Lecture 2: Spectra and localization

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1 Spectra

(Throughout, 'spaces' means pointed topological spaces or simplicial sets – we'll be clear where we need one version or the other.)

The basic objects of stable homotopy theory are **spectra**. Intuitively, a spectrum is the following data:

- a sequence of spaces X_n for $n \in \mathbb{N}$;
- for each n, a map $\Sigma X_n \to X_{n+1}$.

A map of spectra $X \to Y$ is an equivalence class of choices of maps $X_n \to Y_n$ that make the obvious squares commute. Two of these are said to be equivalent if they agree 'cofinally,' meaning roughly that we may ignore what happens for a finite number of values of n.

The classic example is the **suspension spectrum** of a space X, which is given by $(\Sigma^\infty X)_n = X_n$, with the structure maps the identity. With a suitable notion of homotopy theory of spectra, the stable homotopy groups of X as the homotopy groups of its suspension spectrum, and we can likewise use spectra to study phenomena in spaces that only occur after 'enough suspensions.' The homotopy category of spectra, called the **stable homotopy category** is the place where such phenomena live.

Complaint 1.1. Unfortunately, while the stable homotopy category is quite nice to deal with, actual categories of spectra are more ill-behaved, particularly when we introduce smash products. The 'definition' just given is certainly the obvious one, but leaves us with a smash product that is only commutative and associative up to homotopy, a statement (arduously) proved in [1]. Several other categories of spectra exist which are actually monoidal model categories, but at the cost of making the definitions of the objects or homotopies much more complicated. Schwede has a good reference on symmetric spectra [9], which have an action of the symmetric group Σ_n on each X_n, and are the initial object in some category of model categories of spectra.

In fact, Lewis showed [7] that there is no category of spectra satisfying five simple axioms on the smash product and the relationship wtih Spaces_*. For the interested, the axioms are:

1. The smash product is symmetric monoidal.
2. There is an adjunction $\Sigma^\infty : \mathsf{Spaces}_* \rightleftarrows \mathsf{Spec} : \Omega^\infty$.
3. The sphere spectrum, i. e. $\Sigma^\infty S^0$, is the unit for the smash product.
4. Σ^∞ is colax monoidal or Ω^∞ is lax monoidal.
5. There is a natural weak equivalence from $\Omega^\infty \Sigma^\infty X$ to the usual infinite loop space $\operatorname{colim}_n \Omega^n \Sigma^n X$.

The compromise I'll make is to define the category of CW-spectra, invented by Boardman [2] and on which the definitive source is [1]; I'll move quickly to the stable homotopy category, where we'll spend most of our time anyway. The advantages of this are that it's closest to the intuitive 'definition' given above; all the constructions except for the smash product are relatively simple; and we can make the sorts of cellular arguments that we'll need to discuss Bousfield localization. The disadvantages are that I won't really construct the smash product or the model structure, and neither of these is terribly well-behaved.

Keep in mind that you can always choose a category in which they are well-behaved! Theoretically, Lewis-May-Steinberger's category of **S-modules** is probably the best choice for the discussion that follows [8]; if you want to see another construction, [9] is the a great introduction to symmetric spectra for even those with no familiarity with spectra.

Definition 1.2. A **CW-spectrum** X is a sequence $\{X_n\}$ of pointed CW-complexes indexed by $n \in \mathbb{Z}$, with cellular structure maps $\phi_n : \Sigma X_n \to X_{n+1}$. A **subspectrum** $Y \subseteq X$ is a choice of subcomplexes $Y_n \subseteq X_n$ such that $\phi_n(\Sigma Y_n) \subseteq Y_{n+1}$. A subspectrum Y is **closed** if whenever a cell e^m_α of X_n has $\phi_n(\Sigma e^\alpha_m) \subseteq Y_{n+1}$, then $e^m_\alpha \subseteq Y_n$; it is **cofinal** if for all cells $e^\alpha_m \subseteq X_n$, there is a k such that

$$\phi_{n+k-1} \circ (\Sigma\phi_{n+k-2}) \circ \cdots \circ (\Sigma^{k-1}\phi_n)(e^m_\alpha) \subseteq Y_{n+k}.$$

That is, every cell ends up in Y after enough suspensions.

Example 1.3. We already discussed the suspension spectrum of a space. The **sphere spectrum** S is the suspension spectrum of S^0, $(S)_n = S^n$.

Given an abelian group A, the **Eilenberg-Mac Lane spectrum** on A is the spectrum HA with $HA_n = K(A, n)$. The structure maps $\Sigma K(A, n) \to K(A, n+1)$ are adjoint to $K(A, n) \xrightarrow{\sim} \Omega K(A, n+1)$.

Definition 1.4. A **map** $f : X \to Y$ is a choice of cofinal subspectrum $W \subseteq X$ and maps $f_n : W_n \to Y_n$ that commute with the structure maps of X and Y, modulo the equivalence relation that two maps are equivalent if they agree on a cofinal subspectrum of X on which they are both defined.

Note that two maps always have a common cofinal subspectrum of definition, since the intersection of two cofinal subspectra is cofinal. Also, all this works for simplicial sets instead of CW-complexes, though to define homotopy as below, we'll want the simplicial sets involved to be Kan complexes.

Example 1.5. Big Paul gave the example of the **Kan-Priddy map**. For each n, there's a map $\mathbb{R}P^{n-1}_+ \to O(n)$ given by sending a line to reflection in the plane perpendicular to that line. There's also a map $O(n) \to \Omega^n S^n$: an orthogonal transformation gives an automorphism of the sphere. Composing these maps and using an adjunction gives $\Sigma^n \mathbb{R}P^{n-1}_+ \to S^n$, and taking the colimit gives a map of spectra $\Sigma^\infty \mathbb{R}P^\infty_+ \to S$. This map is surjective on homotopy, but doesn't restrict to any map of spaces $\Sigma^n \mathbb{R}P^\infty_+ \to S^n$. Thus we have a map of CW-spectra that can't be fully defined at any stage of the source.

There are two primary reasons to be interested in spectra. The first is that spectra are intimately related to cohomology, as we'll discuss in a bit. The second is that spectra are the natural place to do stable homotopy theory. To this end, we define the stable homotopy category.

Definition 1.6. Let I be the interval $[0, 1]$. The **cylinder** on a spectrum X is the spectrum $(X \wedge I_+)$ with $(X \wedge I_+)_n = X_n \wedge I_+$, and the structure maps given by applying the structure maps of X. If $f, g : X \to Y$ are maps of spectra, a **homotopy** $f \sim g$ is a map $H : X \wedge I_+ \to Y$ with $H|_{X\wedge\{0\}_+} = f$, $H|_{X\wedge\{1\}_+} = g$. (Recall that this equality means the maps agree on a cofinal subspectrum.)

Definition 1.7. The **stable homotopy category** is the category of CW-spectra and homotopy classes of maps. We write $[X, Y]$ for the set of homotopy classes of maps from X to Y. We can make this a graded category by defining $[X, Y]_n = [\Sigma^n X, Y]$. In particular, $\pi_* X = [S, X]_*$, where S is the sphere spectrum.

Now, if X is a space, we have an isomorphism

$$[S, \Sigma^\infty X]_n \cong \pi^S_n X := \mathrm{colim}_{k\to\infty}\, \pi_{n+k}\Sigma^k X.$$

Thus the stable homotopy groups of a space are encoded in the stable homotopy category; likewise, other stable phenomena are expected to live here as well.

In theoretical terms, model categories of spectra are **stable model categories**, in that their homotopy categories are *triangulated*. A s a result, (graded) hom-sets in the stable homotopy category are (graded) abelian groups, and homotopy cofiber sequences and homotopy fiber sequences are the same.

2 Constructions with spectra

Loop space and suspension

Given a spectrum X, its **suspension** is the spectrum given by $(\Sigma X)_n = X_{n+1}$, and its **loop space** is the spectrum given by $(\Omega X)_n = X_{n-1}$. Clearly, these are inverse equivalences on the stable homotopy category, and shift the homotopy groups of a spectrum down or up. We could also construct these by applying the suspension and loop space constructions levelwise to X; the structure maps $\Sigma X_n \to X_{n+1}$ and their adjoints $X_n \to \Omega X_{n+1}$ define homotopy equivalences between the two constructions.

In particular, every spectrum is a double suspension, and the Eckmann-Hilton argument shows that $[X, Y]_* = [X, \Sigma^2\Omega^2 Y]_*$ is always a graded abelian group. Thus, the stable homotopy category is additive!

Cofibers and fibers

In fact, the stable homotopy category is triangulated, meaning not only that suspension is an equivalence, but also that every cofiber sequence is also a fiber sequence. We can construct the cofiber of $f : X \to Y$ by $(Y/X)_n = Y_n \cup CW_n$ for W a cofinal subspectrum of X on which f is defined; one can check that this only depends on f up to homotopy, and that it is well-defined up to homotopy equivalence. We then have a sequence in the stable homotopy category

$$\cdots \to \Omega(Y/X) \to X \to Y \to Y/X \to \Sigma X \to \cdots$$

in which every consecutive triple of terms is both a cofiber sequence and a fiber sequence. In particular, for any spectrum Z we have long exact sequences

$$\cdots \to [X, Z]_{n-1} \to [Y/X, Z]_n \to [Y, Z]_n \to [X, Z]_n \to [Y/X, Z]_{n+1} \to \cdots$$

and

$$\cdots \to [Z, Y/X]_{n+1} \to [Z, X]_n \to [Z, Y]_n \to [Z, Y/X]_n \to [Z, X]_{n-1} \to \cdots .$$

Sums

We can sum or wedge spectra by doing so objectwise: $\left(\bigvee_\alpha X^\alpha\right)_n = \bigvee_\alpha X^\alpha_n$.

Smash product

The smash product of pointed spaces is defined by

$$X \wedge Y = (X \times Y)/(X \times \{*\} \cup \{*\} \times Y);$$

this defines a closed symmetric monoidal structure on the category of pointed spaces (where we'll have to jump through the usual compactness hoops to get 'closed'). In spectra, we'd like to do the same thing. The obvious way to go about it is to define

$$(X \wedge Y)_n = X_n \wedge Y_n,$$

which is clearly a CW-complex if X_n and Y_n are. Unfortunately, *this does not work!* For we'd need a map

$$X_n \wedge Y_n \wedge S^1 \to X_{n+1} \wedge Y_{n+1},$$

but with only the one suspension coordinate, we can only go to $X_{n+1} \wedge Y_n$ or $X_n \wedge Y_{n+1}$.

One way to fix this problem is to have $(X \wedge Y)_{2n} = X_n \wedge Y_n$, $(X \wedge Y)_{2n+1} = X_n \wedge Y_{n+1}$, and alternately increase the level of the X and Y factors. In fact, there are an infinite number of ways to do this, each given by a choice of two disjoint cofinal subsets of $\mathbb{N}$ which tell you when to increase the respective levels of X and Y. After about thirty pages of tediousness [1], one can show that all of these various smash products are equivalent in the stable homotopy category, with these equivalences respecting various associativity, unit, and commutativity isomorphisms.

It is highly recommended that you ignore the details of this construction and black-box the fact that there exists a symmetric monoidal smash product on the stable homotopy category with S as the unit. Needless to say, this does *not* lift to a symmetric monoidal structure on the category of CW-spectra, though as mentioned above, if need be we can work in a model category of spectra on which the smash product is on-the-nose monoidal. One example of a smash product that can and should be explicit is when we one of the factors is the suspension spectrum of a space. In this case we have

$$(X \wedge \Sigma^{\infty} K)_n = X_n \wedge K,$$

where the structure maps use the spectrum structure of X in the obvious way. We often identify spaces with their suspension spectra and thus write this $X \wedge K$, giving an action of the monoidal category of spaces on the stable homotopy category.

Spanier-Whitehead duality

A useful corollary of Brown representability is the following observation. If X is a spectrum, then $[X, Y]_*$ is a covariant functor of Y that clearly satisfies the axioms of a homology theory. Thus there is a spectrum $X\check{}$ and a natural isomorphism

$$[X, Y]_* \cong [S, X\check{} \wedge Y]_*.$$

In particular, $[X, S]_* \cong [S, X\check{}]_*$. It's easy to check that this dual construction commutes with smash products. When X has the homotopy type of a finite CW-spectrum, so does $X\check{}$, and $X\check{}\check{} \simeq X$; in general, this isn't true for non-finite X. Note that $X\check{} \wedge \cdot$ is the internal hom functor right adjoint to $\cdot \wedge X$.

Brown representability

This is the true clincher about spectra. For E a spectrum and X a space, we define

$$E_* X = [S, E \wedge X]_*$$

and

$$E^* X = [X, E]_{-*}.$$

For a pair (A, X), we can just define $E_*(A, X) = E_*(X \cup CA)$.

Proposition 2.1. *The functors E_* and E^{-*} are homology theories on the category of spaces.*

Proof. Long exact sequences follow from the fact that the stable homotopy category is triangulated, as do wedges, once you notice that $X \to X \wedge Y \to Y$ is a split cofiber sequence. Homotopy-invariance is by construction, and excision follows from homotopy-invariance. We also have suspension isomorphisms. □

Surprisingly, every cohomology theory is of this form!

Theorem 2.2 (Brown representability, [3])**.** *Every cohomology theory on the category of spaces is naturally isomorphic to one of the form E^* for some spectrum E.*

The actual statement of Brown representability is more general: it gives conditions for a functor from spaces to sets to be representable by a *space*.

In any case, this theorem allows us to represent the algebraic data of a homology theory topologically, as a homotopy type of spectra. For example, the Eilenberg-Maclane spectrum HA represents cohomology with coefficients in A, S_* is stable homotopy, and there are likewise spectra called $K, MU, E(n), \ldots$. By means of the Atiyah-Hirzebruch spectral sequence, knowing the homotopy groups of these spectra can get you a long way in calculating the cohomology they represent!

Products

The product of cohomology theories is a cohomology theory, so purely formally, products of spectra exist in the stable homotopy category.

Ring and module spectra

A **ring spectrum** is a monoid object in the stable homotopy category. This is a spectrum E with a multiplication map $\mu : E \wedge E \to E$ and a unit map $\eta : S \to E$ such that the associativity diagram

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"'] & E \end{tikzcd}$$

and the unit diagrams

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[rrr, bend right, "1"'] & S \wedge E \arrow[r, "\eta\wedge 1"] & E \wedge E \arrow[r, "\mu"] & E, \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[rrr, bend right, "1"'] & E \wedge S \arrow[r, "1\wedge\eta"] & E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

commute. A ring spectrum E is said to be **commutative** if the commutativity diagram

$$\begin{tikzcd} E \wedge E \arrow[rr, "\text{twist}"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

commutes.

A (left) **module spectrum** over a ring spectrum E is a spectrum F with an action map $\nu : E \wedge F \to F$ such that the associativity diagram

$$\begin{tikzcd} E \wedge E \wedge F \arrow[r, "1\wedge\nu"] \arrow[d, "\mu\wedge 1"'] & E \wedge F \arrow[d, "\mu"] \\ E \wedge F \arrow[r, "\mu"'] & F \end{tikzcd}$$

and the unit diagram

$$\begin{tikzcd} F \arrow[r, "\simeq"] \arrow[rrr, bend right, "1"'] & S \wedge F \arrow[r, "\eta\wedge 1"] & E \wedge F \arrow[r, "\nu"] & F \end{tikzcd}$$

commute.

Remark 2.3. In this and other model categories of spectra, ring and module spectra will often be modelled by objects and structure maps in the actual model category, such that the above diagrams only commute up to homotopy. One thing to keep in mind is that there aren't many 'strict ring spectra,' meaning monoid objects in the original model category. Indeed, the only strict monoid objects in Spaces are products of Eilenberg-Mac Lane spaces of rings. This is the key point in Lewis's argument that there are no nice categories of spectra; the five axioms given end up creating too many strict ring spectra.

Example 2.4. If R is a ring, then the Eilenberg-Mac Lane spectrum HR is a ring spectrum. Recalling that maps $X \to HR$ correspond to elements of $H^0(X; R)$, the unit map is $S \to HR$ corresponding to $1 \in H^0(S^0; R) \cong R$, and the multiplication map is $HR \wedge HR \to HR$ corresponding to the multiplication map in

$$\mathrm{Hom}(R \otimes R, R) \cong H^0(HR \wedge HR; R).$$

Likewise, if M is an R-module, HM is an HR-module spectrum.

The sphere spectrum is also a ring spectrum, with the unit map being $1 : S \to S$ and the multiplication being $1 : S \wedge S \simeq S \to S$. Every spectrum is a module over S.

The primary use of ring and module spectra is to define multiplication on cohomology. Generally speaking, given *any* two spectra E and F, there is a natural external pairing

$$E^p(X) \otimes F^q(X) \cong [X, E]_{-p} \otimes [X, F]_{-q} \to [X, E \wedge F]_{-p-q} \cong (E \wedge F)^{p+q}(X).$$

If $E = F$ is a ring spectrum, post-composing with $E \wedge E \to E$ defines a cup product $E^*(X) \otimes E^*(X) \to E^*(X)$ making $E^*(X)$ a graded ring. If E is a ring spectrum and F an E-module spectrum, post-composing with $E \wedge F \to F$ defines an action of $E^*(X)$ on $F^*(X)$, making $F^*(X)$ a graded $E^*(X)$-module. Similar results exist in homology. For more general results, including a pairing between homology and cohomology and 'slant products' allowing you to 'divide' homology classes by cohomology classes and vice versa, see [1].

A universal coefficient theorem

We conclude with a theorem that will be useful in our discussion of Bousfield localization below.

Theorem 2.5 (Universal coefficient theorem)**.** *Let E be a spectrum, G an abelian group, and let $EG = E \wedge SG$. There exist natural exact sequences*

$$0 \to E_n(X) \otimes G \to (EG)_n(X) \to \mathrm{Tor}(E_{n-1}(X), G) \to 0$$

and

$$0 \to E^n(X) \otimes G \to (EG)^n(X) \to \mathrm{Tor}(E^{n+1}(X), G) \to 0.$$

Proof. Recall that SG is defined by a cofiber sequence

$$\bigvee_\alpha S \to \bigvee_\beta S \to SG$$

corresponding to a free resolution $0 \to R \to F \to G \to 0$ of G with a choice of generators $\{\alpha\}$ and $\{\beta\}$ for R and F. Smashing this with E and X gives a cofiber sequence

$$\bigvee_\alpha E \wedge X \to \bigvee_\beta E \wedge X \to EG \wedge X,$$

and taking graded maps from S gives a long exact sequence

$$\cdots \to \bigoplus_\alpha E_n(X) \to \bigoplus_\beta E_n(X) \to (EG)_n(X) \to \bigoplus_\alpha E_{n-1}(X) \to \cdots.$$

The cokernel of the first map is $E_n(X) \otimes G$, and its kernel is $\mathrm{Tor}(E_n(X), G)$, so the long exact sequence splits into the described short exact sequences. The theorem for cohomology is proved similarly. $\square$

Remark 2.6. Unlike the case of spaces, these sequences do not in general split!

3 Bousfield localization

In the Adams and Adams-Novikov spectral sequences, we have homological data coming from the groups E_*X and E_*Y for some homology theory E_*, and we'd like to compute something like the homotopy classes of maps $[X, Y]_*$. However, it's obvious that we won't be able to compute anything in $[X, Y]_*$ that E can't see; for instance, if $E = H\mathbb{Z}_{(2)}$, we shouldn't expect to discover any of the odd torsion of $[X, Y]_*$. So what we end up doing is replacing Y with an object, called a localization, whose homotopy theory is entirely described by its E_*-homology.

Another way of thinking about this, which is described in more detail below, is that we define a new model category of spectra in which the cofibrations are the same and the weak equivalences are the E_*-equivalences. A localization of an object is then just a fibrant replacement.

Remark 3.1. It's interesting to me that Bousfield localization unites the two common meanings of localization. On the one hand, if $E = HA$ where A is some localization of the ring $\mathbb{Z}$ (for example, $H\mathbb{Z}_{(p)}$ or $H\mathbb{Q}$), then E_*-localizing a spectrum literally localizes its homology and homotopy by tensoring them with A. There's a standard topological way to do this, called 'localization' or 'rationalization,' that you may have seen already, and the construction for a more general E is along the same lines. On the other hand, we're also adding weak equivalences to the model category of spectra, and thus localizing its homotopy category in the sense of inverting maps. [As was pointed out to me, localization of categories is a categorification of localization of rings/monoids.]

Definition 3.2. Let E_* be a homology theory. A spectrum X is called **E_*-acyclic** if $E_*X = 0$. A spectrum X is called **E_*-local** if $[A, X]_* = 0$ for every E_*-acyclic A.

Since an E_*-equivalence is precisely a map with an E_*-acyclic homotopy fiber, X is E_*-local iff every E_*-equivalence $A \to B$ induces an isomorphism $[B, X]_* \cong [A, X]_*$.

Definition 3.3. An **E_*-localization** of a spectrum X is an E_*-equivalence $X \to L_E X$ such that $L_E X$ is E_*-local. An **E_*-localization functor** is a functorial choice of E_*-localizations; thus, we want a functor $L_E : \mathsf{Spec} \to \mathsf{Spec}$ and a natural transformation $1 \Rightarrow L_E$ satisfying the above conditions.

Let's first establish some facts about E_*-local spectra.

Proposition 3.4 (E_*-Whitehead theorem)**.** *If $f : X \to Y$ is an E_*-equivalence of E_*-local spectra, then f is a weak equivalence (and a homotopy equivalence if X is a cell complex).*

Proof. Since both spaces are E_*-local, f induces isomorphisms $[Y, Y]_* \cong [X, Y]_*$ and $[Y, X]_* \cong [X, X]_*$. These lift to homotopy equivalences of mapping spaces, proving that f is a weak equivalence. □

Proposition 3.5. *If E is a ring spectrum and X is a module spectrum over E, then X is E_*-local.*

Proof. Let A be E_*-acyclic and $f : A \to X$ be a map. We can factor f as

$$A \xrightarrow{i \wedge 1} E \wedge A \xrightarrow{1 \wedge f} E \wedge X \xrightarrow{\mu} X$$

where i is the unit map of E and μ the module structure map of X. Since $E_*A = 0$, $E \wedge A$ is contractible, so f is nullhomotopic. Thus $[A, X]_* = 0$. □

Proposition 3.6. *E_*-local spectra are closed under shifts, products, retracts, and cofibers*

Proof. Products and retracts are obvious; cofibers follow from the five lemma. □

Proposition 3.7. *If L_E is any localization functor, then L_E preserves shifts, wedges, and homotopy cofibers.*

We now prove that localization functors exist. Adams attempted to do this by directly localizing the homotopy category, but this procedure is set-theoretically unsound: in general, a localization of a locally small category need not be locally small. To deal with this, we need to be clever with the cardinalities of our spectra, a trick called the 'Bousfield-Smith cardinality argument.' The below is all in [4].

Recall that a subspectrum B of a CW-spectrum X is **closed** if B is a union of cells and any cell of X with some suspension in B is in B; this guarantees that X/B is a CW-spectrum.

Lemma 3.8. *Let X be a CW-spectrum and B a proper closed subspectrum with $E_*(X, B) = 0$, and let κ be an infinite cardinal greater than or equal to $|\pi_* E|$. Then there is a closed subspectrum $W \subseteq X$ with at most κ cells such that W is not contained in B and $E_*(W, W \cap B) = 0$.*

Proof. Let W_1 be any closed subspectrum of X not contained in B and with at most κ cells. Inductively, given W_n, for each class $\alpha \in E_*(W_n, W_n \cap B)$, choose a finite closed subspectrum F_α of X such that α goes to zero in $E_*(W_n \cup F_\alpha, (W_n \cup F_\alpha) \cap B)$, and let W_{n+1} be the union of all W_n with all F_α. If W_n has at most κ cells, then $E_*(W_n)$ has at most κ elements since $|\pi_* E| \le \kappa$; thus by induction, all W_n have at most κ cells. Letting $W = \operatorname{colim} W_n$, it is clear that $E_*(W, W \cap B) = 0$, that W is not contained in B, and that W has at most κ cells. □

Lemma 3.9. *For any E, there exists an E_*-acyclic spectrum A such that a spectrum Y is E_*-local if and only if $[A, Y]_* = 0$.*

Proof. Choose κ as above, and let $\{K_\alpha\}$ be a set of CW representatives for the weak equivalence classes of E_*-acyclic spectra with at most κ cells. Let $A = \bigvee_\alpha K_\alpha$. Clearly if Y is E_*-local, then $[A, Y]_* = 0$. Conversely, if $[A, Y]_* = 0$, then $[A', Y]_* = 0$ for any spectrum A' that can be obtained from A by taking weak equivalences, shifts, wedges, summands, and cofibers. Let $C(A)$ denote this class of spectra; it suffices to show that every E_*-acyclic spectrum is in $C(A)$.

Let X be an E_*-acyclic spectrum; up to weak equivalence, we can take X to be a CW-spectrum. By transfinite induction and the previous lemma, we can construct a sequence

$$0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\gamma = X$$

such that

- each B_λ is an E_*-acyclic closed subspectrum;
- each $B_{\lambda+1}$ is obtained from B_λ by adding a closed subspectrum W_λ as in the previous lemma;
- for λ a limit ordinal, $B_\lambda = \bigcup_{\sigma<\lambda} B_\sigma$.

Now, if $B_\lambda \in C(A)$, there is a cofiber sequence

$$B_\lambda \to B_{\lambda+1} \to K_\alpha,$$

where K_α is weakly equivalent to the E_*-acyclic spectrum $W_\lambda/(W_\lambda \cap B_\lambda)$, and thus a cofiber sequence

$$\Omega K_\alpha \to B_\lambda \to B_{\lambda+1};$$

thus $B_{\lambda+1}$ is also in $C(A)$. Likewise, if λ is a limit ordinal, it is the cofiber of

$$\bigvee_{\sigma<\lambda} B_\sigma \stackrel{1-i}{\to} \bigvee_{\sigma<\lambda} \to B_\lambda,$$

where i is the wedge of $B_\sigma \hookrightarrow B_{\sigma+1}$. By transfinite induction, all B_λ, and in particular X, are in $C(A)$. □

Theorem 3.10. *For any E, there exists a localization functor $X \mapsto [X \to L_E X]$.*

Proof. By the above lemma, all we need is a natural map $X \to L_E X$ such that $[A, L_E X]_* = 0$. As in the small object argument, we can do this by successively coning off all maps from A and using transfinite induction. By construction, A is a wedge of spectra with less than κ cells, each of which should be κ-small, so A is κ-small and the small object argument goes through. This also shows that $A \to L_E X$ is functorial. (For a reference on the small object argument, see e.g. [6][§2.1]). □

Remark 3.11. The interested should know that this process is very general. Given a left proper model category with a set I of generating cofibrations, the **relative I-cell complexes** are the maps that are transfinite compositions of pushouts of coproducts of elements of I – recall that the cofibrations in the model category are precisely the retracts of these. We can run the above argument, with the relative I-cell complexes replacing the relative CW-spectra and any class of maps replacing the E_*-equivalences, so long as we assume:

- smallness conditions on the objects appearing in I, so that the small object argument used above works;
- a somewhat irritating condition called 'compactness' that lets us factor certain maps into relative I-cell complexes through subcomplexes with a bounded cardinality of cells;
- that the maps in I are 'effective monomorphisms,' which means that we can specify a subcomplex of an I-cell complex purely by its cells.

These conditions define a **cellular** model category. An encyclopedic reference on this approach is [5].

In fact, any left proper *combinatorial* model category admits localizations. Since these appear more often and are desirable for other attacks, this is probably the approach you want to use. I don't know whether every cellular model category is combinatorial – if someone has a proof or counterexample, I'd love to see it.

4 Examples of localizations

First and foremost, let's fix the hole discovered at the beginning of the previous section. As we'll surely discuss later, the Adams spectral sequence is in fact a spectral sequence

$$\mathrm{Ext}_{E_*E}(E_*X, E_*Y) \Rightarrow [X, L_E Y]_*.$$

Second, we discuss some specific examples of localizations, specifically localizing with respect to Moore spectra, and localizing connective spectra.

Definition 4.1. Two abelian groups G_1 and G_2 have the same **type of acyclicity** if each prime p is a unit in G_1 iff it is in G_2, and if G_1 is torsion if G_2 is torsion.

In particular, every group has the same type of acyclicity as a localization of $\mathbb{Z}$ (i.e. a subring of $\mathbb{Q}$) or a direct sum of *distinct* rings of the form $\mathbb{Z}/p$. The next proposition shows that when studying localization with respect to Moore spectra, we only need consider these two cases.

Proposition 4.2. *G_1 and G_2 have the same type of acyclicity iff SG_1 and SG_2 give weakly equivalent localization functors.*

Proof. By the universal coefficient theorem discussed above, $(SG)_*(X)$ is an extension of $\mathrm{Tor}(\pi_{n-1}(X), G)$ by $\pi_n(X) \otimes G$. Thus X is $(SG)_*$-acyclic iff $\pi_*(X) \otimes G$ and $\mathrm{Tor}(\pi_*(X), G)$ are both zero. This only depends on the type of acyclicity of G. Clearly the localization functors are equivalent iff the theories have the same acyclic objects. □

Proposition 4.3. *Let G be a localization of $\mathbb{Z}$ and let X be a spectra. Then $L_{SG}(X) \simeq SG \wedge X$, with $\pi_* L_{SG}(X) = G \otimes \pi_* X$.*

Proof. $SG \wedge X$ is a module spectrum over SG, and thus local. By homology with coefficients the map $X \simeq S \wedge X \to SG \wedge X$ is an SG_*-localization. □

In particular, the SG_*-local spectra for such G are precisely those X for which p is a unit in $\pi_*(X)$, for each p that is a unit in G.

Proposition 4.4. *Let $G = \bigoplus_{p \in P} \mathbb{Z}/p$. Then*

$$L_{SG}(X) \simeq \prod_{p \in P} X \wedge (\Omega S\mathbb{Z}/p^\infty)\check{}$$

and if $\pi_* X$ is degreewise finitely generated, then

$$\pi_* L_{SG}(X) = \prod_{p \in P} \mathbb{Z}_p \otimes \pi_* X.$$

In general there's a split short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \to \pi_* L_{SG}(X) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1} X) \to 0.$$

Proof. It suffices to consider one prime p.

$\Omega S\mathbb{Z}/p^\infty$ is the fiber of $S \to S\mathbb{Z}[p^{-1}]$, since $\mathbb{Z}/p^\infty$ is the cokernel of $\mathbb{Z} \to \mathbb{Z}[p^{-1}]$. Thus we have a fiber sequence

$$X \wedge (S\mathbb{Z}[p^{-1}])\check{} \to X^S \to X \wedge (\Omega S\mathbb{Z}/p^\infty)\check{} \to \Sigma X \wedge (S\mathbb{Z}[p^{-1}])\check{}.$$

Now, $\mathbb{Z}/p^\infty$ has the same type of acyclicity as $\mathbb{Z}/p$, so that $X \wedge (\Omega S\mathbb{Z}/p^\infty)\check{}$ is $S\mathbb{Z}/p_*$-local. Meanwhile, $X \wedge (S\mathbb{Z}[p^{-1}])\check{}$ has homotopy groups

$$[S, X \wedge (S\mathbb{Z}[p^{-1}])\check{}]_* \cong [S\mathbb{Z}[p^{-1}], X]_* \cong \pi_* X \otimes \mathbb{Z}[p^{-1}].$$

In particular, p is a unit in these homotopy groups, so $X \wedge (S\mathbb{Z}[p^{-1}])\check{}$ is $S\mathbb{Z}/p_*$-acyclic. Thus $X \to X \wedge (\Omega S\mathbb{Z}/p^\infty)\check{}$ is a $S\mathbb{Z}/p_*$-localization of X, and the conclusion follows. □

One Moore spectrum that pops up more often than you'd think is $S\mathbb{Q}$. By elementary rational homotopy theory, this is homotopy equivalent to $H\mathbb{Q}$!

In addition, since $EG = E \wedge SG$, the above methods let you EG_*-localize as soon as you've managed to E_*-localize. In particular, this only depends on E and the type of acyclicity of G.

One nice thing that comes up is the following.

Proposition 4.5. *Let E, F, X be spectra with $L_F X$ E_*-acyclic. Then the square*

$$\begin{array}{ccc} L_{E\vee F}X & \xrightarrow{f} & L_E X \\ \downarrow g & & \downarrow \\ L_F X & \longrightarrow & L_F L_E X \end{array}$$

is a homotopy pullback square.

Proof. Let P be the homotopy pullback of the square and construct the obvious map $L_{E\vee F}X \to P$. Working in a proper model category of spectra, we get that the map $P \to L_F X$ is an F_*-equivalence, and $P \to L_E X$ is an E_*-equivalence. Also, f and g in the above square are respectively E_*- and F_*-equivalences, because $X \to L_{E\wedge F}X$ is an E_*- and F_*-equivalence and composing this map with f and g gives respectively an E_*- and F_*-localization of X. Thus the maps $P \to L_E X$ and $P \to L_F X$ factor through $L_{E\vee F}X$, giving an isomorphism between P and $L_{E\vee F}X$. □

We'll see one application involving various $K(n)$'s later on. For another, let $F = S\mathbb{Q}$ and $E = \bigvee_p S\mathbb{Z}/p$. Then $E \wedge F$ detects ordinary homology, so $L_{E\vee F}X = X$; also, $L_E X = \prod_p L_{S\mathbb{Z}/p}X$. We get the **Sullivan arithmetic square**

$$\begin{array}{ccc} X & \longrightarrow & \prod_p L_{S\mathbb{Z}/p}X \\ \downarrow & & \downarrow \\ L_{S\mathbb{Q}}X & \longrightarrow & L_{\mathbb{Q}}\left(\prod_p L_{S\mathbb{Z}/p}X\right). \end{array}$$

We end with a note of hope: when dealing with connective X and E, localization is extremely easy!

Theorem 4.6 (Bousfield)**.** *Let E and X be connective, and let $G = \pi_0 E$ (or even a group with the same type of acyclicity, as above). Then $L_E X \simeq L_{SG}X$.*

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