The Kervaire invariant

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1 Introduction

I was inspired to give this talk after seeing Jim Fowler give a talk at the Midwest Topology Seminar at IUPUI, in which he constructed a new class of non-triangulable manifolds. It came as something of a shock to see Fowler using characteristic classes, homotopy groups, and so on to study strange, pathological spaces out of geometry. For me, this was a much-needed reminder that the now-erudite techniques of homotopy theory arose from the study of honest geometric problems, and I decided to leran more of this theory.

Kervaire's 1960 paper [5] turned out to be the perfect place to start. In it, Kervaire constructs the first known manifold admitting no smooth structure (in fact, it doesn't even admit any C^1-differentiable structure!). This was fairly soon after Milnor had started studying exotic spheres, so these ideas were very much on topologists' minds. Moreover, in the course of his proof, he ends up using a lot of the great results of mid-century topology – the Pontryagin-Thom construction, the J-homomorphism, obstruction theory, and various results about the homotopy groups of spheres, among others. In particular, I had to relearn most of what I'd learned from Hatcher and Mosher-Tangora while preparing for this talk, which was a lot of fun.

Kervaire limits his thought to 4-connected 10-dimensional manifolds M, whose cohomology is thus concentrated in degrees 5 and 10. The cup product on the middle cohomology group is a skew-symmetric perfect pairing, and in mod 2 cohomology, Kervaire refines this to a quadratic form. This quadratic form is described up to isomorphism by its dimension and an $\mathbb{F}_2$-valued invariant, the **Kervaire invariant** of the manifold $\Phi(M)$. As was later realized, this invariant can be defined for any *framed* manifold of dimension $4k+2$, and here and elsewhere, I'll try to focus on this more general perspective.

Remark 1. You should think of the Kervaire invariant as an analogue of the **signature** of a $4k$-dimensional manifold, which is by definition the signature of the now-symmetric cup product pairing on its middle cohomology. In fact, L-theory gives a unified perspective on the signature and the Kervaire invariant. Unfortunately, I don't know enough about this to talk about it.

In the rest of the paper, Kervaire constructs a topological manifold M_0 with $\Phi(M_0) = 1$, and shows that $\Phi(M) = 0$ on all differentiable manifolds M. It follows that M_0 admits no differentiable structure. This M_0 is constructed by gluing together two smooth manifolds along a boundary S^9, so as a corollary, one of these copies of S^9 has an exotic smooth structure.

My approach has been to track down and record as many of the fascinating techniques used in papers from this period as I can, and to supplement them with more modern ones whenever I'm able to figure them out. I got stuck on a few proofs, so if you, the reader, are able to clarify anything, fix anything, or give any more general proofs, please let me know!

2 Quadratic forms in characteristic 2 and the Arf invariant

We begin with some algebra.

Definition 2. Let V be a finite-dimensional vector space over a field k. A **quadratic form** Q on V is a function $Q : V \to k$ satisfying

$$Q(ax) = a^2 Q(x) \quad \text{for} \quad a \in k, x \in V,$$

and such that the form

$$B(x, y) = Q(x + y) - Q(x) - Q(y)$$

is bilinear.

If we pick a basis $\{e_1, \ldots, e_n\}$ for V, then Q must take the form

$$Q\left(\sum_i x_i e_i\right) = \sum_{i,j} a_{ij} x_i x_j.$$

In characteristic not equal to 2, we can always complete the square, so we can choose a basis so that

$$Q\left(\sum_i x_i e_i\right) = \sum_i a_i x_i^2.$$

The **discriminant** of Q, defined to be $\prod a_i$, is then a well-defined class in $k^\times/(k^\times)^2$. For example, if Q is a nondegenerate quadratic form over $\mathbb{R}$ of signature (p, q), we can choose p of the a_i to be 1 and q of them to be -1, and the discriminant is $(-1)^q$.

In characteristic 2, things are a little different: the cross-terms won't disappear, and the bilinear form is skew-symmetric, since

$$B(x, x) = Q(2x) - 2Q(x) = 0.$$

However, the quadratic form can be normalized in terms of a symplectic basis.

Proposition 3 (Arf, [1])**.** *Let (V, Q) be a quadratic form over a field k of characteristic 2. Then there is a basis $\{e_1, f_1, \ldots, e_m, f_m, g_1, \ldots, g_n\}$ such that*

$$Q\left(\sum_{i=1}^{m}(x_i e_i + y_i f_i) + \sum_{j=1}^{n} z_j g_j\right) = \sum_{i=1}^{m}(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{n} c_j z_j^2.$$

Proof. First consider the associated bilinear form B. Let $\{g_j\}$ be any basis for $V^\perp$ (which is obviously a direct summand of (V, B)), and pick $e_1 \in \{g_j\}^\perp$. Then there is some f_1 with $B(e_1, f_1) \neq 0$. By scaling f_1, we can assume $B(e_1, f_1) = 1$. Now any vector $v \in \{g_j\}^\perp$ can uniquely be written as

$$v = ae_1 + bf_1 + v', \quad v' \in \{e_1, f_1\}^\perp.$$

Namely, we define $a = B(f_1, v)$ and $b = B(e_1, v)$. It follows that $\{e_1, f_1\}^\perp$ is a direct summand of (V, B). By induction, we can pick a symplectic basis for B. Now letting $a_i = Q(e_i)$, $b_i = Q(f_i)$, and $c_j = Q(g_j)$, it's clear that Q has the required form. □

Definition 4. If Q has the above form, then the **Arf invariant** of Q is $\sum a_i b_i$.

Exercise 5. Show that choosing another symplectic basis for Q changes the Arf invariant by adding an element of k of the form $u^2 + u$.

Let P be the additive subgroup of k consisting of elements of the form $u^2 + u$. Then the Arf invariant is a well-defined element of k/P. In particular, if $k = \mathbb{F}_2$, then the Arf-invariant is defined in $\mathbb{F}_2$.

Theorem 6 (Arf)**.** *Nondegenerate quadratic forms over a field k of characteristic 2 are completely classified by their dimension and their Arf invariant.*

3 The Kervaire invariant

3.1 The Kervaire invariant for framed manifolds

Definition 7. A **framing** on a closed smooth manifold M^n is a trivialization of the normal bundle $\nu(M, i)$ of some smooth embedding $i : M \hookrightarrow \mathbb{R}^{n+k}$. Note that if M is framed and $f : \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ is the inclusion $f(\mathbf{x}) = (\mathbf{x}, 0)$, then $\nu(M, f \circ i) = \nu(M, i) \oplus \mathbb{R}$ inherits a canonical framing: the new copy of $\mathbb{R}$ can point up everywhere. We identify this new framing with the old one.

The main reason to study homotopy groups of spheres if you care about manifolds is the Pontryagin-Thom construction. Let M be a framed manifold with normal bundle $\nu = \nu(M, i : M^n \hookrightarrow \mathbb{R}^{n+k})$. The framing induces an isomorphism between the total space of ν and $M \times \mathbb{R}^k$, and thus a homeomorphism $\mathrm{Th}(\nu) \cong \Sigma^r(M_+)$. This Thom space is obtained from a tubular neighborhood of M, $M \times D^r \subseteq \mathbb{R}^{n+r}$, by collapsing its boundary $M \times S^r$ to a single point. Viewing S^{n+r} as the one-point compactification of $\mathbb{R}^{n+r}$, we get a map $S^{n+r} \to \Sigma^r(M_+)$ by sending everything outside this tubular neighborhood to the basepoint as well. On the other hand, $M_+ \to S^0$ induces $\Sigma^r(M_+) \to S^r$. Thus, the framed manifold M gives an element of $\pi_{n+r}S^r$. Adding a new coordinate, as specified in the definition, corresponds to the suspension map

$$E : \pi_{n+r}S^r \to \pi_{n+r+1}S^{r+1},$$

so we're really looking at a stable homotopy class in $\pi_n S$. Moreover, it's not hard to see that this class is invariant under framed cobordism.

Theorem 8 (Pontryagin)**.** *The map*

$$\Omega_*^{\mathrm{fr}} \to \pi_* S$$

described above is an isomorphism of graded rings.

We can now give Browder's definition of the Kervaire invariant – note this has the advantage of being manifestly invariant under framed cobordism.

Definition 9. Let M^{2n} be a smooth, framed manifold, and let $\alpha \in H^n(M; \mathbb{F}_2)$, viewed as a map $\alpha : M_+ \to K(\mathbb{F}_2, n)$. Composition with the Pontryagin-Thom map $S^{2n+r} \to \Sigma^r(M_+)$ gives a map

$$Q(\alpha) \in \pi_{2n+r}(\Sigma^r(K(\mathbb{F}_2, n))) \cong \pi_{2n}(\Sigma^\infty K(\mathbb{F}_2, n)) \cong \mathbb{F}_2$$

(for r large). The **Kervaire form** is $Q : H^n(M; \mathbb{F}_2) \to \mathbb{F}_2$. The **Kervaire invariant** is the Arf invariant of this quadratic form.

We've used the fact that $\pi_{2n}(\Sigma^\infty K(\mathbb{F}_2, n)) \cong \mathbb{F}_2$, which is not obvious to me but is somewhere in [2]. We also need to check that Q is quadratic, i. e. that $Q(\alpha+\beta) - Q(\alpha) - Q(\beta)$ is bilinear in α and β. This is a question about the effect on stable homotopy groups of the three maps

$$K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} K(\mathbb{F}_2, n),$$

the addition map μ and the two projection maps p_1, p_2. After a single suspension, we get a splitting

$$\Sigma(K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n)) \simeq \Sigma K(\mathbb{F}_2, n) \vee \Sigma K(\mathbb{F}_2, n) \vee \Sigma(K(\mathbb{F}_2, n) \wedge K(\mathbb{F}_2, n)),$$

and p_1 and p_2 are just projections onto the first two factors, while the third factor has its first homotopy group $\pi_{2n+1}(\Sigma(K(\mathbb{F}_2, n) \wedge K(\mathbb{F}_2, n))) = \mathbb{F}_2$, generated by v, say. It's now easy to see that for $\gamma, \delta \in \pi_{2n}\Sigma^\infty K(\mathbb{F}_2, n)$,

$$\mu_*(\gamma \times \delta) = (p_1)_*\gamma + (p_1)_*\delta + (\gamma\delta)v,$$

viewing $\gamma\delta$ as an element of $\mathbb{F}_2$.

3.2 The Kervaire invariant for highly connected topological manifolds

The second definition comes from Kervaire's original paper, where he defines it for 4-connected 10-manifolds. (He additionally considers only PL-manifolds, but I'm sure one can do this for any manifold with a CW-structure using cellular cohomology.) I'm giving the clearer and more general one from the last section of [6]. Of course, this can be generalized still further – for example, [4] interprets it the Kervaire form as a secondary cohomology operation, so that it's well-defined without the connectivity assumptions modulo some indeterminacy.

First, let's recall some the basics of obstruction theory. Suppose we have a CW-complex K, a space Y, and a map $f : K^{(r)} \to Y$ that we'd like to extend to $K^{(r+1)}$. We define the **obstruction cochain** $c(f) \in C^{r+1}(K; \pi_r Y)$ by

$$c(f)(\sigma_{r+1}) = [f(\partial\sigma)] \in \pi_r Y \qquad (\sigma_{r+1} \text{ an } (r+1)\text{-cell}).$$

If g, h are two extensions of f to $K^{(r+1)} \to Y$, we define the **difference cochain** $d(g,h) \in C^{r+1}(K; \pi_{r+1}Y)$ by

$$d(g,h)(\sigma_{r+1}) = [g(\sigma) - h(\sigma)] \in \pi_{r+1}Y.$$

(That is, $g(\sigma)$ and $h(\sigma)$ are r-disks with a common boundary, so they fit together to form an $(r+1)$-sphere in Y.) One then shows that:

1. $c(f) = 0$ iff f extends to $K^{(r+1)}$;
2. $c(f)$ is a cocycle, and its cohomology class is 0 iff there is an extension of $f|_{K^{(r-1)}}$ to $K^{(r+1)}$;
3. $\delta d(g,h) = c(g) - c(h)$.

Let M be a k-connected $2k$-manifold, with $k > 1$. The cohomology of M is concentrated in degrees 0, k, and $2k$, and the cup product is a perfect pairing on $H^k(M)$ (symplectic if k is odd, symmetric if k is even). Pick $\alpha \in H^k(M)$, and consider the problem of defining a map $f : M \to S^k$ with $f^*(i_k) = \alpha$. Since $\pi_r S^k = 0$ for $r < k$, all the obstruction cochains vanish up to degree k. In fact, we might as well have f be trivial on $M^{(k-1)}$, while on M^k we can define it by

$$f(\sigma_k) = \widetilde{\alpha}(\sigma_k) \cdot i_k,$$

where $\widetilde{\alpha}$ is a representative cocycle for α. The next obstruction is in $C^{k+1}(M; \pi_k S^k)$, but note that this is

$$c(f|_{M^{(k)}})(\sigma_{k+1}) = [f|_{M^{(k)}}(\partial\sigma_{k+1})] = [\widetilde{\alpha}(\partial\sigma_{k+1}) \cdot i_k] = [(\delta\widetilde{\alpha})(\sigma_{k+1}) \cdot i_k] = 0$$

since $\widetilde{\alpha}$ is a cocycle. So f extends over $M^{(k+1)}$ by point 1 above. The next and final obstruction is in $C^{2k}(M; \pi_{2k-1}S^k)$. By point 2, we might as well consider its class in $H^{2k}(M; \pi_{2k-1}S^k)$. By point 3, varying $f|_{M^{(2k-1)}}$ does not change this cohomology class. One can also check that this class is natural for manifolds M of the given form (or even for all spaces with cohomology vanishing in between degrees k and $2k$).

Definition 10. Define the **Kervaire class** $c(\alpha) \in H^{2k}(M; \pi_{2k-1}S^k)$ to be this obstruction class. (We will identify this group with $\pi_{2k-1}S^k$ by pairing with $[M]$). As shown below, c is a quadratic form refining the mod 2 cup product pairing; let the **Kervaire invariant** $\Phi(M)$ be its Arf invariant.

In the cases we care about, $c(\alpha)$ takes values in the subgroup of $\pi_{2k-1}S^k$ generated by the Whitehead product $[i_k, i_k]$. Recall that this is the composition

$$S^{2k-1} \to S^k \vee S^k \xrightarrow{\nabla} S^k,$$

where the first map is the attaching map of the $2k$-cell of $S^k \times S^k$.

The EHP sequence is a 2-local exact sequence

$$\cdots \to \pi_{2k+1}S^{k+1} \xrightarrow{H} \pi_{2k+1}S^{2k+1} \xrightarrow{P} \pi_{2k-1}S^k \xrightarrow{E} \pi_{2k}S^{k+1} \to \cdots.$$

Here $H(f)$ is the Hopf invariant of f times the generator $1 \in \pi_{2k+1}S^{2k+1}$, $P(1) = [i_k, i_k]$, and E is the suspension map. If k is odd, then H is nonzero and $\mathrm{im}(P)$ is finite and cyclic. If $k \neq 1, 3, 7$, then $\mathrm{im}(H)$ has index 2, so $[i_k, i_k]$ generates a subgroup of order 2 of $\pi_{2k-1}S^k$.

Lemma 11. *c satisfies*

$$c(\alpha + \beta) = c(\alpha) + c(\beta) + [i_k, i_k]\alpha\beta.$$

Proof. Let Z be the space obtained by killing all the homotopy groups of S^k in degrees at least $2k$. For $\alpha \in H^k(M)$, we can construct $M^{(2k-1)} \to S^k$ as above and extend it to $g : M \to Z$, so that $g^*(e_1) = \alpha$ where e_1 generates $H^k(Z)$. Given α, β, we get $h : M \to Z \times Z$ such that $g^*(e_1 \times 1 + 1 \times e_1) = \alpha + \beta$. As both Z and $Z \times Z$ satisfy the cohomology vanishing condition and c is natural, it suffices to compute $c(e_1 \times 1 + 1 \times e_1)$ in terms of $c(e_1)$.

Observe that

$$(Z \times Z)^{(2k+1)} = Z^{(2k+1)} \times * \cup * \times Z^{(2k+1)} \cup S^k \times S^k.$$

Thus, the obstruction class for $e_1 \times 1 + 1 \times e_1$ is of the form

$$c(e_1 \times 1 + 1 \times e_1) = a \times 1 + 1 \times b + \gamma(e_1 \times e_1), \quad \gamma \in \pi_{2k-1}S^k.$$

By looking at the inclusions of $Z \times *$ and $* \times Z$ and using naturality, we get $a = b = c(e_1)$. By looking at the inclusion of $S^k \times S^k$ and using naturality, we see that γ is the obstruction to the existence of $S^k \times S^k \to S^k$ pulling back i_k to $i_k \times 1 + 1 \times i_k$. Of course, this class is just $[i_k, i_k]$ times the generator of $H^{2k}(S^k \times S^k)$. Thus,

$$c(e_1 \times 1 + 1 \times e_1) = c(e_1) \times 1 + 1 \times c(e_1) + [i_k, i_k](e_1 \times e_1),$$

and the general statement follows by naturality. □

Strangely, for an arbitrary topological manifold satisfying the connectivity condition, it's not true, or at least not obvious to me, that $c(\alpha) \in [i_k, i_k]\pi_{2k-1}S^k \cong \mathbb{F}_2$. However, this is true in two cases we care about. First, if $k = 5$, then $\pi_9 S^5 \cong \mathbb{F}_2$, generated by the Whitehead product.

Second, suppose M is smooth, framed, and satisfies the connectivity condition. Then by the Hurewicz theorem, every class $\alpha \in H^k(M; \mathbb{F}_2)$ has a dual homology class $\alpha^\vee$ represented by an embedded sphere S^k. Let $\mathrm{Th}(S^k)$ be the Thom class of its normal bundle, and i_k the generator of $H^k(\mathrm{Th}(S^k))$. Using excision (this is in [6]), we have $c(i_k) = c(\alpha) \in \pi_{2k-1}S^k$. So the Kervaire form only depends on this embedded sphere. On the other hand, M has a framing, so the normal bundle of S^k is stably trivial. Thus, it's represented by an element of the kernel of

$$\pi_k BSO(k) \to \pi_k BSO,$$

which has order 2 if k is odd and $\neq 1, 3, 7$. This proves that the Kervaire form is valued in $\mathbb{F}_2$ in this case.

As a corollary, we get a nice geometric interpretation of the Kervaire form for framed manifolds. Think of c as operating on $H_k(M; \mathbb{F}_2)$ by Poincaré duality. Then $c(x) = 0$ iff x can be represented by an embedded sphere with trivial normal bundle, and 1 otherwise.

As a second corollary, it's easy to see why this definition agrees with Browder's: the map $S^{r+2k} \to \Sigma^r(K(\mathbb{F}_2, k))$ from Browder's definition can be chosen to factor through $\Sigma^r(S^k)$, where S^k is the embedded sphere above.

4 The Kervaire manifold

Let $p : U \to S^5$ be the unit disk bundle associated to the tangent bundle of S^5, and let $D \subseteq S^5$ be an embedded disk. Over D, U looks like $D^5 \times D^5$. Define W by gluing together two copies of U along this $D^5 \times D^5$ by swapping the two factors. This has corners, but they can be smoothed to make W a smooth manifold.

Note that the boundary of W consists of an $D^5 \times S^4$ and a $S^4 \times D^5$ glued along their boundary $S^4 \times S^4$, via a certain diffeomorphism $f : S^4 \times S^4 \to S^4 \times S^4$. This map can be described explicitly. Note that we've already trivialized the bundle U inside the embedded disk D to perform the gluing, and outside D to identify $\partial W \cong D^5 \times S^4 \cup S^4 \times D^5$. There's a clutching map $f_1 : S^4 \to SO(5)$ patching together these trivializations, and switching the 'sphere' and 'bundle' coordinates of $S^4 \times S^4$ entails rotating the sphere coordinate by f_1 and rotating the bundle coordinate by the opposite of f_1. Therefore, f must take the form

$$f(x, y) = (f_1(f_1(x) \cdot y)^{-1} \cdot x, f_1(x) \cdot y).$$

Lemma 12 (Milnor [7]). *∂W is homeomorphic to S^9.*

Proof. This uses Morse theory and very little about the clutching map f_1. All we need to know is that f_1 factors through $SO(4)$, embedded in $SO(5)$ as the subspace of rotations that fix the last coordinate. To prove this, consider the long exact sequence in homotopy

$$\pi_4 SO(4) \to \pi_4 SO(5) \to \pi_4 S^4 \to \pi_3 SO(4) \to \pi_3 SO(5) \to 0.$$

$\pi_3 SO(5)$ is in the stable range, so it's $\pi_3 SO = \mathbb{Z}$. On the other hand, $SO(4) \cong SO(3) \times S^3 \cong \mathbb{R}P^3 \times S^3$, so $\pi_3 SO(4) \cong \mathbb{Z}^2$. Thus $\pi_4 S^4 \cong \mathbb{Z}$ must get mapped injectively to $\pi_3 SO(4)$ by the connecting map, meaning that $\pi_4 SO(4) \to \pi_4 SO(5)$ is surjective.

We've written $\partial W = D^5 \times S^4 \cup_f S^4 \times D^5$, but now write $\partial W = \mathbb{R}^5 \times S^4 \cup S^4 \times \mathbb{R}^5$ by identifying

$$(tx, y) \simeq (x', t^{-1}y') \quad \text{for} \quad x \in S^4, y \in S^4, t \in (0, \infty),$$

where $(x', y') = f(x, y)$. This has the advantage of giving ∂W a smooth structure. We now define $g : M \to [-1, 1]$ by

$$g(tx, y) = \frac{y_n}{\sqrt{1+t^2}}$$
$$g(x', t^{-1}y') = \frac{y_n}{\sqrt{1+t^2}}.$$

This is well-defined since f_1 factors through $SO(4)$, and clearly smooth. It has two critical points, namely the two points in the first chart with $tx = 0$ and $y_n = \pm 1$, and one checks that these are nondegenerate. By Morse theory, ∂W is homeomorphic to S^9. ☐

Remark 13. In fact, ∂W is PL-homeomorphic to S^9. Also, the above lemma is fairly general, and was used in [7] together with the signature to construct exotic S^{4k-1} for $k = 2, 4, 5, 6, 7, 8$.

The **Kervaire manifold** M_0 is given by attaching a cone to the boundary of W.

Theorem 14. *M_0 has Kervaire invariant 1.*

Proof. First, we compute the cohomology of M_0. Note that $C(\partial W) \cong D^10$, so M_0 has the same homology as W in degrees less than 9. In particular, it's 4-connected, and its H_5 is 2-dimensional, generated by the zero sections in the two copies of U. It's hopefully clear that these intersect once, so if x and y are their duals in $H^5(M_0; \mathbb{F}_2)$, we have $xy = [M_0] \in H^10(M; \mathbb{F}_2)$ (and, of course, $x^2 = y^2 = 0$). Thus, x and y form a symplectic basis, and it remains to evaluate the Kervaire form Q on x and y.

Let $j : S^5 \to M_0$ generate the homology class dual to x. As described, j lands in the smooth manifold W, so we can form the normal bundle of j inside W. Of course, this is just one of the copies of $p : U \to S^5$. As in the previous section, there's an induced map $M_0 \to \mathrm{Th}(p)$, given by sending everything outside U to the basepoint. By the lemma below,

$$\mathrm{Th}(p) \simeq J_2(S^5) \simeq (\Omega S^6)^{(10)} \simeq Z^{(10)},$$

where Z is constructed from S^5 by killing the homotopy groups in degrees at least 10. As a result, we get a map $f : M_0 \to Z$ such that $f^*(e_1) = x$. By the same lemma, the generator e_2 of $H^{10}(Z)$ pulls back to a generator of $H^{10}(\mathrm{Th}(p))$, so $f^*(e_2)$ generates $H^{10}(M_0; \mathbb{Z})$. Thus, $Q(x) = 1$, and likewise $Q(y) = 1$. So $\Phi(M_0) = Q(x)Q(y) = 1$. ☐

Lemma 15. *The Thom space of $p : U \to S^5$ is the 10-skeleton of Z.*

Proof. This relies on the fact that $\pi_9 S^5 = \mathbb{Z}/2$, so that one only has to attach one 10-cell to form $Z^{(10)}$. The attaching map is, of course, $[i_5, i_5] : S^9 \to S^5$. Since TS^5 is not a trivial bundle, and has just a 5-cell and a 10-cell, it must be homotopy equivalent to $Z^{(10)}$. (This can also be identified as the partial James construction $J_2 S^5 = (\Omega S^6)^{(10)}$ by studying the pushout diagrams

$$\begin{array}{ccc} S^9 & \longrightarrow & D^10 \\ \downarrow & & \downarrow \\ S^5 \vee S^5 & \longrightarrow & S^5 \times S^5 \\ \nabla \downarrow & & \downarrow \\ S^5 & \longrightarrow & J_2 S^5, \end{array}$$

from which we see that $J_2 S^5$ is the mapping cone of $[i_5, i_5]$.) ☐

This lemma, by the way, is the best answer as to why dimension 10. For k even, the Hopf invariant on $\pi_{2k+1}S^{k+1}$ is 0, so $\pi_{2k-1}S^k$ is infinite and we don't get the desired $\mathbb{F}_2$. For $k = 1$, 3, or 7, the Hopf invariant is surjective, so $[i_k, i_k]$ is trivial by exactness of the EHP sequence. For $k = 5$, things seem to magically work out. The same technique works at $k = 9$, apparently, though it's more complicated: $\pi_{18}S^9 \cong (\mathbb{Z}/2)^3$.

In the next section, we'll establish that all 4-connected differentiable 10-manifolds have Kervaire invariant 0, proving that M_0 has no differentiable structure.

Corollary 16. *∂W is homeomorphic but not diffeomorphic to S^9.*

If it were, we'd have a smooth manifold, since $C(S^9) \cong D^{10}$ is clearly smooth!

5 Differentiable manifolds of dimension 10 have Kervaire invariant 0

This section contains the proof of the following theorem.

Theorem 17. *If M is a 4-connected differentiable closed 10-manifold, then $\Phi(M) = 0$.*

Lemma 18. *Any such manifold M admits a framing.*

Lemma 19. *The Kervaire invariant is a homomorphism $\Omega^{\text{fr}}_{4k+2} \to \mathbb{F}_2$.*

Proof. We've seen, using Browder's definition, that it's framed cobordism invariant. Let M and N be two framed manifolds. Then the middle cohomology of $M\#N$ is just $H^{2k+1}(M) \oplus H^{2k+1}(N)$, with the two summands orthogonal for the cup product pairing. As a result, we can form a symplectic basis for $H^{2k+1}(M\#N;\mathbb{F}_2)$ by taking a basis for $H^{2k+1}(M;\mathbb{F}_2)$ and one for $H^{2k+1}(N;\mathbb{F}_2)$, and applying naturality of the Kervaire form to the maps $M\#N \to M \vee N \to M$ and $M\#N \to M \vee N \to N$, we see that $\Phi(M\#N) = \Phi(M) + \Phi(N)$. □

In the case of dimension 10, we thus have a homomorphism $\Phi : \pi_{10}S \to \mathbb{F}_2$. Obviously this kills every element of odd order, so it suffices to show that it kills the 2-component of $\pi_{10}S$. This will follow from the following two lemmas.

Lemma 20. *Composition with $\eta \in \pi_1 S$ induces a surjection $\pi_9 S_{(2)} \to \pi_{10}S_{(2)}$.*

Lemma 21. *Any element of $\pi_{10}S$ of the form $\beta\eta$ with $\beta \in \pi_9 S$ can be obtained by applying the Pontryagin-Thom construction to a framed homotopy sphere Σ^{10}.*

Since Σ^{10} has no middle cohomology, its Kervaire invariant is obviously zero, so the theorem will follow.

Proof of Lemma 20. Kervaire proves this at length using Postnikov towers, but it's easy to see from the Adams spectral sequence: if P is the v_1-periodicity element, then Ph_1 and Ph_1^2 are both permanent cycles for degree reasons, and no other groups show up in the 10-stem even on the Adams E_2 page.

In fact, the images of these cycles generate the image of J in $\pi_9 S$ and $\pi_{10}S$ respectively. For what it's worth, $\pi_9 S \cong (\mathbb{Z}/2)^3$ and $\pi_{10}S \cong \mathbb{Z}/6$. □

Proof of Lemma 21. Let $\alpha = \beta\eta$ with $\beta \in \pi_9 S$. Start by considering a framed 9-manifold M represented by β; I claim that M is framed cobordant to a homotopy 9-sphere. Note that any time we have an embedded $S^p \subseteq M$ with trivial normal bundle, we can perform (framed, smooth) surgery to kill the corresponding element in $\pi_p M$. If $p \leq 4$, any such sphere has trivial normal bundle, and if $p \leq 3$, this will only add homotopy elements in the higher degree $8 - p$ (these are essentially the results of [8]). So without loss of generality, M is 3-connected. By Poincaré duality, it suffices to perform surgeries so that M is 4-connected.

For $\lambda \in \pi_4 M$, find a new map $f : S^4 \times D^5 \to M$ representing λ, and prepare the operating theater for surgery along f, creating a new manifold M' in which $\lambda = 0$. We run the risk of introducing a new element λ' into $\pi_4(M')$, represented by $f(* \times \partial D^5)$; this risk is being averted in two steps. (I don't understand this part of the proof.) First, if λ has infinite order, one shows directly that λ' is nullhomotopic in $M - f(S^4 \times D^5)$. Second, if λ has finite order, then one looks at Euler characteristics and sees λ' has infinite order in $H_4(M')$ – so by the first step, one can also surgically remove λ' without otherwise increasing H_4.

So we've shown that β is represented by a framed homotopy 9-sphere Σ^9. By the Pontryagin isomorphism, $\beta\eta$ is represented by $\Sigma^9 \times S^1$. Obviously, the normal bundle of S^1 in $\Sigma^9 \times S^1$ is trivial, so we can again perform surgery to kill the first homology. Therefore, $\beta\eta$ is represented by a framed homotopy 10-sphere. $\square$

6 Further directions

Kervaire manifolds can be constructed in any dimension $4k+2$ by an analogous process: glue together two copies of $D(TS^{2k+1})$ and cone off the boundary. The equivalent of Lemma 12, saying that the boundary is a (possibly exotic) sphere, is true, though one has to use a cleverer proof than mine. Moreover:

Theorem 22 ([4])**.** *The Kervaire invariant is zero on framed manifolds of dimension $8k+2$. Thus, a Kervaire manifold of dimension $8k+2$ admits no differentiable structure. Moreover, the boundary sphere is exotic, and generates the group $bP_{8k+2} \cong \mathbb{Z}/2$ of exotic spheres which bound parallelizable manifolds.*

The **Kervaire invariant** admits several different interpretations. We've already described it in terms of embedded spheres in a framed manifold. In [4], it's defined as a secondary cohomology operation. From Theorem 22, it's clear that there's some link with exotic spheres, and the Kervaire invariant was used this way in [6] to compute the image of

$$\Theta_n/bP_{n+1} \to \pi_n S^0/J.$$

The **Kervaire invariant one problem** asks when there are framed manifolds with Kervaire invariant one. This question was probably introduced by the observation of [4] that the Kervaire invariant was 0 in at least half the relevant dimensions. Next, Browder [3] proved that the Kervaire invariant was 0 in all dimensions not of the form 2^k-2; he also framed the problem in terms of stable homotopy theory, which is how it's been studied ever since. In its least geometrical formulation, the question is about the survival of certain classes in the Adams spectral sequence. Under this formulation and using mod 8 equivariant homotopy theory, Hill, Hopkins, and Ravenel were able to prove that the Kervaire invariant was 0 in dimension 2^k-2 for $k \geq 8$. Framed manifolds of Kervaire invariant one have been constructed in dimension 2^k-2 for $2 \leq k \leq 6$, beginning with [6] in dimensions 6 and 14. Dimension 126 is still open.

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