrations, and the fact that ∞-categories were defined in terms of inner horn filling is very helpful. Below are some examples of the proper use of inner fibrations.

Theorem 6.5. *If $\mathcal{C}$ is an ∞-category and $i : K \to L$ is a monomorphism of simplicial sets, then $i^* : \mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ is an inner fibration. If i is inner anodyne, then i^* is a trivial fibration.*

Proof. A lift exists in the square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Fun}(L,\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(K,\mathcal{C}) \end{tikzcd}$$

if and only if one exists in the square

$$\begin{tikzcd} \Lambda^n_k \times L \coprod_{\Lambda^n_k \times K} \Delta^n \times K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \times L \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

if and only if one exists in the square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n,\mathcal{C}) \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\Lambda^n_k,\mathcal{C}). \end{tikzcd}$$

In particular, the class of maps $K \to L$ for which these lifts exist is of the form $\mathrm{LLP}(S)$, so it's closed under pushouts along arbitrary maps, retracts, and transfinite composition. Thus, it suffices to show that a lift exists when the map $K \to L$ is $\partial\Delta^m \to \Delta^m$, as these maps generate this set. Since $\mathcal{C}$ is an ∞-category, this reduces to showing that

$$\Lambda^n_k \times \Delta^m \coprod_{\Lambda^n_k \times \partial\Delta^m} \Delta^n \times \partial\Delta^m \to \Delta^n \times \Delta^m$$

is inner anodyne. One can explicitly write this map as a composite of pushouts of inner horn attachments, which is done in [3] (or in [5] in the simplest case. □

Corollary 6.6. *If $\mathcal{C}$ is an ∞-category, than* $\mathrm{Fun}(K, \mathcal{C})$ *is an ∞-category for any simplicial set K (giving us the promised functor categories).*

Definition 6.7. If x and y are objects in an ∞-category $\mathcal{C}$, the **mapping space** $\mathrm{Maps}_{\mathcal{C}}(x, y)$ is defined by the pullback

$$\begin{tikzcd}
\mathrm{Maps}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\
* \arrow[r, "{(x,y)}"'] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}).
\end{tikzcd}$$

(As usual, this definition makes sense if $\mathcal{C}$ is replaced by an arbitrary simplicial set.)

Corollary 6.8. *The mapping spaces of an ∞-category are Kan complexes.*

Proof. By the theorem, the map $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^1, \mathcal{C})$ is an inner fibration. Since inner fibrations are closed under pullbacks along arbitrary maps (this is true for any class of maps of the form $\mathrm{RLP}(S)$, the map $\mathrm{Maps}_{\mathcal{C}}(x, y) \to \Delta^0$ is an inner fibration. That is, the mapping space is an ∞-category. To show it's a Kan complex, we'll show it's an ∞-groupoid, i. e. find a left inverse to an arbitrary morphism $f \in \mathrm{Maps}_{\mathcal{C}}(x, y)_1$. This is the same as filling in the Λ^2_0 horn whose sides are f and x_{00}. This is equivalent to finding a lift in the diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\
\Delta^2 \arrow[r, "{(x,y)_{000}}"'] \arrow[ur, dashed] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}).
\end{tikzcd}$$

This, in turn, is equivalent to finding a lift in the diagram

$$\begin{tikzcd}
\Lambda^2_0 \times \Delta^1 \coprod_{\Lambda^2_0 \times \partial\Delta^1} \Delta^2 \times \partial\Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \\
\Delta^2 \times \Delta^1 \arrow[ur, dashed] &
\end{tikzcd}$$

The map on the left can be decomposed into filling a Λ^3_1 horn and two Λ^3_0 horns, whose 01-edges become the isomorphism x_{00} in $\mathcal{C}$, which we can do by the Joyal extension theorem. □

7 Limits and colimits

An initial object in a 1-category $\mathcal{C}$ has a unique map to any other object. In terms of the nerve, if $\emptyset$ is our initial object and x is any other object, then this diagram has a unique lift:

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "{(\emptyset,x)}"] \arrow[d] & N\mathcal{C} \\ \Delta^1 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

In ∞-categories we should replace uniqueness by essential uniqueness, that is, uniqueness up to a contractible space of choices. Thus we make the following definition:

Definition 7.1. An object $\emptyset$ in an ∞-category $\mathcal{C}$ is **initial** if a lift exists in every diagram of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

with $n \geq 1$, such that the vertex 0 of $\partial\Delta^n$ is sent to $\emptyset$.

An object $* \in \mathcal{C}$ is **final** if a lift exists in every one of the above diagrams, with $n \geq 1$, such that the vertex n of $\partial\Delta^n$ is sent to $*$.

The lifting condition for $n = 1$ tells us that there exist maps from $\emptyset$ to any other object. The lifting condition for $n = 2$ tells us that any two maps $\emptyset \to x$ are homotopic, the condition for $n = 3$ tells us that two of these homotopies are homotopic, and so on. As in 1-categories, initial objects in ∞-categories aren't unique – however, they do form a contractible ∞-groupoid.

In 1-categories, we could now define a colimit of a functor F as an initial object of a category of cones under F. To do this for ∞-categories, we have to make a few definitions.

Definition 7.2. Let K and L be simplicial sets. By convention, we take $K_{-1} = L_{-1} = *$. The **join** of K and L is the simplicial set with

$$(K * L)_n = \coprod_{-1 \leq i \leq n} K_i \times L_{n-i-1}$$

and appropriately defined simplicial operators.

For example,

$$\begin{aligned} (K * L)_0 &= K_0 \times * \sqcup * \times L_0 \\ (K * L)_1 &= K_1 \times * \sqcup K_0 \times L_0 \sqcup * \times L_1 \\ (K * L)_2 &= K_2 \times * \sqcup K_1 \times L_0 \sqcup K_0 \times L_1 \sqcup * \times L_2 \\ &\vdots \end{aligned}$$

We should think, for example, of a simplex in $K_1 \times L_0$ as labeled by 0, 1, 2 with 0 and 1 labelling the 1-simplex of K and 2 labelling the 0-simplex of L. Thus, the face operator that forgets the 0 vertex is a map $K_1 \times L_0 \to K_0 \times L_0$ that's the identity on L_0, and the face operator that forgets the 0 vertex on K_1. The face operator that forgets the 2 vertex is a map $K_1 \times L_0 \to K_1 \times *$ that's the identity on K_1. The degeneracy operators, which duplicate vertices, are maps $K_1 \times L_0$ to $K_2 \times L_0$ or $K_1 \times L_1$, depending on which vertex gets duplicated.

You can also think about this in terms of the topological join – intuitively, the space made up as the union of the line segments between points of K and points of L. Thus, given a 1-simplex f in K and a 0-simplex x of L, there's a 2-simplex of the join that joins f to x, with 01-edge f and with 2-vertex x.

In particular, as you might want to check, $\Delta^n * \Delta^m = \Delta^{n+1+m}$. Also, $K * \emptyset = K$ (remember that $\emptyset$ has a unique (-1)-simplex), and the unique map $\emptyset \to L$ gives a canonical inclusion $K * \emptyset \to K * L$.

Definition 7.3. If $p : K \to \mathcal{C}$ is a map of simplicial sets with $\mathcal{C}$ an ∞-category, the **slice category of $\mathcal{C}$ under p** is the simplicial set $\mathcal{C}_{p/}$ (we might also write $\mathcal{C}_{K/}$) whose n-simplices are the maps $K * \Delta^n \to \mathcal{C}$ which restrict to p on $K * \emptyset$. Likewise, the **slice category of $\mathcal{C}$ over p** is the simplicial set $\mathcal{C}_{/p}$ whose n-simplices are the maps $\Delta^n * K \to \mathcal{C}$ that restrict to p on $\emptyset * K$. (As usual, we can replace $\mathcal{C}$ by an arbitrary simplicial set in this definition.)

Proposition 7.4. *If $\mathcal{C}$ is an ∞-category, then all slice categories $\mathcal{C}_{p/}$ and $\mathcal{C}_{/p}$ are ∞-categories.*

Proof. We want to construct lifts in diagrams

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C}_{p/} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Finding such a lift is equivalent to finding a lift in the diagram

$$\begin{tikzcd} K * \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C} \\ K * \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

so to showing that $K * \Lambda^n_k \to K * \Delta^n$ is an inner fibration. I'll just show this when $K = \Delta^0$. In this case, $K * \Delta^n \cong \Delta^{n+1}$, and $K * \Lambda^n_k$ is the union of the codimension-1 faces containing the vertices 0 and $1 + k$. Thus, filling in the inner horn Λ^n_k, we get a Λ^{1+n}_{1+k}, which can then itself be filled in. $\square$

Definition 7.5. A **colimit** of $p : K \to \mathcal{C}$ is an initial object of the slice category $\mathcal{C}_{p/}$. A **limit** of p is a terminal object of $\mathcal{C}_{/p}$.

Remember that these words translate into more classical topology as 'homotopy (co)limit'! It's hard to come up with cute non-abstract examples. But let's consider, for example, the colimit in the ∞-category of spaces of a diagram $* \leftarrow X \to *$. This is a cone over this diagram, i. e., a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3ex, "\Leftarrow"] \arrow[dr, phantom, shift right=3ex, "\Rightarrow"] & * \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

So Y has a map from X and two nullhomotopies of this map, and is 'homotopy initial' in the category of spaces with such structure. Of course, such a space can be modelled by the suspension of X, $\Sigma X = CX \sqcup_X CX$.

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