TMF for number theorists

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1 What is stable homotopy theory?

Stable homotopy theory starts with a question: *what are the homotopy groups of spheres?* We define $\pi_{n+k}S^k$ to be the group of homotopy classes of maps $S^{n+k} \to S^k$. We can **suspend** a sphere by placing it as the 'equator' of a sphere one dimension higher. This construction is actually functorial, so we get maps

$$\Sigma : \pi_{n+k}S^k \to \pi_{n+k+1}S^{k+1}.$$

The 'stable' in 'stable homotopy theory' means stabilization with respect to this suspension operation. Thus, we define

$$\pi_n S = \operatorname{colim}_k \pi_{n+k}S^k,$$

and make these **stable homotopy groups of spheres** $\pi_* S$ the focus of our energy. One of the earliest theorems of this subject, the Freudenthal suspension theorem, implies that this colimit is always attained at a finite stage:

$$\pi_n S = \pi_{n+k}S^k \quad \text{for} \quad k \geq n+2.$$

So if you'd like, we're studying the homotopy groups of sufficiently high-dimensional spheres.

One can, and should, ask why, and I could give you a number of answers. One answer is that the original problem of calculating *all* the homotopy groups of spheres is so intractable that we might as well replace it with a slightly less intractable one. A second answer is that the stable stuff has a lot more structure than the unstable stuff: for example, $\pi_* S$ is a graded ring, since we can compose two maps $\alpha : S^{n+k} \to S^k$ and $\beta : S^{m+j} \to S^j$ only after suspending them:

$$S^{n+k+m+j} \xrightarrow{\Sigma^{m+j}\alpha} S^{k+m+j} \xrightarrow{\Sigma^k \beta} S^{k+j}.$$

A third answer is that, for whatever reason, the reduction has been extraordinarily fruitful. The techniques of unstable homotopy theory have largely stagnated since the 60s, but those of stable homotopy theory continue to develop and offer what at least *feels* like new insight into the nature of space, as well as touching on manifold geometry, group theory, logic, mathematical physics, and as I hope to show in this talk, number theory and algebraic geometry.

Our reduction has worked so well because, starting in the 60s, we allowed it to change the very objects we studied. Rather than studying topological spaces or homotopy types, modern stable homotopy theorists study objects called **spectra**, which are something like 'spaces with the suspension operator inverted.' I don't expect you to know what a spectrum is, and unfortunately, I don't have the time to tell you. Fortunately, spectra are very close to (extraordinary) cohomology theories, which are hopefully a little more familiar. Let me talk about these for a little bit.

2 What is a cohomology theory?

Cohomology theories in topology, much like Weil cohomology theories in algebraic geometry, are functors from spaces to graded abelian groups, subject to a few axioms. I won't give all of them here, but here are some representative examples.

- A cohomology theory E is supposed to be **homotopy-invariant**: if $f, g : X \to Y$ are homotopic maps, then they induce the same map $E^*Y \to E^*X$, and if X and Y are equivalent in the homotopy category, then $E^*Y \cong E^*X$.

- Cohomology theories are **stable** with respect to suspension: $E^*X \cong E^{*+1}\Sigma X$.

- Not an axiom, but a theorem. The **coefficient ring** E^* of E is the E-cohomology of a point. The theorem is that we can sort of recover the E-cohomology of a space from its ordinary cohomology and the E-cohomology of a point, in the sense that there's a spectral sequence

$$H^p(X; E^q) \Rightarrow E^{p+q}X.$$

Perhaps it's best, though, if I just give you some examples of cohomology theories.

- Ordinary cohomology $H\mathbb{Z}$ and cohomology with coefficients in some ring HR are obvious examples. In particular, we like $H\mathbb{Z}_{(p)}$, $H\mathbb{F}_p$, $H\mathbb{Q}$.

- The stable homotopy groups

$$\pi_*X = \operatorname{colim}_n \pi_{n+*}(\Sigma^n X)$$

are an example of a homology theory. The corresponding *cohomology* theory would be stable cohomotopy,

$$\pi^*X = \operatorname{colim}_n [\Sigma^n X, S^{n+*}].$$

- Next up is topological K-theory. $K^0(X)$ is defined as the group of complex vector bundles over X up to isomorphism, and more generally, $K^n(X)$ is defined as $K^0(\Sigma^{-n}X)$. We can do the same thing with real vector bundles, giving a cohomology theory KO.

- Another sort of well-known brand of cohomology theories are the cobordism theories. It's easiest to say this in terms of bordism, the associated homology theory. Here $MO_n(X)$ is the group of smooth n-manifolds with maps to X, modulo the relation that $M \to X$ is equivalent to $N \to X$ if there's an $(n+1)$-manifold $W \to X$ with boundary $M \sqcup N \to X$. We can likewise define MSO oriented cobordism, MU complex cobordism, and so on.

3 Formal group laws

Consider $\mathbb{C}P^\infty$, the classifying space for complex line bundles. Its ordinary cohomology is

$$H\mathbb{Z}^*(\mathbb{C}P^\infty) = \mathbb{Z}[[x]]$$

with x in degree 2. This class x is the universal Chern class, in the sense that a line bundle $V \to X$ is represented by a map $X \to \mathbb{C}P^\infty$, and $c_1(V) \in H^2(X)$ is just the pullback of x along this map. You should think of the $\mathbb{Z}$ as the coefficient ring of $H\mathbb{Z}$. Moreover, there's a multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, representing the tensor product of two line bundles, and this induces a map on cohomology

$$\mathbb{Z}[[x]] \to \mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Z}[[y]].$$

This map is determined by where it sends x, which is just a power series F in two variables. Since the multiplication map of $\mathbb{C}P^\infty$ is a commutative group operation (up to homotopy), this power series has cogroup-like properties:

- $F(x, F(y, z)) = F(F(x, y), z)$,

- $F(x, 0) = F(0, x) = x$ (so the power series is $x + y$ mod terms of degree 2),

- $F(x, y) = F(y, x)$,

- and for some power series i in one variable, $F(x, i(x)) = F(i(x), x) = 0$.

Such a power series is called a **formal group law**. In this case, we know what the power series F is: since $c_1(V \otimes W) = c_1(V) + c_1(W)$, the power series is $F(x, y) = x + y$.

Now you'll notice that the only fact about $H\mathbb{Z}$ that made the above work was that $H\mathbb{Z}^*(\mathbb{C}P^\infty)$ was power series in one variable, generated by the universal Chern class. Many other theories E have similar 'theories of Chern classes'. We say that E is **complex oriented** if this is so, and we get an isomorphism

$$E^*(\mathbb{C}P^\infty) = E^*[[x]],$$

the right-hand E^* now being the coefficient ring of E. Again, we get a formal group law out of E, now over E^* rather than $\mathbb{Z}$.

Another example: complex K-theory is oriented, and has formal group law

$$F(x, y) = x + y + xy = (x + 1)(y + 1) - 1,$$

the multiplicative formal group law.

4 The moduli of formal groups

We can turn the problem on its head, and starting with a formal group law, ask whether there is a complex oriented cohomology theory that carries it. By a theorem of Lazard, the functor that sends R to the set of formal group laws over R is representable, with representing ring

$$L = \mathbb{Z}[x_1, x_2, \dots].$$

To be most honest, though, we should also take isomorphisms of formal group laws into account. There is really a groupoid of formal group laws over a ring R, and so the moduli of formal groups is a stack

$$\mathcal{M}_{\mathrm{fg}} = \mathrm{Spec}(L, W)$$

where

$$W = L[b_1, b_2, \dots].$$

By a theorem of Quillen, and subsequent work of Morava, the universal formal group law over this stack does occur in topology. Namely, the complex cobordism theory MU is complex oriented, has $MU_* \cong L$, and its formal group law is the universal formal group law. The same goes for W and $MU \wedge MU$ (if you don't know what this means, think of it as a tensor product of cohomology theories). To any space X, and in particular to the spheres, we can now associate an algebro-geometric object

$$(MU_*X, (MU \wedge MU)_*X),$$

a quasicoherent sheaf over the moduli of formal groups. And conversely, we can calculate the homotopy of X from the cohomology of this sheaf via a spectral sequence. As you've seen, though, all the rings involved are quite large, so this isn't an effective way to calculate the homotopy of X.

The cohomology theories I'm about to tell you about come about by analyzing the structure of this stack. First localize at a prime p. Write

$$[p]_F(x) = F(\cdots F(F(\overbrace{x, x), x), \dots, x}^{p}) = px + \cdots,$$

the p**-series** of F. Over a field of characteristic p, the first term in this will be of the form ax^{p^n} for some unit a; this n is called the **height** of F. The condition

$$\text{height}\, F \geq n$$

is a closed condition, equivalent to the vanishing of the first $p^n - 1$ coefficients in the p-series. Moreover, any two formal group laws of the same height over an algebraically closed field are isomorphic; not uniquely so, but with a profinite group of automorphisms, called the **Morava stablizer group** $\mathbb{G}_n$. In sum, $\mathcal{M}_{\mathrm{fg},(p)}$ has a descending filtration by the closed 'height $\geq n$' substacks, with a single closed point of 'height exactly n,' whose automorphism group is the Morava stabilizer group.

The program to calculate the p-local stable homotopy groups of a space X – in particular, a sphere – now breaks down as follows.

- Construct cohomology theories that do for the open substacks $\mathcal{M}_{\mathrm{fg},(p)}^{\leq n}$ what MU does for the entire stack. This is possible by the Goerss-Hopkins-Miller theorem; these cohomology theories are the **Johnson-Wilson E-theories** $E(n)$. For a space X, the $E(n)$-cohomology of X (actually, the $E(n)$-localization of X) tells you about the part of X that's seen by formal group laws of height $\leq n$. In fact, we can break things down even further: there are **Morava K-theories** $K(n)$ that tell you about the part of X just seen around the locally closed substack $\mathcal{M}_{\mathrm{fg},(p)}^{=n}$.
- Assemble the homotopy out of X out of the data coming from these E-theories. The Hopkins-Ravenel chromatic convergence theorem says this can be done: consonant with the fact that

  $$\mathcal{M}_{\mathrm{fg},(p)} = \operatorname{colim} \mathcal{M}_{\mathrm{fg},(p)}^{\leq n},$$

  we can write a space X as a homotopy limit of its $E(n)$-localizations,

  $$X = \operatorname{holim} L_{E(n)} X.$$

  (The $E(n)$-localization of X is a space that sees all and only the $E(n)$-data of X, just like the p-completion of a finitely generated abelian group sees all and only its p-power torsion. In the talk, I just called this the '$E(n)$-data' of X, and if you've never seen this before, this is how you should think of it.)
- Assemble the data from the E-theories out of the data of the K-theories. In fact, there are homotopy pullback squares

  $$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{E(n-1)}X \arrow[d] \\ L_{K(n)}X \arrow[r] & L_{E(n-1)}L_{K(n)}X. \end{tikzcd}$$

- Calculate the K-theory data (or just the E-theory data). There's a spectral sequence that does this, starting from the cohomology of the profinite group $\mathbb{G}_n$ acting on something or other. Unfortunately, we're still not done, because this cohomology is still difficult to compute! So the next step is to somehow get a handle on it, typically by replacing formal groups with something more geometric. Let's see how we might do that.

5 Geometric approaches to formal groups

Consider first a formal group of height 0. This means that its p-series takes the form

$$[p]_F(x) = px + \cdots$$

and p is a unit. Since we're supposed to be localized at p, formal groups of height 0 only exist over $\mathbb{Q}$-algebras. Moreover, they are all isomorphic over $\mathbb{Q}$ to the additive formal group, which we recall is the formal group of ordinary cohomology. Thus, the only height 0 cohomology theory we ever need to care about is $H\mathbb{Q}$. The part of $\pi_* S$ detected by height 0 cohomology theories is just its torsion-free quotient; by a theorem of Serre, this is just $\pi_0 S = \mathbb{Z}$.

Next we consider height 1. We've already seen a formal group law of height 1: the multiplicative formal group law, associated to complex K-theory, has p-series

$$[p]_F(x) = x^p$$

over a field of characteristic p. Again, one can prove that this is the only height 1 formal group law, up to isomorphism over $\mathbb{F}_p$. There's a way one can associate an element of $\pi_* S$ to a vector bundle, and the above program at height 1 boils down to asking which elements come from vector bundles in this way. This was done by Frank Adams in the 60s, and uncovered the first infinite family of stable homotopy elements.

In the language of algebraic geometry, the multiplicative formal group law is defined over $\operatorname{Spec}\mathbb{Z}$, so it's classified by a map

$$\widehat{\mathbb{G}_m} : \operatorname{Spec}\mathbb{Z} \to \mathcal{M}_{\mathrm{fg}}.$$

Complex K-theory is a cohomology theory that lives over this map in some sense. But actually, $\widehat{\mathbb{G}_m}$ has a global involution, namely inversion, so we should really think about it as a map

$$\widehat{\mathbb{G}_m} : B\mathbb{Z}/(2) \to \mathcal{M}_{\mathrm{fg}}.$$

We likewise get a cohomology theory over $B\mathbb{Z}/(2)$, which is just the fixed points of K-theory under complex conjugation – that is, it's real K-theory, KO. Complex K-theory is the height 1 E-theory (or K-theory) described above, but KO is a new global object that seems to have sprung up through the stacky picture.

To get at the height 2 points, we again turn the picture on its head. We ask for a stack that covers $\mathcal{M}_{\mathrm{fg}}^{\leq 2}$, and a cohomology theory over this stack, playing the role that KO just played. Then we can recover E-theory and K-theory as a cover of this new cohomology theory, thus using the algebraic geometry to get at the topology.

And fortunately, we do have a good choice for such a stack – namely, the moduli of elliptic curves! Indeed, the completion of an elliptic curve at its group identity is a formal group. If we fix an invariant differential, we get a coordinate for this formal group, turning it into a formal group law. One can show that the height of the formal group is 1 if the curve is ordinary and 2 if it's supersingular. Moreover, the moduli of elliptic curves (with fixed invariant differential) is an étale cover of $\mathcal{M}_{\mathrm{fg}}^{\leq 2}$, which is to say, in a sense, that every formal group law of height below 2 can be obtained as the formal group law attached to an elliptic curve.

For any elliptic curve, we get a complex oriented cohomology theory with the same formal group law, the **elliptic cohomology theory** associated to that curve. These are all local patches of the global cohomology theory living over $\mathcal{M}_{\mathrm{ell}}$, a cohomology theory we call 'topological modular forms,' or TMF. The reason for the name is that the coefficient ring TMF_* is calculated from the cohomology $H^*(\mathcal{M}_{\mathrm{ell}}, \omega^{\otimes *})$, so that modular forms, the H^0 of this ring, partially show up in TMF_*. In fact, TMF_* is a weird Frankenstein monster made up partially of the modular forms ring and partially of the homotopy groups of spheres. But it's calculable because we fully understand the geometry of $\mathcal{M}_{\mathrm{ell}}$, and in turn, it allows us to describe height 2 phenomena more or less completely.

6 Large heights

And now the question is: where next? How to continue the pattern?

$n = 0$	$n = 1$	$n = 2$	$n > 2$
$(H\mathbb{Q})$	KO	TMF	???
$(\operatorname{Spec}\mathbb{Q})$	$B\mathbb{Z}/(2)$	$\mathcal{M}_{\mathrm{ell}}$	???

In each case, the object in the bottom row is

- a stack over $\mathcal{M}_{\mathrm{fg}}$,
- that covers $\mathcal{M}_{\mathrm{fg}}^{\leq n}$,
- whose cohomology can be calculated without too much trouble,
- subject to some algebro-geometric properties (for instance, the map to $\mathcal{M}_{\mathrm{fg}}$ should be flat), allowing you to construct a cohomology theory over that stack.

Another way of phrasing this question is: how do you continue a pattern that starts '(additive group), multiplicative group, elliptic curves, ...'?

Currently, we don't understand how to keep going, but we have a few ideas.

- The current best-worked-out idea is called 'topological automorphic forms' or TAF. An abelian variety of dimension g has a formal group of height up to $2g$, but also of dimension up to g, whereas for homotopy theory we only want 1-dimensional formal groups. The insight of Behrens and Lawson was that, by considering only abelian varieties with enough symmetry, we can cut out a 1-dimensional summand of the formal group. The stack that goes in the bottom row is then a 'PEL Shimura variety of type $U(1, n-1)$,' which parametrizes polarized abelian varieties of dimension n^2 with complex multiplication by an order in a certain central simple algebra. In this case, we actually have the cohomology theory TAF, but as yet it's proved immune to calculation – for instance, even at $n = 3$, we would have to say something about automorphic forms for 9-dimensional abelian varieties of this type.

- If we think of the formal group of an elliptic curve as coming from its H^1, we're led to ask about formal versions of other cohomology groups, which were constructed by Artin and Mazur. The next thing to look at under this point of view would be K3 surfaces, which have a more tractable moduli. This approach has so far foundered at the point of constructing the cohomology theory, since the map $\mathcal{M}_{K3} \to \mathcal{M}_{\text{fg}}$ is flat but not étale. The situation could be resolved by us proving a better cohomology-theory-construction theorem, or by someone finding a good slice of $\mathcal{M}_{K3}$ which does give an étale map.
- Finally, one could look at the formal groups of Jacobians of curves, with elliptic curves being the genus 1 case. Provided one fixes enough symmetries on the curve, as with TAF, one should get a map from the moduli of genus g curves with a marked point to the moduli of formal groups. Work on this approach is still at a minimal level, and I, for one, don't know what's known about formal groups of Jacobians.

To conclude, what I find exciting about this line of study is that it indicates a connection between the geometric mainstays of topology – ordinary cohomology and K-theory – and objects occurring in algebraic geometry – like elliptic curves, and hopefully other things as well. This is an exciting time for homotopy theory, and perhaps there will be a rapprochement between our two fields.

The future beckons!