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Localizations of E-Theory and Transchromatic Phenomena in Stable Homotopy Theory

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ABSTRACT

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Chromatic homotopy theory studies a parametrization of stable homotopy theory in terms of algebraic objects called formal groups. Transchromatic homotopy theory is specifically concerned with the behavior of spaces and cohomology theories as these formal groups change in height. We pursue a central transchromatic object, the $K(n-1)$-localization of a height n Morava E-theory E_n. We give a modular description of the coefficients of $L_{K(n-1)}E_n$ in terms of deformations of formal groups together with extra data about the $(n-1)$th Lubin-Tate coordinate. We use this to describe co-operations and power operations in this transchromatic setting. As an application, we construct exotic multiplicative structures on $L_{K(1)}E_2$, not induced from the ring structure on E_2 by $K(1)$-localization.

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List of abbreviations

For the sake of reference, here is some notation introduced elsewhere in the paper:

k is a field of characteristic p, generally perfect, often even finite. W denotes the Witt vectors functor. Λ is the completed Laurent series ring $Wk((u_{n-1}))^{\wedge}_p$.

CLN is the category of complete local noetherian rings, and CLN_A of complete local noetherian A-algebras. Gpd is the category of groupoids. Other categories are written in $\mathsf{sans\text{-}serif}$, but typically identifiable from their names.

Γ is a formal group over k of finite height n.

$E = E_n = E(k, \Gamma)$ is the Morava E-theory for (k, Γ); this is even periodic, with periodicity class u in degree 2 and $\pi_0 E$ non-canonically isomorphic to $Wk[[u_1, \ldots, u_{n-1}]]$. LE is the localization $L_{K(n-1)}E_n$. At a certain point, F is used for E_{n-1}.

BP is the Brown-Peterson spectrum with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where $|v_i| = 2(p^i - 1)$; the map $BP \to E$ sends $v_n \mapsto u^{p^n-1}$, $v_i \mapsto 0$ for $i > n$, and and $v_i \mapsto u^{p^i-1}u_i$ for $i < n$. I_n is the ideal $(p, v_1, \ldots, v_{n-1})$, or its image in any BP_*-module.

Notation like $E^{\wedge}_* E$ denotes completed homology $\pi_* L_{K(n)}(E \wedge E)$. Which n is intended varies, but is generally clear from context.

$\mathbb{G}^u$ is the universal deformation formal group of Γ, defined over E_0. In particular, there is a canonical isomorphism $\mathbb{G}^u \otimes_{E_0} k \cong \Gamma$. We write $\mathbb{H}$ for the height $n-1$ formal group $\mathbb{G}^u \otimes k((u_{n-1}))$. $\mathbb{G}$ will generally be used for other formal groups.

We will write Def_Γ for the deformations functor of Γ, and $\mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}$ for the functor of deformations of $\mathbb{H}$ augmented with Λ-algebra structure; both are defined more explicitly below.

In chapter 6, ψ^p and θ are certain operations on θ-algebras, and $\mathbb{T}$ is a monad acting on the completed E-theory of an E_∞ ring spectrum.

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CHAPTER 1

Introduction

All of which will perhaps but for a supersubtle way of pointing the plain moral that a young embroiderer of the canvas of life soon began to work in terror, fairly, of the vast expanse of that surface, of the boundless number of its distinct perforations for the needle, and of the tendency inherent in his many-coloured flowers and figures to cover and consume as many as possible of the little holes. The development of the flower, of the figure, involved thus an immense counting of holes and a careful selection among them. That would have been, it seemed to him, a brave enough process, were it not the very nature of the holes so to invite, to solicit, to persuade, to practise positively a thousand lures and deceits. The prime effect of so sustained a system, so prepared a surface, is to lead on and on; while the fascination of following resides, by the same token, in the presumability somewhere of a convenient, of a visibly-appointed stopping-place.

– Henry James, *The Art of the Novel*

As Hillis Miller has demonstrated, the figure is incoherent: the covering of the holes in the canvas by means of embroidery requires a simultaneous puncturing with a needle.

– Sheila Teahan, "The afterlife of figures"

We begin with an old story about complex vector bundles. A complex line bundle V over a space X is classified by a map $X \to \mathbb{C}P^\infty$. V also has a first Chern class, which is an element $c_1(V) \in H^2(X;\mathbb{Z})$; in fact, this is the image of a specific element

$$x \in H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[[x]], |x| = 2$$

under the map $H^*(\mathbb{C}P^\infty) \to H^*(X)$.

The tensor product of complex line bundles is classified by a grouplike multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Taking cohomology and applying the Künneth formula, we get a map

$$H^*(\mathbb{C}P^\infty) \to H^*(\mathbb{C}P^\infty) \otimes_{\mathbb{Z}} H^*(\mathbb{C}P^\infty)$$

$$\mathbb{Z}[[x]] \to \mathbb{Z}[[x]] \otimes \mathbb{Z}[[y]] \cong \mathbb{Z}[[x,y]].$$

In fact, this map sends x to $x+y$, which is equivalent to the very classical statement that, for line bundles V and W, $c_1(V \otimes W) = c_1(V) + c_1(W)$. From this statement and the splitting principle follows the rest of the theory of Chern classes.

There is also a theory of Chern classes for complex K-theory, but the tensor product formula is more complicated. One has

$$c_1^K(V \otimes W) = c_1(V) + c_1(W) + \beta c_1(V)c_1(W),$$

where $\beta \in K^{-2}(*)$ is the Bott periodicity class. For a general cohomology theory A equipped with Chern classes – a **complex oriented cohomology theory** – there is a tensor product formula that is a *power series* in two variables the A-cohomology of a point. This associates to A an algebraic structure called a **formal group**.

Over the last fifty years or so, homotopy theorists have discovered a series of surprising and productive relationships between complex oriented cohomology theories and their formal groups. This story begins with Quillen, who proved in [**Qu69**] that the complex cobordism theory MU, which carries the universal theory of Chern classes, also carries the universal formal group (see Theorem 2.1.2 for a precise statement). This result pointed the way to computations of stable homotopy groups using the MU-based Adams spectral sequence (see [**Rav04**] for a survey of this work).

It also indicated that the stable homotopy category might be more algebraic than previously thought. So, in work like [**Mora89**] and [**La76**], mathematicians began to construct cohomology theories from formal groups, a narrative that has run through [**GH05**] to the modern theory of topological modular forms [**TMF14**] and their higher generalizations [**BL10**]. On the other hand, the classification of formal groups was shown to be linked to deep structural properties of the stable homotopy category as a whole [**DHS88, HSt99**].

To explain this last point, localized at a fixed prime p, a formal group has a positive integer invariant called the **height**, which says how many times formal multiplication by p factors through the mod p Frobenius. Informally, height is a measure of the complexity of a formal group or a cohomology theory – p-complete K-theory has height 1, elliptic cohomology theories can have heights up to 2, and so on. Now, there are topologically defined **chromatic localization** functors L_n and $L_{K(n)}$, which pare down a space X to retain just the information about X visible to complex oriented cohomology theories, respectively of height *at most* n and height *exactly* n (see Section 2.2.5). These functors are central to modern stable homotopy theory, for a number of reasons.

(1) By the thick subcategory theorem of [**HSt99**], the functors $L_{K(n)}$ are the most precise functors of this form available. It is impossible to systematically shave more cohomological information away from a space without destroying the thing.

(2) By the chromatic convergence theorem [**Rav92**], the stable homotopy of a finite complex X, such as the sphere, can be recovered from all of its L_n-localizations by a limit procedure.

(3) By general facts about localization, there are homotopy pullback squares

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X. \end{tikzcd}$$

Thus, the stable homotopy of a finite complex can be recovered from its $K(n)$-localizations, together with transchromatic attaching data connecting localizations of different heights.

(4) Finally, $K(n)$-local objects exhibit various periodicity phenomena, in their homotopy groups and in various spectral sequences, of wavelength $2(p^n - 1)$. These periodicity phenomena were observed in the Adams spectral sequence far before $K(n)$-localization was understood properly, and are the source of the name 'chromatic homotopy theory': Ravenel thought of the chromatic localizations as a prism the 'white light' of a space into its various 'colors'.

In principle, then, computations like the stable homotopy groups of spheres are reduced to a series of periodic, $K(n)$-local calculations, followed by a transchromatic assembly procedure. In practice, this is all extremely difficult. The stable homotopy groups of spheres are now extremely well understood $K(1)$-locally; understood, but not well, $K(2)$-locally; and not understood at all at higher heights. Moreover, the transchromatic assembly procedure is largely missing both computational and conceptual understanding.

This thesis largely aims to study the algebraic geometry behind transchromatic homotopy theory. This is connected to the algebraic problem of what happens to a formal group as it changes height.

Before we describe the contents of this thesis, we need to give its other main ingredient, which is Morava E-theory. This is a complex oriented cohomology theory constructed from the algebraic geometry of deformations of formal groups. In [**LT66**], Lubin and Tate proved that a height n formal group Γ over a perfect, characteristic p field k has a universal deformation which lives over the ring $Wk[[u_1, \ldots, u_{n-1}]]$. The parameters u_i control the height of the deformation: for example, inverting u_{n-1} forces the height to be at most $n-1$. By a theorem of Goerss, Hopkins, and Miller [**GH04**], there is an essentially *unique* complex oriented, $K(n)$-local, E_∞ ring spectrum, called **Morava E-theory** $E = E(k, \Gamma)$, with

$$\pi_* E = Wk[[u_1, \ldots, u_{n-1}]][u^{\pm 1}], |u| = 2, |u_i| = 0,$$

and with formal group the universal deformation defined by Lubin and Tate.

This theorem suggests that the relationship between stable homotopy theory and the algebraic geometry of formal groups is extremely close when localized at a single prime and height. In particular, basically all topological facts about E-theory should be expressible in terms of formal groups. For example:

(1) The profinite group $\mathbb{G}_n$ of automorphisms of the field k and the formal group Γ acts on the Lubin-Tate ring, and this extends to an action of $\mathbb{G}_n$ on E_n by E_∞ maps. By a theorem of Devinatz and Hopkins ([**DH04**], and see Theorem 5.1.1

in this document),

$$\pi_* L_{K(n)}(E_n^{\wedge(s+1)}) \cong \mathrm{Hom}_{cts}(\mathbb{G}_n^{\times s}, \pi_* E_n).$$

This means that the $K(n)$-local E_n-based Adams spectral sequence for the sphere takes the form

$$E_2^{st} = H^s(\mathbb{G}_n, \pi_t E_n) \Rightarrow \pi_{t-s} L_{K(n)} S.$$

Moreover, $L_{K(n)}S$ is the homotopy fixed points of $\mathbb{G}_n$ acting on E_n, in a sense described by [**DH04**]. This also means that the E_*-comodule structure on the completed E-homology of a space or spectrum is just a continuous $\mathbb{G}_n$-action.

(2) The completed E-homology of a $K(n)$-local E_∞ ring spectrum carries power operations, which are parametrized by isogenies of deformations of the formal group Γ ([**AHS04**], [**Re09**], and see chapter 6 below).

(3) The E-cohomology of $B\mathbb{Z}/p^k$ is just the ring of functions on the p-torsion points of $\mathbb{G}^u$. One can use this fact to construct a character map from $E^0 BG$, for G a finite group, with image in a ring of 'generalized class functions' on conjugacy classes of n-tuples of commuting elements of p-power order. This map becomes an isomorphism after base changing to a certain ring of level structures, which in particular forces the height of both sides to decrease below n ([**HKR00**], [**Sta13**]).

Of the transchromatic objects straddling heights $n-1$ and n, one of the most basic is the $K(n-1)$-localization of a height n E-theory, known in this document as $L_{K(n-1)}E_n$

or just LE. This is even periodic, with

$$LE_0 = Wk[[u_1, \dots, u_{n-1}]][u_{n-1}^{-1}]^{\wedge}_{(p,u_1,\dots,u_{n-2})} \cong Wk((u_{n-1}))^{\wedge}_{p}[[u_1, \dots, u_{n-2}]],$$

a complete local ring with residue field $k((u_{n-1}))$, over which there is a naturally defined height $n-1$ formal group $\mathbb{H} = \mathbb{G}^u \otimes k((u_{n-1}))$. This looks very much like the height $n-1$ E-theory associated to $\mathbb{H}$ – the only problem being that the field $k((u_{n-1}))$ is not perfect, so that the Lubin-Tate theorem does not apply.

This thesis discusses the relationship between LE and formal groups, starting with the idea that LE_0 classifies deformations of a height $n-1$ formal group together with some extra data. But there are several different options for what this extra data is! First, one could view LE_0 as carrying not a formal group but a p-divisible group. This is the result of base changing the p-power torsion subgroups of $\mathbb{G}^u$ to LE_0; it fits into an exact sequence

$$0 \to \mathbb{G}^{for} \to \mathbb{G} \to \mathbb{G}^{et} \to 0$$

where $\mathbb{G}^{for}$ is the formal group of LE, and $\mathbb{G}^{et}$ is a Galois twist of a constant group scheme of the form $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$. Homotopy theorists have long thought that this p-divisible group structure should manifest itself in transchromatic phenomena, in particular due to the use of p-divisible groups of abelian varieties in defining topological automorphic forms ([**Lu09**], [**Lu18a**], [**Lu18b**], [**BL10**]). However, this still remains elusive; the best work on this subject is still [**Sta13**], which uses p-divisible groups like these to produce character maps.

Second, one could replace the field $k((u_{n-1}))$ with its perfect closure $k((u_{n-1}^{1/p^\infty}))$. Then $\mathbb{H} \otimes k((u_{n-1}^{1/p^\infty}))$ is a formal group over a perfect field, so has an associated height $n-1$ E-theory E^{perf} which has a map from LE. One could then try to interpret topological facts about LE as facts about E^{perf}, together with 'descent data' along this inseparable map of fields. This viewpoint is not explored here, although the author guesses that it is implicit in the following story. In particular, the map $k((u_{n-1})) \to k((u_{n-1}^{1/p^\infty}))$ lifts to many different maps

$$Wk((u_{n-1}))^\wedge_p \to W(k((u_{n-1}^{1/p^\infty}))),$$

which can each be used to define a different map $LE \to E^{\mathrm{perf}}$. This unexpected choice might be to blame for the variety of E_∞ structures on LE discussed in section 5.

Third, one could simply view $\mathbb{H}$ as a height $n-1$ formal group, and the extra data as knowledge about the last Lubin-Tate coordinate u_{n-1}. This is the most naïve but the most workable approach, and the one we take here. After two background chapters (chapters 2 and 3), we discuss basic properties of LE in chapter 4, and prove the following:

Theorem 1.0.1 (Theorem 4.2.2, Theorem 4.3.8)**.** *The ring LE_0 classifies deformations of $\mathbb{H}$, together with a choice of last Lubin-Tate coordinate u_{n-1}.*

To be more precise, a deformation of $\mathbb{H}$ over a complete local ring R takes as part of its data a map $k((u_{n-1})) \to R/\mathfrak{m}$, and the extra data here is a lift of this to a continuous map $Wk((u_{n-1}))^\wedge_p \to R$. There are two different versions of this theorem because there are two applicable notions of continuity. $Wk((u_{n-1}))^\wedge_p$ is a p-adic ring, and one can consider p-adically continuous maps. However, the residue field $k((u_{n-1}))$ has its own topology as well. One can enforce a higher version of continuity, known as 'pipe-continuity', that takes this into account.

In chapter 5, we use this result to study completed co-operations for localized E-theory. We begin by proving the Devinatz-Hopkins theorem that $\pi_* L_{K(n)}(E \wedge E) = \mathrm{Hom}_{cts}(\mathbb{G}_n, E_*)$ from the point of view of formal groups (Theorem 5.1.1). Although the results in the localized case are not quite so clean, we are able to prove the following.

Theorem 1.0.2 (Theorem 5.3.1, Theorem 5.2.4)**.** *Continuous maps from $\pi_* L_{K(n-1)}(E \wedge E)$ into an LE_*-algebra R_* with R_0 complete local represent pairs of a continuous map $Wk((u_{n-1}))_p^\wedge \to R_0$, and an isomorphism of formal groups over $R_0/\mathfrak{m}$.*

Continuous maps from $\pi_* L_{K(n-1)}(E_{n-1} \wedge E_n)$ into an $(E_{n-1})_*$-algebra R_* with R_0 complete local represent pairs of a continuous map $Wk((u_{n-1}))_p^\wedge \to R_0$, and an isomorphism of formal groups over $R_0/\mathfrak{m}$.

This is to be used in the final chapter 6 on power operations and E_∞ structures. We first survey Rezk's theory of power operations on the E-theory of E_∞ ring spectra. For an E_∞-ring spectrum X, $E_0^\wedge X$ has an algebraic structure describable in terms of isogenies of formal groups, known as a $\mathbb{T}$-algebra. Conversely, given a $\mathbb{T}$-algebra, there is conjecturally (Conjecture 6.2.8) an obstruction theory for realizing it as the E-theory of an E_∞-ring spectrum, with obstructions living in certain $\mathbb{T}$-algebra André-Quillen cohomology groups. By combining the height $n-1$ version of this with the previous result, we obtain the following.

Theorem 1.0.3 (Theorem 6.4.1)**.** *The possible* $\mathbb{T}$*-algebra structures on* $(E_{n-1})_0^\wedge E_n$ *are in bijection with Frobenius lifts on* $Wk((u_{n-1}))_p^\wedge$*. Thus, there are non-isomorphic* $\mathbb{T}$*-algebra structures on* $(E_{n-1})_0^\wedge E_n$*.*

At $n = 2$, the obstruction theory is known to exist, and the obstruction groups can actually be computed. Thus, we obtain the following transchromatic variant on the Goerss-Hopkins-Miller theorem.

Theorem 1.0.4 (Theorem 6.5.2, Corollary 6.5.8)**.** *There are non-isomorphic* E_∞ *structures on* $L_{K(1)}E_2$*.*

Besides serving as an instructive calculation in $K(1)$-local obstruction theory, this result, and the methods used to prove it, point the way towards studying the transchromatic behavior of highly structured ring spectra and their power operations. In particular, these exotic E_∞-algebras are $K(1)$-localizations of $K(2)$-local *spectra*, and are E_∞*-algebras*, but only one of them – the $K(1)$-localization of the canonical E_∞-algebra E_2 – is a $K(1)$-localization of a $K(2)$-local E_∞-algebra. It is likely that, as is the case here, $K(n)$-local power operations generally satisfy integrality conditions that can be distorted on their $K(n-1)$-localizations.

CHAPTER 2

Background on formal groups

2.1. Formal groups, formal group laws, and complex orientations

2.1.1. Complex oriented ring spectra and formal groups

Recall that a complex-oriented ring spectrum is a ring spectrum A with a factorization of the unit

$$\begin{tikzcd} S^0 \arrow[d, leftrightarrow, "u"] \arrow[dr] & \\ \Sigma^{-2}\Sigma^\infty_+\mathbb{C}P^\infty & A. \end{tikzcd}$$

This forces an isomorphism

$$A^*(\mathbb{C}P^\infty) = (A_{-*})[[x]]$$

with $x \in A^2(\mathbb{C}P^\infty)$, induced by the orientation map u. The grouplike E_∞-space multiplication on $\mathbb{C}P^\infty$ (classifying the tensor product of complex line bundles) induces a map

$$A^*(\mathbb{C}P^\infty) \to A^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong A^*(\mathbb{C}P^\infty)\widehat{\otimes}_{A_*}A^*(\mathbb{C}P^\infty)$$

or

$$A_*[[x]] \to A_*[[x, y]].$$

The image of x under this map is a power series of the form

$$x + y + \cdots + a_{ij}x^i y^j + \cdots ,$$

which we call $x +_{\mathbb{G}_A} y$. As the multiplication on $\mathbb{C}P^\infty$ is a group multiplication up to homotopy, the map $A_*[[x]] \to A_*[[x,y]]$ is a cogroup map, and $x +_{\mathbb{G}_A} y$ is an example of a formal group law, defined as follows.

Definition 2.1.1. A formal group law F over a ring R is a power series $x +_F y \in R[[x,y]]$ satisfying the following properties:

(1) $(x +_F y) +_F z = x +_F (y +_F z)$,

(2) $x +_F y = y +_F x$,

(3) $x +_F 0 = x$,

(4) and there exists a power series $[-1]_F(x) \in R[[x]]$ such that $x +_F [-1]_F(x) = 0$.

(As it turns out, the existence of inverses follows formally from the other properties; see [**Rav04**, A2.1.2].)

A **homomorphism** of formal group laws $\phi : F \to F'$ is a power series $\phi(x) \in R[[x]]$ such that

$$\phi(x +_F y) = \phi(x) +_{F'} \phi(y).$$

An **isomorphism** is such a power series that is invertible under composition. This means that it must satisfy $\phi(x) = ux + \cdots$, where $u \in R$ is a unit; it is a **strict isomorphism** if $u = 1$.

There is a ring spectrum MU, the **complex cobordism spectrum**, which carries the universal complex orientation: homotopy classes of multiplicative maps $MU \to A$

correspond to complex orientations of R. By an amazing theorem due to Quillen, the relationship between complex orientations and formal groups is no accident, but part of the nature of complex cobordism itself.

Theorem 2.1.2 ([**Qu69**]). *The ring MU_* represents formal group laws – that is, graded ring maps $MU_* \to R$ correspond to formal group laws over R. Likewise, MU_*MU represents strict isomorphisms of formal group laws.*

A complex-orientable spectrum admits many different complex orientations, and thus many different formal group laws, with any two formal group laws related by an isomorphism. It is worthwhile to reformulate the theory of formal group laws in a coordinate-free way.

Definition 2.1.3. Let X be a scheme and consider pointed formal schemes over X, $(\mathbb{G} \to X, e \in X(\mathbb{G}))$. A **coordinate** on a pointed formal scheme $(\mathbb{G}, e)$ over X is an isomorphism of pointed formal schemes over X

$$(\mathbb{G}, e) \cong (\widehat{\mathbb{A}}^1_X, 0).$$

A (1-dimensional, commutative) **formal group** over X is a commutative group object $\mathbb{G}$ in the category of formal schemes over X, such that after passing to a Zariski cover, the underlying formal scheme of $\mathbb{G}$, pointed by its identity section, admits a coordinate. A **homomorphism** (resp. **isomorphism**) of formal groups is a homomorphism (resp. isomorphism) of group objects in formal schemes over X.

Of course, a formal group law over a ring R defines a formal group over the affine scheme $\operatorname{Spec} R$. On the other hand, a formal group $\mathbb{G} \to X$ is defined by a formal group law over any open affine where it admits a coordinate.

2.1.2. Base change

A map of rings $f : R_1 \to R_2$ induces base change functors

$$\mathsf{FG}_{R_1} \to \mathsf{FG}_{R_2}, \quad \mathsf{FGL}_{R_1} \to \mathsf{FGL}_{R_2}$$

which we write as $\mathbb{G} \mapsto \mathbb{G} \otimes_{R_1} R_2$ or as $\mathbb{G} \mapsto f^*\mathbb{G}$.

These functors are easiest to describe in terms of formal group laws. If $\mathbb{G}$ has the formal group law

$$x +_{\mathbb{G}} y = x + y + \cdots + \sum a_{ij} x^i y^j,$$

then $f^*\mathbb{G}$ has the formal group law

$$x +_{f^*\mathbb{G}} y = x + y + \cdots + \sum f(a_{ij}) x^i y^j.$$

There is a subtlety regarding base changes of formal groups. One would like to define $f^*\mathbb{G} = \mathbb{G} \times_{\operatorname{Spec} R_1} \operatorname{Spec} R_2$. Since base change preserves products, this is actually a sheaf of groups on $\operatorname{Spec} R_2$. However, its underlying ring of functions is Zariski-locally

$$R_1[[x]] \otimes_{R_1} R_2.$$

This is equal to $R_2[[x]]$ when R_2 is *finite* over R_1. Instead, one observes that $\mathbb{G}$ is a filtered colimit of finite R_1-schemes, locally of the form $\operatorname{Spec} R_1[x]/(x^n)$, and defines

$$f^*(\mathbb{G}) = \lim \operatorname{Spec} R_1[x]/(x^n) \times_{\operatorname{Spec} R_1} \operatorname{Spec} R_2.$$

This is a formal group, and in fact the schemes $\operatorname{Spec} R_1[x]/(x^n)$ can be defined in a coordinate-free way. See [**Go08**, Remark 1.30] or [**Me72**].

2.1.3. Invariant differentials

Definition 2.1.4. Let $\pi : \mathbb{G} \to X$ be a formal group over X and let $e : X \to \mathbb{G}$ be the identity section of $\mathbb{G}$. The **bundle of invariant differentials** of $\mathbb{G}$ is the line bundle on X

$$\omega_{\mathbb{G}} := e^* \Omega_{\mathbb{G}/X}.$$

If $\mathbb{G}$ is a formal group with a coordinate over an affine scheme $\operatorname{Spec} R$, given by a formal group law F, then the R-module of invariant differentials is free and has a canonical generator. Writing $F(x, y) = x +_F y$, this is given by

$$\eta_F := \frac{dx}{F_x(0, x)}.$$

Moreover, an isomorphism of formal group laws over R, $\phi : F \to F'$, induces a map $d\phi : \omega_{F'} \to \omega_F$ on invariant differentials, which is just multiplication by $\phi'(0)$.

2.1.4. Moduli of formal groups

Definition 2.1.5. The **moduli of formal groups** is the functor $\mathcal{M}_{\mathrm{fg}} : \mathsf{Rings} \to \mathsf{Gpd}$ such that $\mathcal{M}_{\mathrm{fg}}(R)$ is the groupoid of formal groups $\mathbb{G}$ over R and their isomorphisms.

We also define a functor $\mathcal{M}'_{\mathrm{fg}} : \mathsf{Rings} \to \mathsf{Gpd}$. The objects of $\mathcal{M}'_{\mathrm{fg}}(R)$ are pairs $(\mathbb{G}, \eta)$, where $\mathbb{G}$ is a formal group over R and η is a nonvanishing section of the bundle of invariant differentials. The isomorphisms of $\mathcal{M}'_{\mathrm{fg}}(R)$ from $(\mathbb{G}, \eta)$ to $(\mathbb{G}', \eta')$ are isomorphisms of formal groups over R, $\phi : \mathbb{G} \to \mathbb{G}'$, such that $d\phi(\eta') = \eta$.

Theorem 2.1.6. *The Hopf algebroid (MU_*, MU_*MU) represents the functor $\mathcal{M}'_{\mathrm{fg}}$.*

Proof. This is a restatement, in fancier language, of Theorem 2.1.2. Let $\mathbb{G}$ be a formal group over a ring R, with an invariant differential η. After passing to a Zariski cover, $\mathbb{G}$ admits a coordinate x and thus a formal group law F. Moreover, by scaling this coordinate via an isomorphism of the form $x \mapsto ux$, we can make it so that

$$\eta = \frac{dx}{F_x(0, x)}.$$

Any other coordinate for $\mathbb{G}$ with this property is related to F by a strict isomorphism.

Likewise, let $\phi : (\mathbb{G}, \omega) \to (\mathbb{G}', \omega')$ be an isomorphism of formal groups which preserves the chosen invariant differentials. After passing to a Zariski cover and choosing coordinates as above, we see that ϕ must be a strict isomorphism between the induced formal group laws.

Thus, define $\mathcal{M}_{\mathrm{fgl}}$ to be the Zariski sheafification of the functor sending R to the groupoid of formal group laws and their strict isomorphisms over R. The above argument shows that $\mathcal{M}'_{\mathrm{fg}} \simeq \mathcal{M}_{\mathrm{fgl}}$. But $\mathcal{M}_{\mathrm{fgl}}$ is presented by (MU_*, MU_*MU), by Quillen's theorem. $\square$

Localized at p, there is a standard way to simplify one's formal group laws. One defines the nth Frobenius series for the formal group law F by th

$$F_n(x) = x^{1/n} +_F \zeta x^{1/n} +_F \cdots +_F \zeta^{n-1} x^{1/n},$$

where ζ is a primitive nth root of unity. *A priori*, this is only defined in $R_*[[x^{1/n}]][\zeta]$, but the commutativity of the formal group law guarantees it is actually in $R_*[[x]]$.

Definition 2.1.7. A formal group law F is *p-typical* if $F_n(x) = 0$ for every n prime to p.

Theorem 2.1.8 (Cartier)**.** *Every formal group law over a p-local ring has a canonical isomorphism to a p-typical one, which is the identity if the formal group law is already p-typical.*

Theorem 2.1.9. *Localized at a prime p, MU has a ring spectrum summand, the Brown-Peterson spectrum BP, such that BP_* represents p-typical formal group laws, and BP_*BP represents strict isomorphisms of p-typical formal group laws.*

Corollary 2.1.10. *The Hopf algebroid (BP_*, BP_*BP) presents $(\mathcal{M}'_{\mathrm{fg}})_{(p)}$.*

Proof. This follows from Cartier's theorem 2.1.8 and Theorem 2.1.6. $\square$

Most of the spectra we will be dealing with are even periodic, justifying the following even periodic variants.

Definition 2.1.11. The spectra MUP and BPP are the even periodic ring spectra defined by $MUP = MU[u^{\pm 1}]$ and $BPP = BP[u^{\pm 1}]$, where $|u| = 2$.

Theorem 2.1.12. *The Hopf algebroids* (MUP_0, MUP_0MUP) *and* (BPP_0, BPP_0BPP) *respectively present* $\mathcal{M}_{\mathrm{fg}}$ *and* $(\mathcal{M}_{\mathrm{fg}})_{(p)}$.

Proof. We sketch the proof for MUP; the one for BPP is almost identical. MU_* is concentrated in nonnegative, even degrees. Thus, there is an isomorphism of rings $MU_* \cong MUP_0$, sending an element $x \in MU_{2j}$ to $u^{-j}x$. This means that MUP_0 also represents formal group laws.

The map $MU \to MUP$ induces a flat map on homotopy groups, so

$$MUP_*MUP = MUP_* \otimes MU_*MU \otimes_{MU_*} MUP_* = MU_*MU[u^{\pm 1}, \overline{u}^{\pm 1}].$$

Thus,

$$MUP_0MUP \cong MU_*MU[w^{\pm 1}],$$

where $w = u\overline{u}^{-1}$. Thus, a map $MUP_0MUP \to R$ classifies a strict isomorphism of formal group laws over R, $\phi : F \xrightarrow{\sim} F'$, together with a unit $w \in R$. This is equivalent to the data of a not necessarily strict isomorphism $\phi_w : F \xrightarrow{\sim} F'_w$, where

$$x +_{F'_w} y = w(w^{-1}x + w^{-1}y)$$

and

$$\phi_w(x) = w\phi(x).$$

The unit w can be recovered as the coefficient of x in $\phi(x)$. One then has to check that these isomorphisms commute with the Hopf algebroid structure. Finally, one notes that the groupoid of formal groups and isomorphisms is Zariski-locally equivalent to the groupoid of formal group laws and isomorphisms. □

2.2. Formal groups over fields

2.2.1. Characteristic zero

Proposition 2.2.1. *Let K be a field of characteristic zero, and let $\mathbb{G}$ be a formal group over k. Then $\mathbb{G}$ is uniquely strictly isomorphic to the additive formal group $\widehat{\mathbb{G}}_a$.*

Proof. Choose a coordinate x for $\mathbb{G}$ and an invariant differential $\eta = \eta(x)\,dx$. Then the power series

$$\log_{\mathbb{G}}(T) = \int_x \eta(x)\,dx$$

is an isomorphism $\mathbb{G} \to \mathbb{G}_a$, which is strict if and only if $\eta(x) = 1 + \cdots$. See [**Rav04**, Appendix 2] for details. □

2.2.2. Positive characteristic, height, and the Frobenius

In characteristic p, the situation is not so simple.

Definition 2.2.2. The p**-series** of a formal group law F is the power series

$$[p]_F(x) = \overbrace{x +_F x +_F \cdots +_F x}^{p}.$$

Over a characteristic p field k, this has the form

$$[p]_F(x) = ux^{p^n} + \cdots$$

for some $1 \leq n \leq \infty$, called the **height** of F. The height is invariant under changes of coordinate, and thus depends only on the underlying formal group of F.

By convention, the height is said to be ∞ if $[p]_F(x) = 0$, and zero over fields of characteristic p. However, we are almost exclusively interested in finite, nonzero heights here.

It will be worthwhile to introduce another point of view. Any characteristic p scheme X has a Frobenius endomorphism $\sigma_X : X \to X$, defined as the identity on points and $x \mapsto x^p$ on structure sheaves. We then have a diagram

(2.1)
$$\begin{tikzcd}
\mathbb{G} \arrow[dr, "\mathrm{Frob}_{\mathbb{G}}"] \arrow[drr, bend left, "\sigma_{\mathbb{G}}"] \arrow[ddr, bend right] & & \\
& \mathbb{G}^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{G} \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "\sigma_k"'] & \operatorname{Spec} k.
\end{tikzcd}$$

Note that, by Section 2.1.2, $\mathbb{G}^{(p)}$ is a formal group if σ_k is finite – for example, if k is perfect. In this case, the map $\mathrm{Frob}_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}^{(p)}$ is a homomorphism of formal groups called the **relative Frobenius**.

Iterating this procedure, one gets maps

$$\mathrm{Frob}^n : \mathbb{G} \overset{\mathrm{Frob}}{\to} \mathbb{G}^{(p)} \overset{\mathrm{Frob}}{\to} \cdots \overset{\mathrm{Frob}}{\to} \mathbb{G}^{(p^n)}.$$

The height of $\mathbb{G}$ is the maximal n such that $[p] : \mathbb{G} \to \mathbb{G}$ admits a factorization of the form

$$\mathbb{G} \xrightarrow{\mathrm{Frob}^n} \mathbb{G}^{(p^n)} \to \mathbb{G}.$$

For a formal group over a general ring R, we view height as a function on $\operatorname{Spec} R$ or $\operatorname{Spf} R$:

$$\text{height}(\mathbb{G}) : P \text{ prime ideal of } R \mapsto \text{height}(\mathbb{G} \otimes_R \operatorname{Frac}(R_P)).$$

In particular, the height of a formal group over a complete local ring R is identified with its height over the residue field of R.

Proposition 2.2.3. *The height of* $\mathbb{G}$ *is upper semicontinuous on* $\operatorname{Spec} R$.

Proof. In other words, the set of points on R where $\text{height}(\mathbb{G}) \leq n$, for any n, is Zariski-open. After passing to a Zariski cover, we may assume that $\mathbb{G}$ has a formal group law F, with p-series

$$[p]_F(x) = px + \sum b_i x^i.$$

Then $\text{height}(\mathbb{G}) \leq n$ at a prime ideal P if and only if the coefficient b_{p^n} is not in P. This is an open condition. $\square$

We now discuss the relationship of height with BP-theory.

Proposition 2.2.4 (cf. [**Go08**, 2.42])**.** *The* p*-series of a* p*-typical formal group law over a* p*-local ring* R *can be written in the form*

$$[p]_F(x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots$$

where $v_i \in R$. Conversely, this expression uniquely determines the p-typical formal group law.

By Theorem 2.1.9, BP_* carries the universal p-typical formal group law. In fact,

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots],$$

and the universal p-typical formal group law over BP_* has p-series given by the above formula. Thus, one can think of height as pulled back from a filtration on $\operatorname{Spec} BP_*$. Formal groups have height at least n precisely on the closed subscheme of $\operatorname{Spec} BP_*$ where $p, v_1, v_2, \ldots, v_{n-1}$ vanish, and height at most $n-1$ on its open complement.

2.2.3. Automorphisms

Definition 2.2.5. Let Γ be a height n formal group over a field k. The **Morava stabilizer group** $\operatorname{Aut}(k, \Gamma)$ is the group of pairs (τ, g), where $\tau : k \xrightarrow{\sim} k$ is an automorphism, and g is an isomorphism of formal groups over k, $g : \Gamma \xrightarrow{\sim} \tau^*\Gamma$.

Equivalently, one can write $(\tau, g) \in \operatorname{Aut}(k, \Gamma)$ as a commutative square

$$\begin{tikzcd}
\Gamma \arrow[r, "\sim"] \arrow[d] & \Gamma \arrow[d] \\
\operatorname{Spec} k \arrow[r, "\tau^*"'] & \operatorname{Spec} k
\end{tikzcd}$$

where the horizontal maps are isomorphisms, the top one a (non-k-linear) isomorphism of formal groups. Then $g : \Gamma \to \tau^*\Gamma$ is the map induced by the universal property of the pullback.

It is common to write $\mathbb{G}_n = \mathrm{Aut}(k, \Gamma)$. If k is a finite field containing $\mathbb{F}_{p^n}$ (which implies that $\mathrm{Aut}_k(\Gamma) = \mathrm{Aut}_{\overline{\mathbb{F}_p}}(\Gamma)$), then we have

$$\mathrm{Aut}(k, \Gamma) = \mathrm{Gal}(k/\mathbb{F}_p) \ltimes \mathrm{Aut}_k(\Gamma).$$

The composition law can then be written

$$(\tau_2, g_2)(\tau_1, g_1) = (\tau_2\tau_1, \tau_2^*(g_1)g_2).$$

The odd variance of this formula is a result of our choice to write τ as a map of fields (that is, rings) and g as a map of formal groups (that is, affine formal schemes).

2.2.4. The moduli stack of formal groups

We have arrived at a sort of picture of the moduli stack. For each prime p, $(\mathcal{M}_{\mathrm{fg}})_{(p)}$ admits a filtration

$$(\mathcal{M}_{\mathrm{fg}})_{(p)} = \mathcal{M}_{\mathrm{fg}}^{\geq 0} \supseteq \mathcal{M}_{\mathrm{fg}}^{\geq 1} \supseteq \mathcal{M}_{\mathrm{fg}}^{\geq 2} \supseteq \cdots$$

where each $\mathcal{M}_{\mathrm{fg}}^{\geq n}$, the moduli of height at least n formal groups over p-local rings, is a closed substack of codimension n. For $n \geq 2$, the open complement of $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ in $\mathcal{M}_{\mathrm{fg}}^{\geq (n-1)}$ consists of a single geometric point – the unique isomorphism class of height n formal group over an algebraically closed field – with isotropy group $\mathrm{Aut}(k, \Gamma)$. The open complement of $\mathcal{M}_{\mathrm{fg}}^{\geq 1}$ in $\mathcal{M}_{\mathrm{fg}}^{\geq 0}$ is just $\mathrm{Spec}\,\mathbb{Q}$. The intersection $\bigcap_{n=1}^{\infty} \mathcal{M}_{\mathrm{fg}}^{\geq n}$ also has a single geometric point – the additive formal group in characteristic p, with p-series $[p](x) = 0$. Finally, the stack admits a presentation

$$\begin{tikzcd} \operatorname{Spec} BP_*BP \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} BP_* \arrow[d] \\ \operatorname{Spec} BP_* \arrow[r] & (\mathcal{M}_{\mathrm{fg}})_{(p)} \end{tikzcd}$$

and the height filtration pulls back to a filtration of $\operatorname{Spec} BP_*$, with $(\operatorname{Spec} BP_*)^{\leq n}$ given by the vanishing of $p, v_1, \dots, v_{n-1}$.

2.2.5. Chromatic localization

Several deep theorems of Devinatz, Hopkins, Smith, and Ravenel [**DHS88, HSm98, Rav92**] relate the moduli of formal groups to structure on the category of spectra. While we will not discuss this in any detail here, it is important at least to state that there are functors $X \mapsto L_n X$ (resp. $X \mapsto L_{K(n)}X$) on the category of spectra, and natural maps $X \to L_n X$ (resp. $X \mapsto L_{K(n)}X$), inducing isomorphisms on A-homology whenever A is a p-local complex oriented cohomology theory of height at most n (resp. of height exactly n). These functors are Bousfield localizations with respect to spectra constructible from BP. L_n is localization with respect to a spectrum $E(n)$ with

$$E(n)_* = BP_*[v_n^{-1}]/(v_{n+1}, v_{n+2}, \dots) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}],$$

and $L_{K(n)}$ is localization with respect to a spectrum $K(n)$ with

$$K(n)_* = E(n)_*/(p, v_1, \dots, v_{n-1}) = \mathbb{F}_p[v_n^{\pm 1}].$$

These functors are particularly important for analyzing the homotopy groups of spheres or other finite spectra, for the following reason.

Theorem 2.2.6 (Hopkins-Ravenel chromatic convergence theorem, [**Rav92**])**.** *If X is a finite p-local spectrum, then $X \simeq \operatorname{holim} L_n X$.*

Moreover, there are homotopy pullback squares

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X. \end{tikzcd}$$

Thus, in principle, the homotopy type of a finite spectrum can be computed from its $K(n)$-localizations and some 'transchromatic' information about attaching maps between localizations at different chromatic heights. In practice, every part of this is extremely difficult. If these brief few paragraphs have piqued the reader's interest, they are highly encouraged to learn more from resources such as [**Rav04, Rav92, Go08, Lu10**].

2.2.6. Moduli of isomorphisms

Definition 2.2.7. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two formal groups over rings a ring A. The **moduli of isomorphisms** from $\mathbb{G}_1$ to $\mathbb{G}_2$ is the functor $\operatorname{Iso}(\mathbb{G}_1, \mathbb{G}_2) : \mathsf{Alg}_A \to \mathsf{Sets}$ given by

$$\operatorname{Iso}(\mathbb{G}_1, \mathbb{G}_2)(B) = \{\phi : \mathbb{G}_1 \otimes_A B \xrightarrow{\sim} \mathbb{G}_2 \otimes_A B \text{ an isomorphism of formal groups }\}.$$

This functor is representable by an A-scheme: this is part of the fact that $\mathcal{M}_{\text{fg}}$ is an algebraic stack. In fact, if $\mathbb{G}_1$ and $\mathbb{G}_2$ are both equipped with coordinates, then the Hopf algebroid presentation of $\mathcal{M}_{\text{fg}}$ gives a representation of $\text{Iso}(\mathbb{G}_1, \mathbb{G}_2)$ by an affine scheme,

$$\text{Iso}(\mathbb{G}_1, \mathbb{G}_2) = \text{Spec}(A \otimes_{A \otimes A} (A \otimes^{\mathbb{G}_1}_{MUP_0} MUP_0 MUP \otimes^{\mathbb{G}_2}_{MUP_0} A)),$$

or, if A is p-local,

$$\text{Iso}(\mathbb{G}_1, \mathbb{G}_2) = \text{Spec}(A \otimes_{A \otimes A} (A \otimes^{\mathbb{G}_1}_{BPP_0} BPP_0 BPP \otimes^{\mathbb{G}_2}_{BPP_0} A)).$$

Note that the maps used to define the tensor product are those classifying formal group laws for $\mathbb{G}_1$ and $\mathbb{G}_2$.

Theorem 2.2.8. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be height n formal groups over a field k of characteristic p. Then $\text{Iso}(\mathbb{G}_1, \mathbb{G}_2)$ is pro-étale over k.*

See [**Go08**, 5.29] for a proof. We note the following obvious corollary, which we will use repeatedly:

Corollary 2.2.9. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be height n formal groups over a field k of characteristic p, and let R be a a k-algebra which is complete with respect to an ideal I. Then $\text{Iso}_R(\mathbb{G}_1, \mathbb{G}_2) \to \text{Iso}_{R/I}(\mathbb{G}_1, \mathbb{G}_2)$ is an isomorphism.*

Proof. If I is nilpotent, this is just the infinitesimal criterion of étaleness. In general, we write $R = \lim R/I^n$ and note that the moduli of isomorphisms commutes with limits. $\square$

2.3. Deformations of formal groups

The Lubin-Tate theorem is originally stated in the following form [**LT66**]:

Theorem 2.3.1 (Lubin-Tate)**.** *Let Γ be a formal group of finite height n over a field k of characteristic $p > 0$, let A be a ring with an ideal I such that $A/I \cong k$, and let $i : A \to R$ be a map to a complete local noetherian ring R sending I into the maximal ideal of R, thus inducing a map on residue fields $i : k \to R/\mathfrak{m}$. Then there is a formal group $\mathbb{G}^u$ over $A[[u_1, \ldots, u_{n-1}]]$ such that, for any formal group $\mathbb{G}$ over R with an isomorphism $\alpha : \Gamma \otimes_k R/\mathfrak{m} \xrightarrow{\sim} \mathbb{G} \otimes_R R/\mathfrak{m}$, there is a unique continuous A-algebra map $f : A[[u_1, \ldots, u_{n-1}]] \to R$ and a unique isomorphism $f^*\mathbb{G}^u \xrightarrow{\sim} \mathbb{G}$ that reduces to α over $R/\mathfrak{m}$.*

We can restate this in more modern terms as follows.

Definition 2.3.2. Let Γ be a formal group over k and R a complete local noetherian ring with maximal ideal $\mathfrak{m}$. A **deformation** of Γ over R is a triple

$$(\mathbb{G}, \bar{i}, \alpha),$$

where $\mathbb{G}$ is a formal group over R, $\bar{i}$ is an inclusion $k \to R/\mathrm{m}$, and α is an isomorphism $\Gamma \otimes_k^{\bar{i}} R/\mathfrak{m} \xrightarrow{\sim} \mathbb{G} \otimes_R R/\mathrm{m}$.

A **$\star$-isomorphism** $\phi : (\mathbb{G}_1, \overline{i_1}, \alpha_1) \to (\mathbb{G}_2, \overline{i_2}, \alpha_2)$ is the requirement that $\overline{i_1} = \overline{i_2}$ and an isomorphism $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ of formal groups over R, such that the square

$$\begin{tikzcd}
\Gamma \otimes_k R/\mathfrak{m} \arrow[r, "\alpha_1"] \arrow[d, "1"] & \mathbb{G}_1 \otimes_R R/\mathfrak{m} \arrow[d, "\phi"'] \\
\Gamma \otimes_k R/\mathfrak{m} \arrow[r, "\alpha_2"'] & \mathbb{G}_2 \otimes_R R/\mathfrak{m}
\end{tikzcd}$$

commutes.

Finally, let A be as above, and let CLN_A be the category of complete local noetherian A-algebras and continuous maps. If Γ is a formal group over k, then let

$$\mathsf{Def}_\Gamma^A : \mathsf{CLN}_A \to \mathsf{Gpd}$$

be the functor that sends an A-algebra R to the groupoid of deformations of Γ over R such that $\bar{i} : A/I \to R/\mathfrak{m}_R$ is the reduction of the A-algebra structure map, and $\star$-isomorphisms.

Corollary 2.3.3 (Equivalent form of Theorem 2.3.1)**.** *The functor* Def_Γ^A *on* CLN_A *is pro-represented by* $\operatorname{Spf} A[[u_1, \ldots, u_{n-1}]]$.

Definition 2.3.4. Let CLN be the category of complete local rings. Let Def_Γ be the sheaf of groupoids on CLN that sends $R \in \mathsf{CLN}$ to the groupoid of deformations of Γ over R and $\star$-isomorphisms.

Theorem 2.3.5. *If k is perfect, then for any $R \in \mathsf{CLN}$ and $i : k \to R/\mathfrak{m}$, there is a unique continuous map completing the diagram*

$$\begin{tikzcd} Wk \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k \arrow[r] & R/\mathfrak{m}. \end{tikzcd}$$

We will prove this in Section 3.2.

Corollary 2.3.6. *If k is perfect, then Def_Γ is pro-represented by* $\operatorname{Spf} Wk[[u_1, \ldots, u_{n-1}]]$.

Proof. Given $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$, Theorem 2.3.5 implies that R has a unique continuous Wk-algebra structure such that $(\mathbb{G}, i, \alpha)$ is an object of $\mathsf{Def}_\Gamma^{Wk}(R)$. Thus, its $\star$-isomorphism class is represented by a unique map $Wk[[u_1, \ldots, u_{n-1}]] \to R$, and it admits no nontrivial $\star$-automorphisms. Since $\star$-isomorphisms as objects of $\mathsf{Def}_\Gamma(R)$ are the same as $\star$-isomorphisms as objects of $\mathsf{Def}_\Gamma^{Wk}(R)$, this completes the proof. $\square$

Remark 2.3.7. The ring $Wk[[u_1, \ldots, u_{n-1}]]$ is called the **Lubin-Tate ring** for (k, Γ). As a result of Corollary 2.3.6, it carries a deformation $(\mathbb{G}^u, 1, \alpha^u)$ which is a universal deformation of Γ, in the sense that any other deformation $(\mathbb{G}, i, \alpha)$ over $R \in \mathsf{CLN}$ is uniquely $\star$-isomorphic to the base change of $\mathbb{G}^u$ along a unique map $Wk[[u_1, \ldots, u_{n-1}]] \to R$.

In fact, this can be made fairly explicit. If Γ has the Honda formal group law over k, with p-series

$$[p]_\Gamma(x) = x^{p^n}$$

(for some chosen coordinate x), then we can choose a coordinate on $\mathbb{G}^u$ such that

$$[p]_{\mathbb{G}^u}(x) = px +_{\mathbb{G}^u} u_1 x^p +_{\mathbb{G}^u} \cdots +_{\mathbb{G}^u} u_{n-1} x^{p^{n-1}} +_{\mathbb{G}^u} x^{p^n}.$$

The isomorphism $\alpha^u : \Gamma \xrightarrow{\sim} \mathbb{G}^u \otimes k$ matches these two coordinates.

In other words, a deformation of a height n formal group is specified, up to $\star$-isomorphism, by deformations of the coefficients of $x^p, \ldots, x^{p^{n-1}}$ in its p-series, and the Lubin-Tate parameters keep track of these deformations. This should indicate that the parameters $u_1, \ldots, u_{n-1}$ themselves are very non-canonical, and are related to the choice of coordinate on the formal group Γ.

Remark 2.3.8. There is a left action of $\mathrm{Aut}(k, \Gamma)$ on Def_Γ, defined as follows. Given $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$ and $(\tau : k \to k, g : \Gamma \to \tau^*\Gamma) \in \mathrm{Aut}(k, \Gamma)$ (see Definition 2.2.5), define

$$(\tau, g)(\mathbb{G}, i, \alpha) = (\mathbb{G}, i \circ \tau, \alpha g^{-1} : \Gamma \otimes_k^{i\tau} R/\mathfrak{m} \xrightarrow{g^{-1}} \Gamma \otimes_k^{i} R/\mathfrak{m} \xrightarrow{\alpha} \mathbb{G} \otimes_R R/\mathfrak{m}).$$

2.4. Morava E-theory

If Γ is a height n formal group over a perfect field k, its universal deformation is classified by a map $Wk[[u_1, \ldots, u_{n-1}]] \to \mathcal{M}_{\mathrm{fg}}$. This map is flat, which ultimately reduces to the fact that $(p, u_1, \ldots, u_{n-1})$ is a regular sequence [**Go08**]. The existence of **Morava E-theory** then follows from the Landweber exact functor theorem, as follows.

Theorem 2.4.1. *There is a complex orientable, even periodic ring spectrum $E = E(k, \Gamma)$ such that $\pi_0 E = Wk[[u_1, \ldots, u_{n-1}]]$ and the formal group of E (rescaled to degree zero) is the universal deformation of Γ.*

Some of these spectra can be constructed more homotopically. One defines a ring spectrum $BP\langle n\rangle$ as the quotient of BP by the ideal $(v_{n+1}, v_{n+2}, \ldots)$ [**EKMM**]. The Johnson-Wilson E-theory $E(n)$ is then $BP\langle n\rangle[v_n^{-1}]$. This admits an étale extension in ring spectra, $\widetilde{E(n)}$, with

$$\pi_*\widetilde{E(n)} = \pi_* E(n)[u]/(u^{(p^n-1)} - v_n) = \mathbb{Z}_{(p)}[u_1, \ldots, u_{n-1}][u^{\pm 1}]$$

where $|u| = 2$ and $|u_i| = 0$, and $u_i = v_i u^{1-p^i}$. Finally, the $K(n)$-localization of $\widetilde{E(n)}$ is a Morava E-theory. Specifically, it is the Morava E-theory for the Honda formal group of height n over $\mathbb{F}_p$, with

$$[p]_\Gamma(x) = x^{p^n}.$$

Both constructions lack a certain something. Both the Landweber exact functor theorem, and the process of quotienting ring spectra by ideals, only produce ring objects in the stable homotopy category. This does not let us use power operations or most of the machinery of derived algebraic geometry. More advanced obstruction theory techniques imply the following, for which one should see [**GH04**] and [**Re98**].

Theorem 2.4.2 (Goerss-Hopkins-Miller)**.** *There is a unique E_∞ ring spectrum, up to E_∞ homotopy equivalence, whose underlying ring spectrum is $E(k, \Gamma)$. Moreover, the space of E_∞ endomorphisms of this spectrum is homotopy equivalent to the discrete group* $\mathrm{Aut}(k, \Gamma)$.

There is thus a canonical way to lift very locally defined formal groups, namely the universal deformations of finite height formal groups over perfect fields, from algebraic geometry into homotopy theory. In this very local setting, the relationship between stable homotopy theory and algebraic geometry is as close as it can possibly be. Conversely, the Goerss-Hopkins-Miller theorem suggests that *every* topological fact about E-theory should mean something in terms of formal groups. This prophecy has been answered, for example, by [**Re09**] and [**Sta13**].

CHAPTER 3

Complements on completion

3.1. Derived completion

The coefficient ring of E-theory is complete and local, and so one would like to think of the E-homology and cohomology of a finite complex as valued in E_*-modules which are *complete* for the maximal ideal I_n. However, problems arise from the fact that the category of complete modules is not closed under colimits.

First, while an infinite direct product of complete modules is complete, an infinite direct sum is not – thus, while the E-cohomology of an infinite wedge of spheres is complete, its E-homology is not. This suggests replacing E-homology with the better-behaved *completed homology* functor

$$X \mapsto E^{\wedge}_* X = \pi_* L_{K(n)}(E \wedge X).$$

This functor is better-behaved in several ways: for example, while E_*E is quite complicated, $E^{\wedge}_*E$ is pro-free over E_* and has the simple Hopf algebra description $E^{\wedge}_*E \cong \mathrm{Hom}_{cts}(\mathbb{G}_n, E)$.

Second, the cokernel of a map of complete modules may not be complete. Thus, the cohomology of a spectrum with infinitely many cells may not be complete.

As it turns out, E-cohomology and completed E-homology are valued in a wider category of *L-complete E_*-modules*. That is, they are fixed by the zeroth left derived functor of completion, which is not the completion functor itself, as this functor is not right exact. Here, we first review the relevant definitions, and then show that LE-homology is valued in a similar category. The reader is directed to [**HSt99**], [**BF15**], and [**GM92**] for more detailed information.

Definition 3.1.1. Let R be a ring and I an ideal in R. We write $M^\wedge$ for the I-adic completion of an R-module M, and L_sM for the sth derived functor of I-adic completion applied to M. There are natural maps $M \to L_0M \to M^\wedge$. A module M is **L-complete** if the natural map $M \to L_0M$ is an isomorphism, and **complete** if the map $M \to M^\wedge$ is an isomorphism.

(In particular, complete objects will always be separated and complete with respect to some ideal.)

In what follows, we will assume that *R is Noetherian and I is generated by a regular sequence of length n.* This is less general than the hypotheses in [**GM92**] (they are able to weaken the Noetherian condition), and more general than those in [**HSt99**] and [**BF15**] (they both assume that R is local and I its maximal ideal). Nevertheless, the proofs in [**HSt99**] and [**BF15**] do not rely on I being maximal.

The following results are collected from [**GM92**].

Proposition 3.1.2. *Suppose that R is Noetherian and I is generated by a finite regular sequence of length n. Then:*

(1) $L_sM = 0$ for $s \geq n+1$.

(2) If M is finitely generated, then the natural map $M^\wedge \to L_0M$ is an isomorphism.

(3) If M is I-adically complete, or of the form L_sN for some N and s, then it is L-complete.

(4) If M is L-complete, then $L_sM = 0$ for $s > 0$.

(5) There are natural exact sequences

$$0 \to \lim^1_k \operatorname{Tor}^R_{s+1}(R/I^k, M) \to L_sM \to \lim_k \operatorname{Tor}^R_s(R/I^k, M) \to 0.$$

Let $\mathsf{Mod}^{\wedge}_R$ be the category of L-complete R-modules. The following results are collected from [**HSt99**, Appendix A]; though stated for the case where R is local and I is its maximal ideal, they are true in this more general setting with the same proof.

Proposition 3.1.3. *There is an adjunction $L_0 : \mathsf{Mod}_R \leftrightarrows \mathsf{Mod}^{\wedge}_R : i$, where i is the inclusion. The category $\mathsf{Mod}^{\wedge}_R$ is an exact subcategory of Mod_R, closed under extensions and limits in Mod_R. Furthermore, if $\cdots \to M_2 \to M_1 \to M_0$ is a diagram in $\mathsf{Mod}^{\wedge}_R$, then $\lim^1 M_i \in \mathsf{Mod}^{\wedge}_R$.*

The main idea in [**GM92**] is a duality between derived completion and local cohomology, as follows. For $x \in R$, let $K^\bullet(x)$ be the cochain complex $R \xrightarrow{x} R$, concentrated in degrees 0 and 1. Given a sequence of elements $\underline{x} = (x_0, \ldots, x_{n-1})$, let

$$K^\bullet(\underline{x}) = K^\bullet(x_0) \otimes \cdots \otimes K^\bullet(x_{n-1}).$$

There are natural maps $K^\bullet(x^r) \to K^\bullet(x^{r+1})$, given by the diagrams

$$\begin{tikzcd} R \arrow[r, "x^r"] \arrow[d, "1"'] & R \arrow[d, "x"] \\ R \arrow[r, "x^{r+1}"'] & R. \end{tikzcd}$$

We let $K^\bullet(\underline{x^\infty}) = \operatorname{hocolim}_r K^\bullet(\underline{x^r})$.

Theorem 3.1.4 ([**GM92**, Theorem 2.5])**.** *If I is generated by the regular sequence $x_0, \ldots, x_{n-1}$, then there is a quasi-isomorphism*

$$L_\bullet M \simeq \operatorname{Hom}(K^\bullet(\underline{x^\infty}), M).$$

This shows why the derived completion is concentrated in degrees at most n, and gives an explicit model for computing it.

Greenlees and May thus call the derived completion 'local homology', as the complex $K^\bullet(\underline{x^\infty})$ also computes local *cohomology*:

$$H_I^*(M) = H^*(K^\bullet(\underline{x^\infty}) \otimes M).$$

In particular,

$$H_I^*(R) = H^*(K^\bullet(\underline{x^\infty})).$$

A composite functor spectral sequence implies the following.

Corollary 3.1.5. *There is a spectral sequence*

$$E_2^{pq} = \operatorname{Ext}_R^p(H_I^q(R), M) \Rightarrow L_{q-p}M.$$

3.2. Witt vectors and Cohen rings

Notation 3.2.1. In this section, we write CLN for the category of complete local Noetherian rings and continuous maps, and we let k be a field of characteristic p.

We start by defining the Witt vectors, which are ubiquitous in p-adic deformation theory, as well as being important in the study of Frobenius lifts. For more information, see [**Hes08**] or [**Rab14**].

Definition 3.2.2. The nth **ghost component** is the polynomial

$$\mathbf{w}_n(a_0, \ldots, a_n) = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n.$$

For any ring R, these assemble to a map

$$\mathbf{W} = \prod \mathbf{w}_n : R^{\mathbb{N}} \to R^{\mathbb{N}}.$$

The Witt vectors of R, $W(R)$, are the set $R^{\mathbb{N}}$ equipped with a functorial ring structure that makes $\mathbf{W}$ a ring homomorphism. If R is a $\mathbb{Q}$-algebra, one can recover an element of the domain from its ghost components, so this is clearly well-defined and $W(R) \cong R^{\mathbb{N}}$ as rings. If R has characteristic p, then the ghost components only see the zeroth coordinate, and are useless for defining a ring structure. The key word is 'functorial': the right ring structure can be obtained by looking at maps from other rings. If R is p-torsion-free, the ghost component map is at least injective, meaning that at most one compatible ring structure exists. That one actually does is a consequence of the following fact, which we will not prove.

Lemma 3.2.3 (Dwork). *Suppose that R is a p-torsion-free ring which admits a Frobenius lift (a ring homomorphism $\psi^p : R \to R$ such that $\psi^p(x) \cong x \bmod p$). Then a sequence $(a_i) \in R^{\mathbb{N}}$ is in the image of $\mathbf{W}$ if and only if $a_n \cong \psi^p(a_{n-1}) \bmod p^n$, for each n.*

Since ψ^p is a ring homomorphism, the set of (a_i) satisfying this condition is obviously closed under addition and multiplication. Thus, in the universal case $R = \mathbb{Z}[x_0, x_1, \ldots, y_0, y_1, \ldots]$, there are polynomials S_n and P_n with

$$S_n(x_0, \ldots, x_n, y_0, \ldots, y_n) = \mathbf{w}_n(x_0, \ldots, x_n) + \mathbf{w}_n(y_0, y_1, \ldots, y_n),$$

$$P_n(x_0, \ldots, x_n, y_0, \ldots, y_n) = \mathbf{w}_n(x_0, \ldots, x_n)\mathbf{w}_n(y_0, \ldots, y_n).$$

Definition 3.2.4. The **Witt vectors** of R, $W(R)$, are the set $R^{\mathbb{N}}$ equipped with addition and multiplication

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (S_0(\underline{a}, \underline{b}), S_1(\underline{a}, \underline{b}), \ldots),$$

$$(a_0, a_1, \ldots)(b_0, b_1, \ldots) = (P_0(\underline{a}, \underline{b}), P_1(\underline{a}, \underline{b}), \ldots).$$

The **truncated Witt vectors** $W_n(R)$ are the set $R^{\{0,\ldots,n-1\}}$ with addition and multiplication given by the analogous formulas.

This ring comes with a lot of additional structure, which we will mention briefly. There is a ring map $F : W(R) \to W(R)$, the **Frobenius**, such that if (a_i) has ghost components $(\mathbf{w}_0, \mathbf{w}_1, \ldots)$, then $F(a_i)$ has ghost components $(\mathbf{w}_1, \mathbf{w}_2, \ldots)$. This is a lift of the Frobenius map on $W(R)/pW(R)$.

There is an additive map $V : W(R) \to W(R)$, the **Verschiebung**, defined by $V(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$. This satisfies $FV = p$ and $V(xF(y)) = V(x)y$. There is also a multiplicative map $[\cdot] : R \to W(R)$, the **Teichmüller lift**, defined by $[a] = (a, 0, 0, \dots)$. Eveyr element of $W(R)$ has a unique expansion of the form $\sum_{n=0}^{\infty} V^n[b_n]$, with $b_n \in R$.

If k is a perfect field of characteristic p, then $W(k)$ is a very well-behaved ring. This will be the primary place where we use the Witt vectors.

Proposition 3.2.5 (cf. [**Rab14**]). *If k is a perfect field of characteristic p, then $W(k)$ is a p-torsion-free complete local ring with maximal ideal $VW(k) = pW(k)$ and residue field k. Every element of $W(k)$ has a unique expansion of the form $\sum_{n=0}^{\infty} p^n[b_n]$, with $b_n \in R$.*

The Witt vectors of a perfect characteristic p field enjoy the universal property of Theorem 2.3.5. It will help to prove a slightly more general version of this, as follows.

Theorem 3.2.6. *Let k be a perfect field of characteristic p and let R be a ring which is complete with respect to an ideal I that contains p. Then for any map $i : k \to R/I$, there exists a unique continuous map completing any diagram of the form*

$$\begin{tikzcd} Wk \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k \arrow[r, "i"'] & R/I. \end{tikzcd}$$

Proof. This proof originally goes back to Cartier, and one should consult [**Se79**, §II.4-6]. Recall that elements of Wk for k perfect can be uniquely written in the form $\sum [a_n]p^n$, where $[a_n]$ is the Teichmüller lift of $a_n \in k$. The idea is that there is a unique multiplicative lift $\tau : k^\times \to R^\times$. One is then forced to send $\sum [a_n]p^n$ to $\sum \tau(a_n)p^n$, which converges by completeness of R.

Regard k as a subring of R/I. For each $a \in k$, define

$$U_n(a) = \{x^{p^n} : x \in R,\, x \equiv a^{p^{-n}} \pmod{I}\}.$$

Here $a^{p^{-n}}$ is the unique p^nth root of a in k. We have $U_{n+1}(a) \subseteq U_n(a)$. Moreover, if x^{p^n} and y^{p^n} are elements in $U_n(a)$, then $x \equiv y \bmod I$, and thus $x^{p^n} \equiv y^{p^n} \bmod I^{n+1}$ using the binomial theorem and the fact that $p \in I$. By completeness of R, there is a unique element in $\bigcap_{n \geq 0} U_n(a)$. Call this $\tau(a)$.

One now observes that $\tau(a^p) = \tau(a)^p$, and that τ is the unique section $k^\times \to R^\times$ with this property. Indeed, if τ' also has this property, then

$$\tau'(a) = \tau'(a^{p^{-n}}) \in U_n(a) \text{ for all } n,$$

so $\tau'(a) = \tau(a)$. Thus, there is *at most one* multiplicative section. But τ is also multiplicative, because $U_n(a) \cdot U_n(b) \subseteq U_n(ab)$. $\square$

Remark 3.2.7. Theorem 2.3.5 is just the case when I is a maximal ideal of a complete local ring R. One can see this is as a trivial case of deformation theory. Indeed, define the functor of **deformations of nothing**, $\mathsf{Def} : \mathsf{CLN} \to \mathsf{Sets}$, by

$$\mathsf{Def}(R) = \{i : k \to R/\mathfrak{m}\}.$$

Then we have proved that $\mathsf{Def}(R) \cong \mathrm{Hom}_{cts}(Wk, R)$.

However, the above theorem proves slightly more. A map $k \to R/I$ gives deformations of nothing over numerous complete local rings, namely the completions of the localizations of R at maximal ideals containing I. The theorem implies that these deformations of nothing assemble over $\operatorname{Spf} R$ to give a unique deformation of nothing over everything, which is really something.

If k is a non-perfect characteristic p field, its ring of Witt vectors is harder to get a handle on. There is still a surjection $\mathbf{w}_0 : W(k) \to k$ with kernel $VW(k)$, but this ideal need not be principal, and $V(1) \neq p$. In addition, the universal property of Theorem 2.3.5 is not satisfied. However, there is still a weak version. (Topologists wishing to know more should also consult the last section of [**AMS98**]).

Definition 3.2.8. A **Cohen ring** for a characteristic p field k is a complete DVR with residue field k and uniformizer p.

Example 3.2.9. The Witt vectors of a perfect field k are a Cohen ring for k. For an imperfect field, we have $px = V(1)x = V(F(x))$, so the set of multiples of p is in general a proper subset of the maximal ideal VWk. Thus, the Witt vectors are not a Cohen ring in this case.

Example 3.2.10. If k is perfect, the ring $\Lambda = W\mathbb{F}_q((x))^\wedge_p$ is a Cohen ring for the field $k((x))$.

Theorem 3.2.11. *Every characteristic p field k has a Cohen ring.*

Proof. We follow the Zorn's lemma argument in [**Stacks**, Tag 0323]. Given a field k of characteristic p, consider the category $\mathcal{C}$ of pairs (k_1, C_1), where k_1 is a subfield of k

and $C_1 \twoheadrightarrow k_1$ is a Cohen ring. A map $(k_1, C_1) \to (k_2, C_2)$ is an inclusion $k_1 \subseteq k_2$, together with a map $C_1 \to C_2$ making the square

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_2 \arrow[d] \\ k_1 \arrow[r, hook] & k_2 \end{tikzcd}$$

commute. Although this category will turn out to be essentially small, one can make it small by requiring all Cohen rings to be completions of quotients of subrings of a fixed polynomial ring over $\mathbb{Z}_p$ generated by $|k|$ many indeterminates. The constructions below will demonstrate that only this many indeterminates are ever necessary.

The category $\mathcal{C}$ is nonempty, because $\mathbb{F}_p$ has the Cohen ring $\mathbb{Z}_p$.

If k_2 is generated over k_1 by a single element $\alpha \in k$, and k_1 has the Cohen ring C_1, then we define a Cohen ring C_2 for k_2 as follows. If α is transcendental over k_1, then

$$C_2 = C_1[x]^\wedge_p.$$

The ideal (p) is a prime in $C_1[x]$, and is height 1 by Krull's Hauptidealsatz. Thus, C_2 is a complete, one-dimensional, local ring, and so it is a Cohen ring for k_2 (via the map that sends x to α).

If α is algebraic over k_1, satisfying the minimal polynomial

$$f(T) = T^d + c_1 T^{d-1} + \cdots + c_d,$$

then let

$$C_2 = C_1[x]/(x^d + \tilde{c}_1 x^{d-1} + \cdots + \tilde{c}_d)^\wedge_p,$$

where $\tilde{c}_i$ is any lift of c_i to C_1. This again has a map to k_2 under C_1 that sends x to α. Any maximal ideal of C_2 will intersect C_1 in a prime ideal, so again, C_2 is complete and local with maximal ideal (p), and thus a Cohen ring for k_2.

Finally, suppose given a filtered diagram $F : I \to \mathcal{C}$ with $F(i) = (k_i, C_i)$. Define $k' = \bigcup_{i \in I} k_i$, $\tilde{C}' = \operatorname{colim}_{i \in I} C_i$, and $C' = (\tilde{C}')^\wedge_p$. Observe that if x is not a multiple of p in $\tilde{C}'$, and x is in the image of some C_i, then x isn't a multiple of p in C_i, so it's a unit in C_i and thus a unit in $\tilde{C}'$. It follows that $\tilde{C}'$ is a local ring with maximal ideal (p), thus a DVR, and so C' is a Cohen ring. The map $C' \to k'$ is surjective, and so its kernel must be (p). Thus, R' is a Cohen ring for k'. It is clear that $(k', C') = \operatorname{colim}_I F$.

Thus, $\mathcal{C}$ is nonempty and admits small filtered colimits. By a categorical version of Zorn's lemma [**KS06**, Theorem 9.4.2], it has a weakly terminal object, i. e. an object (k', C') such that any map $(k', C') \to (k'', C'')$ has a left inverse. In particular, for any map $(k', C') \to (k'', C'')$, $k' = k''$. By what we've shown, if k' is any proper subfield of k, then there exists a map $(k', C') \to (k'(\alpha), C'')$ where $\alpha \notin k'$. Thus, we must have $k' = k$, which means that k has a Cohen ring. $\square$

Proposition 3.2.12. *If C is a Cohen ring for k, then the map $\mathbb{Z}/p^n\mathbb{Z} \to C/p^nC$ is formally smooth for each n.*

Proof. When $n = 1$, $\mathbb{F}_p \to C/pC = k$ is a separably generated field extension, thus in particular formally smooth (see [**Stacks**, Tag 0322]). In general, observe that C is p-torsion-free and thus flat over $\mathbb{Z}_p$, and in fact faithfully flat because p is not invertible in C. Thus, $C/p^{n+1}C$ is flat over $\mathbb{Z}/p^{n+1}\mathbb{Z}$. The ideal (p^n) squares to zero in $\mathbb{Z}/p^{n+1}$, and by induction, we can assume that $\mathbb{Z}/p^n\mathbb{Z} \to C/p^nC$ is formally smooth. The same follows for $n+1$ by [**Stacks**, Tag 031L]. $\square$

Corollary 3.2.13. *If C is a Cohen ring for k, R is a ring which is complete with respect to an ideal I containing p, and $i : k \to R/I$ is an inclusion, then there exists a map completing the diagram*

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k \arrow[r, "i"'] & R/I. \end{tikzcd}$$

Proof. Starting with the map $C/p = k \to R/I$, we use formal smoothness to inductively construct lifts

$$\begin{tikzcd} \mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d] & R/I^n \arrow[d] \\ C/p^nC \arrow[r] \arrow[ur, dashed] & R/I^{n-1} \end{tikzcd}$$

along the square-zero extensions $R/I^n \to R/I^{n-1}$. By completeness, these assemble to a map $C \to R$ with the desired property. $\square$

Corollary 3.2.14. *Any two Cohen rings for k are (non-uniquely) isomorphic.*

Proof. Let C_1 and C_2 be Cohen rings for k. By the previous corollary, there exists a map $f : C_1 \to C_2$ reducing to the identity on k. If $f(x) = 0$, then x must be divisible by p, and writing $x = px_1$ and proceeding inductively, we see that x is divisible by all powers of p, so is zero by completeness. Thus, f is injective. If $y \in C_2$, then there is an $x_0 \in C_2$ such that $f(x_0) - y$ is divisible by p, so is equal to some py_1. Proceeding inductively again and using completeness, we see that f is surjective. $\square$

3.3. Pipe phenomena

We record some basic definitions concerning pipe rings here, and refer to [**MGPS**] and [**Ka00**] for further details. Throughout this section, n will always be a nonnegative integer or -1, and r will be a nonnegative half-integer or -1.

Definition 3.3.1. Let $\mathcal{C}$ be a category. The category of **ind-objects** in $\mathcal{C}$, $\mathsf{Ind}(\mathcal{C})$, has as its objects diagrams $I \to \mathcal{C}$ where I is a filtered category. We will generally write $\operatorname{colim}_{\alpha \in I} X_\alpha$ for such a diagram – bearing in mind that the diagram is what is meant, and not its colimit, which may not exist in $\mathcal{C}$. The morphism sets are defined by

$$\mathrm{Hom}_{\mathsf{Ind}(\mathcal{C})}(\operatorname*{colim}_{\alpha} X_\alpha, \operatorname*{colim}_{\beta} Y_\beta) = \lim_{\alpha} \operatorname*{colim}_{\beta} \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Likewise, the category $\mathsf{Pro}(\mathcal{C})$ of **pro-objects** has as its objects diagrams $\lim_{\alpha \in I} X_\alpha$, where I is a cofiltered category. The morphism sets are defined by

$$\mathrm{Hom}_{\mathsf{Pro}(\mathcal{C})}(\lim_{\alpha} X_\alpha, \lim_{\beta} Y_\beta) = \lim_{\beta} \operatorname*{colim}_{\alpha} \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

There are functors $\mathcal{C} \to \mathsf{Ind}(\mathcal{C})$ and $\mathcal{C} \to \mathsf{Pro}(\mathcal{C})$, sending an object X to the diagram $\{*\} \to \mathcal{C}$ with image X. Moreover, $\mathsf{Ind}(\mathcal{C})$ has all filtered colimits, and $\mathsf{Pro}(\mathcal{C})$ has all filtered limits.

Definition 3.3.2. The categories of r-**pipes** are defined by:

- $\mathsf{Pipes}_{-\frac{1}{2}}$ is the category of finite sets,
- $\mathsf{Pipes}_0 = \mathsf{Pro}(\mathsf{Pipes}_{-\frac{1}{2}})$ is the category of profinite sets,
- for n a nonnegative integer, $\mathsf{Pipes}_{n+\frac{1}{2}} = \mathsf{Ind}(\mathsf{Pipes}_n)$,

- and for n a nonnegative integer, $\mathsf{Pipes}_{n+1} = \mathsf{Pro}(\mathsf{Pipes}_{n+\frac{1}{2}})$.

The categories Pipes_r have finite products. We write $\mathsf{PipeRings}_r$ for the category of ring objects in Pipes_r. There are product-preserving, fully faithful functors $\mathsf{Pipes}_r \to \mathsf{Pipes}_{r+\frac{1}{2}}$, which induce functors $\mathsf{PipeRings}_r \to \mathsf{PipeRings}_{r+\frac{1}{2}}$. We write $\mathsf{Pipes} = \operatorname{colim} \mathsf{Pipes}_r$ and $\mathsf{PipeRings} = \operatorname{colim}_r \mathsf{PipeRings}_r$.

There is a terminal pipe 1, and a realization functor $\mathsf{Pipes} \to \mathsf{Sets}$ (and $\mathsf{PipeRings} \to \mathsf{Rings}$) defined by

$$X \mapsto \operatorname{Re}(X) = \operatorname{Hom}_{\mathsf{Pipes}}(1, X).$$

More concretely, the realization functor takes colimits and limits along the diagrams defining X, in the category of sets. That is, the realization of a $(-\frac{1}{2})$-pipe is itself, and for higher values of pipiness,

$$\operatorname{Re}(\operatorname*{colim}_{\alpha} X_\alpha) = \operatorname*{colim}_{\alpha} \operatorname{Re}(X_\alpha), \quad \operatorname{Re}(\lim_{\alpha} X_\alpha) = \lim_{\alpha} \operatorname{Re}(X_\alpha).$$

This functor is in general neither injective on objects, nor faithful, but it is under certain conditions.

Definition 3.3.3. Every $(-\frac{1}{2})$-pipe and 0-pipe is **fine**. For $n \geq 0$, an $(n+\frac{1}{2})$-pipe

$$Y = \operatorname*{colim}_{\beta} Y_\beta$$

is **fine** if each Y_β is a fine n-pipe, and the maps on realizations

$$\underline{Y_\beta} \to \underline{Y}$$

are injective. An $(n+1)$-pipe

$$Y = \lim_{\alpha} Y_\alpha$$

is **fine** if each Y_α is a fine $(n+\frac{1}{2})$-pipe.

Every $(-\frac{1}{2})$-pipe is **cofine**. A 0-pipe is **cofine** if it is a limit of finite sets along surjective maps. For $n \geq 0$, an $(n+\frac{1}{2})$-pipe

$$X = \operatorname*{colim}_{\lambda} X_\lambda$$

is **cofine** if each X_λ is a cofine n-pipe. An $(n+1)$-pipe

$$X = \lim_{\mu} X_\mu$$

is **cofine** if each X_μ is a cofine $(n+\frac{1}{2})$-pipe, and the maps on realizations

$$\underline{X} \to \underline{X_\mu}$$

are surjective.

Remark 3.3.4. Note that the usage of 'cofine' and 'fine' is reversed from [**Ka00**], and agrees with [**MGPS**]; the definition corrects a reversal of 'ind' and 'pro' found in [**MGPS**].

Lemma 3.3.5 ([**MGPS**], Lemma 5)**.** *If X is cofine and Y is fine, then the realization map*

$$\mathrm{Hom}_{\mathsf{Pipes}}(X,Y) \to \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Re}(X), \mathrm{Re}(Y))$$

is injective.

Definition 3.3.6. There is a natural topology on the realization of a pipe, given by taking the limits and colimits in the category of topological spaces rather than the category of sets. To be precise:

- If X is a $(-\frac{1}{2})$-pipe, $\mathrm{Re}(X)$ is discrete;
- if X is a 0-pipe, $\mathrm{Re}(X)$ has its profinite topology (which is its limit topology for the limit diagram of $(-\frac{1}{2})$-pipes);
- if $X \in \mathsf{Pipes}_{n+\frac{1}{2}}$ with $X = \operatorname{colim}_\beta X_\beta$ where X_β are n-pipes, then $\mathrm{Re}(X) = \operatorname{colim}_\beta \mathrm{Re}(X_\beta)$ has the colimit topology, i. e., it is topologized as a quotient of the disjoint union $\coprod \mathrm{Re}(X_\beta)$;
- if $X \in \mathsf{Pipes}_{n+1}$ with $X = \lim_\mu X_\mu$ where $X_\mu \in \mathsf{Pipes}_{n+\frac{1}{2}}$, then $\mathrm{Re}(X) = \lim_\mu \mathrm{Re}(X_\mu)$ has the limit topology, i. e., it is topologized as a subspace of $\prod \mathrm{Re}(X_\mu)$.

Lemma 3.3.7. *If X is cofine, Y is fine, and $r \leq \frac{1}{2}$, then the realization map*

$$\mathrm{Hom}_{\mathsf{Pipes}}(X, Y) \to \mathrm{Hom}_{\mathsf{Spaces}}(\mathrm{Re}(X), \mathrm{Re}(Y))$$

is an isomorphism.

Proof. Without loss of generality, both X and Y are r-pipes for some common r. Injectivity is guaranteed by Lemma 3.3.5, so it suffices to prove surjectivity, which we do by induction on r. If $r = -1$, then the statement is obvious. If $r = 0$, then suppose given a continuous map

$$f : \mathrm{Re}(X) \to \mathrm{Re}(Y)$$

of profinite spaces, and presentations $X = \lim X_\lambda$, $Y = \lim Y_\mu$. Let $\pi_\mu : \mathrm{Re}(Y) \to \mathrm{Re}(Y_\mu)$ be the projection; for fixed μ and each $y \in \mathrm{Re}(Y_\mu)$, $f^{-1}\pi_\mu^{-1}(y)$ is open in $\mathrm{Re}(X)$, and is thus a union of sets of the form $\pi_\lambda^{-1}(x)$ for x in a fixed $\mathrm{Re}(X_\lambda)$. Thus, $\mathrm{Re}(X) \to \mathrm{Re}(Y_\mu)$ factors through $\mathrm{Re}(X_\lambda)$. It follows that

$$\mathrm{Hom}_{\mathsf{Spaces}}(\mathrm{Re}(X), \mathrm{Re}(Y)) = \lim_\mu \operatorname*{colim}_\lambda \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Re}(X_\lambda), \mathrm{Re}(Y_\mu)) = \mathrm{Hom}_{\mathsf{Pipes}}(X, Y).$$

For $r = \frac{1}{2}$, write $X = \operatorname{colim} X_\lambda$ and $Y = \operatorname{colim} Y_\mu$. Given a map $f : \mathrm{Re}(X) \to \mathrm{Re}(Y)$, observe that $\mathrm{Re}(X_\lambda)$ is compact (because profinite) and so $f(\mathrm{Re}(X_\lambda))$ is also compact. Thus its cover by $\bigcup_\mu \mathrm{Re}(Y_\mu)$ factors through some finite stage $\mathrm{Re}(Y_\mu)$, and by the case $r = 0$ there is a map of 0-pipes $X_\lambda \to Y_\mu$. Thus, f can be lifted to a map of $\frac{1}{2}$-pipes $X \to Y$. $\square$

Example 3.3.8. The canonical examples are coefficients of iterated localizations of E-theory,

$$\pi_0 L_{K(h_t)} \cdots L_{K(h_1)} E_n, \quad 0 \leq h_t < \cdots < h_1 < n.$$

On coefficients, each localization inverts in a parameter u_{n_i} in the regular sequence $(p, u_1, \ldots, u_{n-1})$, which is a filtered colimit, and then completes at the lower parameters $(p, u_1, \ldots, u_{n_i - 1})$, which is a cofiltered limit. Thus, the above ring is the realization of an m-pipe ring, which is moreover fine and cofine [**MGPS**, Lemma 33].

In situations like these, where the diagrams involved are clear, we will not distinguish the pipe ring, considered as an iterated diagram, from its realization, considered as a topological ring. We will say that a map between realizations of pipe rings is **pipe-continuous** if it is the realization of a map of pipe rings.

In particular, we specialize to the 1-pipe rings

$$\pi_0 L_{K(n-1)} E_n = Wk[[u_1, \ldots, u_{n-1}]][u_{n-1}^{-1}]^{\wedge}_{(p,u_1,\ldots,u_{n-2})} = Wk((u_{n-1}))^{\wedge}_p[[u_1, \ldots, u_{n-2}]].$$

The topology on their realizations is explicitly described as follows [**Morr12**]. We can write elements of this ring as

$$\sum_{i \in \mathbb{Z}} a_i u_{n-1}^i, \quad a_i \in Wk[[u_1, \ldots, u_{n-2}]],$$

where a_i goes to 0 in the maximal ideal topology on $Wk[[u_1, \ldots, u_{n-2}]]$ as i goes to $-\infty$. For each $i \in \mathbb{Z}$, choose an open neighborhood U_i of 0 in $Wk[[u_1, \ldots, u_{n-2}]]$ such that

- $U_i = Wk[[u_1, \ldots, u_{n-2}]]$ for $i >> 0$, and
- all U_i contain some fixed power of the maximal ideal.

Then the set

$$\left\{ \sum a_i u_{h-1}^i : a_i \in U_i \right\}$$

is an open neighborhood of 0 in $\pi_0 L_{K(n-1)} E_n$, and the topology on $\pi_0 L_{K(n-1)} E_n$ is the coarsest one invariant under translation such that these are open neighborhoods of 0.

3.3.1. Local rings and Cohen rings

Definition 3.3.9. For $r \leq r'$, an r**-ideal** of an r'-pipe ring R is the kernel of a surjection $R \to S$ where S is an r'-pipe ring.

Definition 3.3.10. A **complete local 0-pipe ring** is a cofine, fine, profinite complete local ring. A **complete local** $(n+\frac{1}{2})$**-pipe ring** is a cofine and fine $(n+\frac{1}{2})$-pipe ring of the form

$$R[x^{-1}] = \operatorname{colim}\left(R \xrightarrow{x} R \xrightarrow{x} \cdots\right) = \operatorname*{colim}_m x^{-m}R$$

where R is a complete local n-pipe ring and x is a non-zero-divisor of R. A **complete local** $(n+1)$**-pipe ring** is a cofine and fine $(n+1)$-pipe ring formed by a limit of $(n+\frac{1}{2})$-pipe rings of the form $R/\mathfrak{m}^k$, where R is a complete local $(n+\frac{1}{2})$-pipe ring and $\mathfrak{m}$ is a maximal $(n+\frac{1}{2})$-ideal of R.

An $(n+\frac{1}{2})$**-pipe field** is a pipe ring of the form $K = R/\mathfrak{m}$, where R is a complete local $(n+1)$-pipe ring. Note that K is fine and cofine and $\underline{K}$ is a field.

If K is an $(n+\frac{1}{2})$-pipe field, a **Cohen** n**-pipe ring** for K is a complete local $(n+1)$-pipe ring C with maximal $(n+\frac{1}{2})$-ideal pC, such that $\underline{C}$ is a Cohen ring for $\underline{K}$.

Example 3.3.11. Let $K = k((u))$, where k is a finite field. This is a fine and cofine $\frac{1}{2}$-pipe ring, the colimit of the 0-pipes $u^{-m}k[[u]]$. A Cohen 1-pipe ring for k is

$$\Lambda = \lim_n W_n k((u)) = \lim_n \operatorname*{colim}_m u^{-m} W_n k[[u]].$$

Proposition 3.3.12. *If $K = k((u))$ where k is a finite field, $\underline{\Lambda}$ is a Cohen ring and R is a complete local 1-pipe ring with a map of $\frac{1}{2}$-pipe rings $k \to R/\mathfrak{m}$, then there exists a map filling in the diagram*

$$\begin{tikzcd} \Lambda \arrow[r, dashed] \arrow[d] & R \arrow[d] \\ k \arrow[r] & R/\mathfrak{m}. \end{tikzcd}$$

Proof. Since $\underline{\Lambda}$ is a Cohen ring for $\underline{k}$, there exists such a map filling in the analogous diagram of realizations. We construct this map by inductively filling in the diagrams of realizations of $\frac{1}{2}$-pipe rings,

$$\begin{tikzcd} \mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d] & \underline{R/\mathfrak{m}^n} \arrow[d] \\ \underline{\Lambda/p^n\Lambda} \arrow[r] \arrow[ur, dashed] & \underline{R/\mathfrak{m}^{n-1}} \end{tikzcd}$$

By Lemma 3.3.5 and Lemma 3.3.7, it suffices to show that we can always fill in these diagrams with *continuous* maps.

Furthermore, the $\frac{1}{2}$-pipe ring structure on $\underline{\Lambda/p^n\Lambda}$ is obtained from the 0-pipe ring $\mathbb{Z}_q/p^n[[u]]$ by inverting the element u. Thus, to find a continuous lift

$$\underline{\Lambda/p^n\Lambda} \to \underline{R/\mathfrak{m}^n}$$

it suffices to find a map of realizations of $\frac{1}{2}$-pipe rings

$$\underline{\mathbb{Z}_q/p^n[[u]]} \to \underline{R/\mathfrak{m}^n},$$

lifting

$$\underline{\mathbb{Z}_q/p^n[[u]]} \to \underline{R/\mathfrak{m}^{n-1}},$$

such that the map factors through some stage in the colimit diagram for $R/\mathfrak{m}^n$, and such that the image of u becomes invertible in $\underline{R/\mathfrak{m}^n}$. However, this last condition is automatic as long as the map reduces to a map $\underline{k} \to \underline{R/\mathfrak{m}}$. Finally, we have $R = S[x^{-1}]$ for some complete local profinite ring S, and so it suffices to inductively find continuous lifts

$$\begin{tikzcd} \mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d] & S/\mathfrak{m}^n \arrow[d] \\ \mathbb{Z}_q/p^n[[u]] \arrow[r] \arrow[ur, dashed] & S/\mathfrak{m}^{n-1} \end{tikzcd}$$

However, $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_q/p^n[[u]]$ is formally smooth on underlying rings and continuous, and thus it is formally smooth as a map of topological rings, by [**Stacks**, Tag 07EA]. Thus, the desired lifts exist. □

Remark 3.3.13. Though this could have been proved in more generality, fields of the form $\mathbb{F}_q((u))$ are essentially the only $\frac{1}{2}$-pipe fields of characteristic p.

3.4. Three notions of continuity

To sum up, we consider the ring $\Lambda = Wk((x))_p^\wedge$, which will play a key role in the story to follow. This ring admits three different kinds of topology and, correspondingly, three notions of continuity.

(1) The p-adic topology on Λ is a ring topology in the most conventional sense: an adic topology for a maximal ideal. A p-adically continuous map $\Lambda \to R$, where R is a complete local ring, descends to a map $k((x)) \to R/\mathfrak{m}$. However, this map need not be continuous in any reasonable sense, even if $R/\mathfrak{m}$ has a nice topology. Since $k((x))$ has infinite transcendence degree over $k(x)$, there are many discontinuous maps out of it.

(2) We could also consider Λ as a 1-pipe ring and look at maps into other 1-pipe rings R. Assuming that R is fine and cofine and $R = \lim R/p^n$ with each

R/p^n a $\frac{1}{2}$-pipe ring, a pipe-continuous map $\Lambda \to R$ is a diagram of $\frac{1}{2}$-pipe maps $W_n k((x)) \to R/p^n$. By Lemma 3.3.7, this is equivalently a diagram of compatible mpas of topological rings $W_n k((x)) \to R/p^n$. Here $W_n k((x))$ has the topology making $x^r W_n k[[x]]$ an basis of open neighborhoods of 0. A convergent sequence in $Wk((x))^\wedge_p$ is one that converges x-adically mod p^n for every n: for example, x^r converges to zero as $r \to \infty$ in the pipe topology, but not p-adically.

(3) Finally, there is an inclusion $Wk[[x]] \to Wk((x))^\wedge_p$, and we can give $Wk((x))^\wedge_p$ the coarsest ring topology making $Wk[[x]]$, with its maximal ideal topology, an open subring. That is, a basis of neighborhoods of 0 is given by $(p, x)^r Wk[[x]]$. This topology has more and smaller open sets than either of the other two topologies considered. For example, the sequence $p^r x^{-r}$ converges to zero p-adically and in the pipe topology, but not in this topology. Note that this topology contains $Wk((x))$ as an open subring, and a continuous map $Wk((x)) \to R$ is equivalent to a continuous map $Wk[[x]] \to R$ sending x to a unit. However, extending this to a continuous map $Wk((x))^\wedge_p \to R$ involves some choices with no simple description. This is the topology Torii uses to study localizations of E-theory [**To11**].

CHAPTER 4

Localized E-theory

4.1. Description of $L_{K(n-1)}E_n$ as ring and cohomology theory

Our primary concern in this paper is the spectrum $L_{K(n-1)}E_n$, where $n \geq 2$. We abbreviate this spectrum by LE. We begin by describing its coefficient ring.

Proposition 4.1.1. *The coefficient ring LE_* is even periodic, with*

$$LE_0 = Wk[[u_1, \ldots, u_{n-1}]][u_{n-1}^{\pm 1}]^{\wedge}_{(p,u_1,\ldots,u_{n-2})}.$$

Proof. By [**Rav84**], BP satisfies the telescope conjecture, in the sense that there is an equality of Bousfield classes $\langle BP \wedge T(n-1)\rangle = \langle BP \wedge K(n-1)\rangle$, where $T(n)$ is a v_{n-1}-telescope of a finite type $n-1$ spectrum. As E_n is a BP-module, it also satisfies the telescope conjecture. By [**Hov93**, Theorem 1.5.4],

$$L_{K(n-1)}E_n = \operatorname{holim} S/(p^{i_0}, v_1^{i_1}, \cdots, v_{n-2}^{i_{n-2}}) \wedge v_{n-1}^{-1}E_n,$$

where the limit is over type $(n-1)$ generalized Moore spectra. We observe that

$$(v_{n-1}^{-1}E_n)_*S/(p^{i_0}, v_1^{i_1}, \cdots, v_{n-2}^{i_{n-2}}) = E_*[u_{n-1}^{-1}]/(p^{i_0}, u_1^{i_1}, \cdots, u_{n-2}^{i_{n-2}}),$$

which is even periodic with

$$(v_{n-1}^{-1}E_n)_0S/(p^{i_0}, v_1^{i_1}, \cdots, v_{n-2}^{i_{n-2}}) = Wk[[u_1, \ldots, u_{n-1}]][u_{n-1}^{\pm 1}]/(p^{i_0}, u_1^{i_1}, \cdots, u_{n-2}^{i_{n-2}}).$$

The transition maps in the diagram are surjective, so there is no $\lim^1$ and the result is still even periodic. The limit on π_0 is the completion $Wk[[u_1, \dots, u_{n-1}]][u_{n-1}^{\pm 1}]^{\wedge}_{(p,u_1,\dots,u_{n-2})}$, as desired. □

Proposition 4.1.2. *We have*

$$LE_0 = Wk((u_{n-1}))^{\wedge}_p[[u_1, \dots, u_{n-2}]].$$

Proof. Elements of both rings can be identified as certain possibly infinite formal sums

$$\sum \{a_I u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^{i_{n-1}} : a_I \in Wk, i_j \in \mathbb{N} \text{ for } 1 \leq j \leq n-2, i_{n-1} \in \mathbb{Z}.\}$$

Such a sum is in LE_0 if and only if its reduction modulo each power of $(p, u_1, \dots, u_{n-2})$ is in $k((u_{n-1}))$. In other words, the exponents i_{n-1} appearing in all nonzero terms with $i_0, \dots, i_{n-2}$ less than some fixed i are bounded below. On the other hand, such a sum is in $Wk((u_{n-1}))^{\wedge}_p[[u_1, \dots, u_{n-2}]]$ if and only if the terms with each fixed $i_1, \dots, i_{n-2}$ add up to an element of $u_1^{i_{n-1}} \cdots u_{n-2}^{i_{n-2}} Wk((u_{n-1}))^{\wedge}_p$. That is, the exponents i_{n-1} appearing in the nonzero terms with fixed $i_1, \dots, i_{n-2}$, and with i_0 less than some fixed i, are bounded below. Since there are only finitely many choices of $i_1, \dots, i_{n-2}$ less than any fixed i, the two conditions are in fact equivalent. □

4.1.1. Completed homology

The spectrum LE defines completed homology and cohomology theories:

$$LE^* X = \pi_* F(X, LE),$$

$$LE^{\wedge}_*X = \pi_* L_{K(n-1)}(LE \wedge X) = \pi_* L_{K(n-1)}(E \wedge X).$$

These are valued in the category $\mathsf{Mod}^{\wedge}_{LE_*}$ of graded LE_*-modules which are L-complete for the ideal $I_{n-1} = (p, \ldots, u_{n-2})$, as we now prove.

Proposition 4.1.3. *The functors LE^* and $LE^{\wedge}_*$ from HoTop to Mod_{LE_*} naturally factor through $\mathsf{Mod}^{\wedge}_{LE_*}$.*

Proof. The homology of the sphere is complete, and thus L-complete. Since $\mathsf{Mod}^{\wedge}_{LE_*}$ is an abelian category closed under extensions, the same follows for any finite complex. Now let X be an arbitrary spectrum and write X as a filtered colimit of its finite subcomplexes X_α. Then $LE^*X = \lim LE^*X_\alpha$, which is also L-complete.

Finally, the completed homology of X is

$$LE^{\wedge}_*X = \pi_* L_{K(n-1)}(E_n \wedge L_{n-1}X) = \pi_* \operatorname{holim}(E[u_{n-1}^{-1}]/(p_0^i, \ldots, u_{n-2}^{i_{n-2}}) \wedge L_{n-1}X).$$

There is a Milnor exact sequence

$$0 \to \lim{}^1 \pi_{k+1}(E[u_{n-1}^{-1}]/(p_0^i, \ldots, u_{n-2}^{i_{n-2}}) \wedge L_{n-1}X) \to$$
$$LE^{\wedge}_kX \to \lim \pi_k(E[u_{n-1}^{-1}]/(p_0^i, \ldots, u_{n-2}^{i_{n-2}}) \wedge L_{n-1}X) \to 0.$$

Each term in the limit diagram is torsion to a power of I_{n-1} and thus L-complete. Since L-complete modules are closed under extensions, limits and $\lim^1$, $LE^{\wedge}_kX$ is L-complete for each k. $\square$

Proposition 4.1.4. *If X is finite, then $LE^{\wedge}_*X = LE_*X$, which is complete in the ordinary sense.*

Proof. If X is finite, then $LE \wedge X$ is in the thick subcategory generated by LE. In particular, it is $K(n-1)$-local. It follows that $LE_*X = LE_*^\wedge X$, a finite L-complete LE_*-module. This is also complete in the ordinary sense, by Proposition 3.1.2. □

Proposition 4.1.5. *If LE_*X is free over LE_*, then $LE_*^\wedge X$ is its I_{n-1}-completion.*

Proof. In the Milnor exact sequence for $LE_*^\wedge X$, we have $\pi_*(LE/I \wedge X) \cong \pi_*(LE \wedge X)/I$, where I is an ideal in LE_0. Thus, the transition maps in the towers are surjective, the $\lim^1$ term vanishes, and the lim term is the ordinary completion of LE_*X. □

4.2. Augmented deformations

In Proposition 4.1.1, we showed that $LE_0 \cong Wk((u_{n-1}))_p^\wedge[[u_1, \ldots, u_{n-2}]]$. We write Λ for the coefficient ring $Wk((u_{n-1}))_p^\wedge$; the previous section showed that this is a Cohen ring for $k((u_{n-1}))$.

We will also write $\mathbb{H}^u$ for the base change of the universal deformation formal group $\mathbb{G}^u$ over E_0 to LE_0, and $\mathbb{H}$ for its base change to the residue field $k((u_{n-1}))$. By the discussion in Remark 2.3.7, if we started with the Honda formal group law with p-series

$$[p]_\Gamma(x) = x^{p^n},$$

then $\mathbb{H}$ has a coordinate with p-series

$$[p]_\mathbb{H}(x) = u_{n-1}x^{p^{n-1}} +_\mathbb{H} x^{p^n}.$$

In particular, its height is $n-1$.

Definition 4.2.1. Let $\mathbb{H}$ be a formal group over $k((u_{n-1}))$. An **augmented deformation** of $\mathbb{H}$ over $R \in \mathsf{CLN}$ is a triple $(\mathbb{G}, i, \alpha)$, where:

- $\mathbb{G}$ is a formal group over R,
- $i : \Lambda \to R$ is a local ring homomorphism, inducing a map $\bar{i} \to R/\mathfrak{m}_R$,
- and $\alpha : \mathbb{H} \otimes^{\bar{i}}_{k((u_{n-1}))} R/\mathfrak{m}_R \xrightarrow{\sim} \mathbb{G} \otimes_R R/\mathfrak{m}_R$ is an isomorphism of formal groups over $R/\mathfrak{m}_R$.

A $\star$**-isomorphism** of augmented deformations, $\phi : (\mathbb{G}_1, i_1, \alpha_1) \to (\mathbb{G}_2, i_2, \alpha_2)$, is the requirement that $i_1 = i_2$ and a map $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ of formal groups over R, such that the square

$$\begin{array}{ccc} \Gamma \otimes_{k((u_{n-1}))} R/\mathfrak{m} & \xrightarrow{\alpha_1} & \mathbb{G}_1 \otimes_R R/\mathfrak{m} \\ \downarrow 1 & & \phi \downarrow \\ \Gamma \otimes_{k((u_{n-1}))} R/\mathfrak{m} & \xrightarrow[\alpha_2]{} & \mathbb{G}_2 \otimes_R R/\mathfrak{m} \end{array}$$

commutes.

Let

$$\mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}} : \mathsf{CLN} \to \mathsf{Gpd}$$

be the functor that sends R to the groupoid of augmented deformations of $\mathbb{H}$ and $\star$-isomorphisms.

Theorem 4.2.2. *The functor* $\mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}}$ *is represented by* LE_0*. That is, there is a natural equivalence*

$$\mathrm{Hom}_{cts}(LE_0, R) \simeq \mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}}(R),$$

where $LE_0 = Wk((u_{n-1}))^{\wedge}_p[[u_1, \ldots, u_{n-2}]]$ *is given its maximal ideal topology (cf. Section 3.4)*

Proof. This is more or less an immediate consequence of the Lubin-Tate theorem. We define a putative universal deformation in $\mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}(LE_0)$ by the formal group $\mathbb{H}^u$, the map $1 : \Lambda \to \Lambda$, and the canonical isomorphism $\alpha^u : \mathbb{H} \xrightarrow{\sim} \mathbb{H}^u \otimes k((u_{n-1}))$. A continuous map of complete local rings $f : LE_0 \to R$ then induces an augmented deformation of $\mathbb{H}$, namely the base change of $(\mathbb{H}^u, 1, \alpha^u)$ along f. We note that the underlying map i of this base change is $f|_\Lambda$, which is local because f is.

On the other hand, suppose given $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}(R)$. We must exhibit a unique continuous map $f : LE_0 \to R$ and a unique $\star$-isomorphism between $(\mathbb{G}, i, \alpha)$ and an augmented deformation of the above form. Since i is local, we may regard R as a Λ-algebra of the form given in Theorem 2.3.1, and $(\mathbb{G}, \bar{i}, \alpha)$ as an object of $\mathsf{Def}_{\mathbb{H}}^{\Lambda}(R)$. Theorem 2.3.1 now implies that $(\mathbb{G}, i, \alpha)$ is uniquely isomorphic, as an object of $\mathsf{Def}_{\mathbb{H}}^{\Lambda}(R)$, to a base change of the universal deformation along a unique local ring map

$$\Lambda[[u_1, \ldots, u_{n-2}]] \to R$$

compatible with i. Equivalently, it is uniquely isomorphic, as an object of $\mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}(R)$, to a base change of $(\mathbb{H}^u, 1, \alpha^u)$ along a unique local ring map

$$\Lambda[[u_1, \ldots, u_{n-2}]] \to R.$$

$\square$

Again, there are graded versions of all these notions, and we also have an equivalence of functors from even periodic complete local rings to groupoids:

$$\mathrm{Hom}_{cts}(LE_*, R_*) \simeq \mathsf{Def}_{\mathbb{H}}^{\mathrm{aug},*}(R).$$

4.3. Pipe formal groups and pipe deformations

As we noted in Example 3.3.8, LE_0 is a complete local 1-pipe ring. This is slightly more structure than a complete local ring: for example, pipe maps out of LE_0 induce continuous maps on the residue field $k((u_{n-1}))$. In this section, we repeat the above story and prove the analogue of Theorem 4.2.2 for the pipe ring LE_0.

4.3.1. Pipe formal groups

Before proving the pipe analogue of Proposition Theorem 4.2.2, we need to define formal groups and their deformations in this pipe setting.

Definition 4.3.1. If R is an n-pipe ring, define the $(n+1)$-pipe ring

$$R[[x_1, \ldots, x_d]] = \lim_k R[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^k.$$

A **formal group law** over R is a commutative group structure on the functor

$$\widehat{\mathbb{A}}^1_R = \mathrm{Hom}_{R/\mathsf{PipeRings}_{n+1}}(R[[x]], \cdot);$$

equivalently, it is a map of $(n+1)$-pipe R-algebras

$$F : R[[x]] \to R[[x_1, x_2]]$$

satisfying the usual properties of a formal group law. An **isomorphism of formal group laws** $f : F \to G$ is an isomorphism of $(n+1)$-pipe rings

$$f : R[[x]] \to R[[x]]$$

such that $f(F(x,y)) = G(f(x), f(y))$. We write $\mathsf{FGL}(R)$ for the category of formal group laws and their isomorphisms over R.

A **formal group** over R is a commutative group structure on a functor

$$\mathsf{PipeRings}_{n+1} \to \mathsf{Sets}$$

which is isomorphic to $\widehat{A}^1_R$. We write $\mathsf{FG}(R)$ for the category of formal groups and their isomorphisms over R.

Clearly, there is a map of groupoids $\mathsf{FGL}(R) \to \mathsf{FG}(R)$ which forgets about the coordinate; since any formal group can be given a coordinate, this map is an equivalence. Classically, we define formal groups so that they only locally admit coordinates; here, lacking good notions of algebraic geometry and primarily interested in local phenomena anyway, we won't make this distinction.

Proposition 4.3.2. *The realization of $R[[x_1, \ldots, x_d]]$ is the power series ring $\underline{R}[[x_1, \ldots, x_d]]$. Thus, realization induces a functor $\mathsf{FGL}(R) \to \mathsf{FGL}(\underline{R})$. If R is fine and cofine, this functor is faithful.*

Proof. This is clear from the definitions. For the last part, one should note that if R is fine and cofine, so is $R[[x_1, \ldots, x_d]]$. □

Proposition 4.3.3. *A map $i : R \to S$ of pipe rings induces a functor $i^* : \mathsf{FGL}(R) \to \mathsf{FGL}(S)$.*

Proof. More generally, a map $i : R \to S$ of pipe rings induces a base change functor from the category of pipe rings of the form $R[[x_1, \ldots, x_d]]$ to the category of pipe rings of the form $S[[x_1, \ldots, x_d]]$. Importantly, this holds even for maps between pipe rings of different levels. To prove this, it suffices to show that the category of pipe rings of the form $R[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^k$ is functorial in R. If R is an n-pipe ring, then a map between two n-pipe rings of this form is in particular a map between finite free n-pipe modules over R, and it suffices to observe that the category of finite free pipe modules is functorial in R. □

Remark 4.3.4. If R is a limit of $(n-\frac{1}{2})$-pipe rings,

$$R = \lim\{R_\alpha : \alpha \in I\},$$

then we can also define

$$R[[x_1, \ldots, x_d]] = \lim\{R_\alpha[x_1, \ldots, x_d]/(x_1^{e_1}, \ldots, x_d^{e_d}) : (\alpha, e_1, \ldots, e_d) \in I \times \mathbb{N}^d\},$$

which is an n-pipe ring rather than an $(n+1)$-pipe ring. This distinction is immaterial for the purposes of dealing with formal groups: for any formal group in this 'n-pipe' sense, the multiplication is continuous over the n-pipe ring R with respect to the x_i, and so the formal group is canonically the realization of a formal group in the '$(n+1)$-pipe' sense we have opted to use.

Example 4.3.5. The formal group $\mathbb{G}^u$ over $\pi_0 E_n$ is a formal group in the 0-pipe sense. As a result, its base change over $\pi_0 L_{K(n-1)} E_n$ is a 1-pipe formal group, and the reduction $\mathbb{H}$ to the residue field $k((u_{n-1}))$ is a $\frac{1}{2}$-pipe formal group.

Definition 4.3.6. The **height** of a formal group over a pipe field k is the height of its realization over $\underline{k}$.

4.3.2. Deformations of formal groups

Definition 4.3.7. Let $\mathbb{H}$ be a formal group over an $(n-\frac{1}{2})$-pipe field K. A **deformation** of $\mathbb{H}$ over a complete local n-pipe ring R is:

- a formal group $\mathbb{G}$ over R,
- a map $\bar{i} : K \to R/\mathfrak{m}$ of $(n-\frac{1}{2})$-pipe rings,
- and an isomorphism of formal groups over $R/\mathfrak{m}$,

$$\alpha : \bar{i}^*\mathbb{H} \xrightarrow{\sim} \mathbb{G} \otimes_R R/\mathfrak{m}.$$

An **isomorphism** of deformations, $\phi : (\mathbb{G}_1, \bar{i}_1, \alpha_1) \xrightarrow{\sim} (\mathbb{G}_2, \bar{i}_2, \alpha_2)$, is the requirement $\bar{i}_1 = \bar{i}_2$ and an isomorphism of formal groups $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ such that the square

$$\begin{tikzcd} \bar{i}_1^*\mathbb{H} \arrow[r, "\alpha_1"] \arrow[d, "="'] & \mathbb{G}_1 \otimes_S S/\mathfrak{m} \arrow[d, "\phi"] \\ \bar{i}_2^*\mathbb{H} \arrow[r, "\alpha_2"'] & \mathbb{G}_2 \otimes_S S/\mathfrak{m} \end{tikzcd}$$

commutes.

We write $\mathsf{Def}_{\mathbb{H}}$ for the functor from complete local n-pipe rings to groupoids, sending R to the groupoid of deformations of $\mathbb{H}$ and their isomorphisms. If Λ is a Cohen ring for K, we write $\mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}$ for the same functor where

- $\bar{i}$ is equipped with a lift to an n-pipe ring map $i : \Lambda \to R$, and
- isomorphisms of deformations $(\mathbb{H}_1, i_1, \alpha_1) \to (\mathbb{H}_2, i_2, \alpha_2)$ include the requirement that $i_1 = i_2 : \Lambda \to S$.

Theorem 4.3.8. *If K is the $\frac{1}{2}$-pipe field $k((u_{n-1}))$, Λ is the 1-pipe ring $Wk((u_{n-1}))^{\wedge}_{p}$, and $\mathbb{H}$ is the base change of the Lubin-Tate formal group $\mathbb{G}^u$ to K, then $\mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}}$ is represented on the category of complete local 1-pipe rings by $LE_0 = \Lambda[[u_1, \ldots, u_{n-2}]]$.*

Proof. Let R be a complete local 1-pipe ring. By definition, R is of the form $(S[x^{-1}])^{\wedge}_{\mathfrak{m}}$, where S is a profinite complete local ring and $\mathfrak{m}$ is a maximal $\frac{1}{2}$-ideal of $S[x^{-1}]$. Suppose given $(\mathbb{G}, i, \alpha) \in \mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}}(R)$. By Proposition 4.3.3, $\mathbb{G}$ descends to a compatible system of formal groups $\mathbb{G} \otimes R/\mathfrak{m}^r$ over the $\frac{1}{2}$-pipe rings $R/\mathfrak{m}^r$, and α descends to a compatible system of isomorphisms over these same rings. The map $i : \Lambda \to R$ lifts a map $\bar{i} : k((u_{n-1})) \to R/\mathfrak{m}$, which means that i sends the ideal (p) into $\mathfrak{m}$. Thus, there are induced $\frac{1}{2}$-pipe maps $W_r k((u_{n-1})) \to R/\mathfrak{m}^r$.

Taking the realization of all this data, we get a compatible system of deformations in the usual, non-pipe sense,

$$(\mathbb{G}_r, \bar{i}, \alpha_r) = (\mathbb{G} \otimes \underline{R/\mathfrak{m}^r}, \bar{i}, \alpha \otimes \underline{R/\mathfrak{m}^r}) \in \mathsf{Def}_{\mathbb{H}}(\underline{R/\mathfrak{m}^r}),$$

together with a lift of $\bar{i}$ to

$$i_r : Wk((u_{n-1})) \to W_r k((u_{n-1})) \to \underline{R/\mathfrak{m}^r}.$$

By Theorem 4.2.2, for each r there is a unique continuous map $f_r : LE_0 \to \underline{R/\mathfrak{m}^r}$ and a unique isomorphism from the pushforward of the universal augmented deformation along f_r to the augmented deformation $(\mathbb{G}_r, i_r, \alpha_r)$. Moreover, f_r factors through LE_0/p^r.

Finally, the diagram

$$\begin{tikzcd}
LE_0/p^r \arrow[r, "f_r"] \arrow[d] & \underline{R/\mathfrak{m}^r} \arrow[d] \\
LE_0/p^{r-1} \arrow[r, "f_{r-1}"'] & \underline{R/\mathfrak{m}^{r-1}}
\end{tikzcd}$$

commutes.

It remains to show that each f_r is a $\frac{1}{2}$-pipe ring map, meaning that they assemble to a 1-pipe ring map $LE_0 \to R$. This follows from the lemma below. $\square$

Lemma 4.3.9. *Let R and R' be complete local 1-pipe rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$. Let f be a ring map between their realizations such that $f(\mathfrak{m}) \subseteq \mathfrak{n}$, and such that f induces a continuous map on residue $\frac{1}{2}$-pipe fields. Then f is the realization of a 1-pipe ring map.*

Proof. The hypotheses imply that f induces compatible maps of realizations, f_r : $\underline{R/\mathfrak{m}^r} \to \underline{R'/\mathfrak{n}^r}$. By Lemma 3.3.7, it suffices to prove that each f_r is continuous, for then they lift to compatible maps of $\frac{1}{2}$-pipe rings, thus defining a map of 1-pipe rings realizing to f. We prove this by induction on r, the case $r = 1$ being a hypothesis. Assume that f_r is continuous, and let U be a neighborhood of 0 in $R'/\mathfrak{n}^{r+1}$ such that U is closed under addition and multiplication and contains $\mathfrak{n}^r U$. Such neighborhoods are cofinal in the filtered poset of neighborhoods of 0. Since f_r is continuous, there is a neighborhood $V \ni 0$ in $R/\mathfrak{m}^{r+1}$ such that

$$f_{r+1}(V) \subseteq U + \mathfrak{m}^r R/\mathfrak{m}^{r+1}.$$

In fact, Definition 3.3.10 of complete local pipe rings says that $R/\mathfrak{m}^{r+1} = S[x^{-1}]/\mathfrak{m}^{r+1}$ where S is a profinite complete local ring, say with maximal ideal $\mathfrak{p}$, and $x \in \mathfrak{p}$. Thus, we can take V to be an ideal of the form $\mathfrak{p}^s S/\mathfrak{m}^{r+1}$. If x, $y \in V$,

$$f_{r+1}(xy) \in (U + \mathfrak{m}^r R/\mathfrak{m}^{r+1})^2 \subseteq U.$$

Thus, $f_{r+1}^{-1}(U)$ contains all products of pairs of elements of V. Since U is closed under addition, it even contains all sums of products of pairs of elements of V. But this is another open ideal, $\mathfrak{p}^s S/\mathfrak{m}^{r+1}$. Thus, f_{r+1} is continuous. □

We conclude this chapter by relating the above to the work of Mazel-Gee, Peterson, and Stapleton.

Definition 4.3.10. Let $\mathsf{StagedDef}_1$ be the category of maps of pipe rings $R_0 \to R_1$, where R_0 is a complete local 0-pipe ring and R_1 is a complete local 1-pipe ring of the form $R_1 = R_0[x^{-1}]^{\wedge}_{\mathfrak{m}}$. Let Γ be a height n formal group over a finite field k, and define

$$\mathsf{Def}_\Gamma : \mathsf{StagedDef}_1 \to \mathsf{Gpd}$$

as follows. An object of $\mathsf{Def}_\Gamma(R_0 \to R_1)$ is a $(\mathbb{G}_0, i, \alpha) \in \mathsf{Def}_\Gamma(R_0)$, such that $\mathbb{G}_0 \otimes_{R_0} R_1$ is height $n-1$ over the residue field of R_1. An isomorphism in $\mathsf{Def}_\Gamma(R_0 \to R_1)$ is an isomorphism in $\mathsf{Def}_\Gamma(R_0)$.

Corollary 4.3.11. *The functor* Def_Γ *is represented on* $\mathsf{StagedDef}_1$ *by the diagram* $E_0 \to LE_0$.

Proof. A commutative square

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & LE_0 \arrow[d] \\ R_0 \arrow[r] & R_1 \end{tikzcd}$$

defines objects $(\mathbb{G}_0, i_0, \alpha_0) \in \mathsf{Def}_\Gamma(R_0)$ and $(\mathbb{G}_1, i_1, \alpha_1) \in \mathsf{Def}_\Gamma^{\mathrm{aug}}(R_1)$ via the vertical maps, together with an isomorphism $\mathbb{G}_0 \otimes_{R_0} R_1 \cong \mathbb{G}_1$. In particular, $\mathbb{G}_0 \otimes_{R_0} R_1$ has height $n-1$ over the residue field of R_1. Conversely, given an object in $\mathsf{Def}_\Gamma(R_0 \to R_1)$, the underlying $(\mathbb{G}_0, i_0, \alpha_0) \in \mathsf{Def}_\Gamma(R_0)$ is represented by a unique continuous map $E_0 \to R_0$. Since $\mathbb{G}_0 \otimes_{R_0} R_1$ has height $n-1$, the map from E_0 to the realization $\underline{R_1}$ factors through LE_0. On residue fields, $k((u_{n-1})) \to \underline{R_1/\mathfrak{m}} = R_0[x^{-1}]/\mathfrak{m}$ factors through a continuous map $k[[u_{n-1}]] \to R_0/\mathfrak{m}$, and is thus continuous. By Lemma 4.3.9, there is an induced 1-pipe map $LE_0 \to R_1$ making the diagram commute. $\square$

Of course, this is a very special case of the main theorem of [**MGPS**], which proves that the diagram of pipe rings

$$\pi_0 E_n \to \pi_0 L_{K(h_1)} E_n \to \cdots \to \pi_0 L_{K(h_t)} \cdots L_{K(h_1)} E_n,$$

for $h_t < \cdots < h_1 < n$, represents a similar moduli problem, of deformations of a height n formal group over a complete local ring, such that the deformation drops to the specified heights over specified pipe localizations. The representation theorem proved here, Theorem 4.3.8, describes what happens to one of these moduli problems when we forget about the map from $\pi_0 E_n$.

CHAPTER 5

The E-theory of E-theory

5.1. Co-operations for E-theory

The completed E-homology $E_*^\wedge X = \pi_* L_{K(n)}(E \wedge X)$ of a space or spectrum X is naturally a complete comodule for a coalgebra of co-operations $E_*^\wedge X$. This has a surprisingly simple form. In this section, we write $\mathbb{G}_n = \mathrm{Aut}(k, \Gamma)$.

Theorem 5.1.1 ([**DH04**]). *If $E = E(k, \Gamma)$ where Γ is the height n Honda formal group over a finite field k containing $\mathbb{F}_p$, there is an isomorphism*

$$E_*^\wedge E \cong \mathrm{Hom}_{cts}(\mathbb{G}_n, E_*),$$

where $\mathrm{Hom}_{cts}(\mathbb{G}_n, E_*)$ is the E_*-algebra of continuous set maps $\mathbb{G}_n \to E_*$.

As an immediate application, in the $K(n)$-local E-based Adams spectral sequence

$$E_2 = \mathrm{Ext}^*_{E_*^\wedge E}(E_*, E_*^\wedge X) \Rightarrow \pi_* L_{K(n)} X,$$

we can rewrite the E_2 page as group cohomology $H^*_{cts}(\mathbb{G}_n, E_*^\wedge X)$.

For the sake of inspiring the arguments below, we give a proof of this theorem. It will be more convenient to work more generally than with the Honda formal group, so let Γ be a height n formal group of characteristic p over a perfect field, and let $E = E(k, \Gamma)$. Let $\mathsf{Alg}^\wedge_{E_*}$ be the category of even periodic, I_n-adically complete E_*-algebras, and let $\mathsf{Alg}^\wedge_{E_0}$

be the category of I_n-adically complete E_0-algebras. The proof will show that $E_*^\wedge E$ and $\mathrm{Hom}_{cts}(\mathbb{G}_n, E_*)$ represent the same functor on $\mathsf{Alg}_{E_*}^\wedge$.

Lemma 5.1.2. *The ring $E_*^\wedge E$ is an object in $\mathsf{Alg}_{E_*}^\wedge$.*

Proof. Since E is Landweber exact,

$$E_*E = \pi_*(E \wedge E) = E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_*$$

is even periodic and flat over E_* (where, for definiteness, we use the left E_*-module structure). The theory $E \wedge E$ is L_n-local, which means that

$$L_{K(n)}(E \wedge E) = \operatorname*{holim}_I E \wedge E \wedge S/I = \operatorname*{holim}_I E \wedge E/I$$

where I ranges over ideals $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ such that the associated Moore spectrum exists. Again, the homotopy groups of the objects in the limit diagram are

$$E_*(E/I) = E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_*/I.$$

Thus, the maps in the diagram are surjective on homotopy groups, so that

$$\pi_* L_{K(n)}(E \wedge E) = \lim_I E_*(E/I) = (E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_*)^\wedge_{I_n}.$$

As I_n and its powers are images of invariant ideals in BP_*BP, it doesn't matter whether we complete with respect to the I_n coming from the left or right E_*-module structure. Clearly, this is an object of $\mathsf{Alg}_{E_*}^\wedge$. $\square$

Lemma 5.1.3. *For $R_* \in \mathsf{Alg}^{\wedge}_{E_*}$, pre-composition with the completion map $E_*E \to E^{\wedge}_*E$ induces an isomorphism*

$$\mathrm{Hom}_{\mathsf{Alg}^{\wedge}_{E_*}}(E^{\wedge}_*E, R_*) \cong \mathrm{Hom}_{E_*}(E_*E, R_*).$$

Moreover, for any map $E^{\wedge}_*E \to R_*$ in $\mathsf{Alg}^{\wedge}_{E_*}$, the composition

$$E_* \xrightarrow{\eta_R} E_*E \to R_*$$

is also continuous.

Proof. Let $R_* \in \mathsf{Alg}^{\wedge}_{E_*}$ and give R_* the I_n-adic topology. As we saw in the proof of the previous lemma, I_n is the image of an invariant ideal in BP_*BP. Thus, any map

$$f : E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_* \to R_*$$

extending the given map $E_* \to R_*$ has $f(I_n \cdot E_*E) \subseteq I_nR_*$. In particular, R_* is also complete with respect to $I_n \cdot E_*E$, so that f factors uniquely through the completion $E^{\wedge}_*E$. Moreover, $f(\eta_R(I_n \cdot E_*))$ is also in I_nR_*, so that the map $E_* \to R_*$ coming from the right unit is also continuous. $\square$

Let $\mathsf{Alg}^{\wedge}_{E_*,loc}$ be the full subcategory of $R \in \mathsf{Alg}^{\wedge}_{E_*}$ such that R_0 is complete local. Then the map $E_0 \to R_0$ classifies an object $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R_0)$.

Proposition 5.1.4. *Let $R_* \in \mathsf{Alg}^{\wedge}_{E_*,loc}$, and let $(\mathbb{G}, i, \alpha)$ be the deformation of Γ classified by $E_0 \to R_0$. Then the set of maps $\mathrm{Hom}_{\mathsf{Alg}^{\wedge}_{E_*}}(E^{\wedge}_*E, R_*)$ is naturally isomorphic to the set* *of pairs (j,γ), where j is a map $k \to R_0/\mathfrak{m}$, and γ is an isomorphism of formal groups over $R/\mathfrak{m}$, $\gamma : \Gamma \otimes^i_k R/\mathfrak{m} \to \Gamma \otimes^j_k R/\mathfrak{m}$.*

Proof. By Lemma 5.1.3,

$$\mathrm{Hom}_{\mathsf{Alg}^{\wedge}_{E_*}}(E^{\wedge}_* E, R_*) \cong \mathrm{Hom}_{E_*}(E_*E, R_*) \cong \mathrm{Hom}_{E_0}(E_0E, R_0).$$

We also have

$$E_0E = E_0 \otimes_{BPP_0} BPP_0BPP \otimes_{BPP_0} E_0.$$

The Hopf algebroid (BPP_0, BPP_0BPP) presents the moduli of p-local formal groups, so there is a pullback square of stacks

$$\begin{tikzcd} \mathrm{Spec}\, E_0E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\, E_0 \arrow[d] \\ \mathrm{Spec}\, E_0 \arrow[r] & \mathcal{M}_{\mathrm{fg}}. \end{tikzcd}$$

Again, Lemma 5.1.3 implies that, for any E_0-map $E_0E \to R_0$, the map $E_0 \to R_0$ coming from the right unit is also continuous, and thus classifies a deformation. So we have a homotopy pullback of groupoids,

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Alg}^{\wedge}_{E_*}}(E^{\wedge}_* E, R_*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Def}_{\Gamma}(R_0) \arrow[d] \\ \{*\} \arrow[r, "\mathbb{G}"'] & \mathcal{M}_{\mathrm{fg}}(R_0). \end{tikzcd}$$

An object in the pullback is given by another deformation $(\mathbb{G}', j, \beta) \in \mathsf{Def}_{\Gamma}(R_0)$ and an isomorphism $\phi : \mathbb{G} \to \mathbb{G}'$. An isomorphism in the pullback is an isomorphism ψ :

$(\mathbb{G}', j, \beta) \to (\mathbb{G}'', j, \delta)$ such that the obvious triangle involving ϕ commutes. Now, there is an isomorphism of deformations $\phi^{-1} : (\mathbb{G}', j, \beta) \to (\mathbb{G}, j, \beta \circ \phi^{-1})$; this is the only isomorphism from $(\mathbb{G}', j, \beta)$ to a deformation whose underlying formal group is exactly $\mathbb{G}$. It follows that the connected components of the pullback groupoid are contractible, as expected, and correspond to pairs

$$(j, \beta : \Gamma \otimes^j R_0/\mathfrak{m} \xrightarrow{\sim} \mathbb{G} \otimes R_0/\mathfrak{m}).$$

Equivalently, they correspond to pairs

$$(j : k \to R_0/\mathfrak{m}, \gamma = \beta^{-1}\alpha : \Gamma \otimes^i R_0/\mathfrak{m} \xrightarrow{\sim} \Gamma \otimes^j R_0/\mathfrak{m}).$$

□

Example 5.1.5. The group $\mathrm{Aut}(k, \Gamma)$ acts on this set by pre-composing with the map j and post-composing with the isomorphism γ. However, this action need not be transitive. For example, let Γ be the height 1 formal group over the perfect field $K = \overline{\mathbb{F}_p}((u^{1/p^\infty}))$ with p-series

$$[p]_\Gamma(x) = ux^p.$$

Let $L = K \otimes_{\overline{\mathbb{F}_p}} K = \overline{\mathbb{F}_p}((u^{1/p^\infty}, v^{1/p^\infty}))$, and let R_0 be the algebraic closure of L. There are two maps $j_1, j_2 : K \to R_0$, respectively sending u to u and to v. The base changes of Γ along these maps are isomorphic over R_0 via

$$\phi(x) = (u/v)^{1/(p-1)}x.$$

This isomorphism is not induced by an element of $\mathrm{Aut}(k, \Gamma)$.

We now specialize to the case where Γ is the height n Honda formal group over a finite field k containing $\mathbb{F}_{p^n}$, with $[p]_\Gamma(x) = x^{p^n}$. In this case, the formal group is algebraic enough to prevent the above subtlety from occurring. (In fact, the following argument works in slightly more generality: one can take $[p]_\Gamma(x) = ux^{p^n}$ for $u \in k^\times$, which at least implies that Frob_Γ is central in $\mathrm{End}_k(\Gamma)$.) The author thanks Paul Goerss for pointing out this subtlety and the following method of addressing it.

Proof of Theorem 5.1.1. First, we need to construct a continuous E_*-algebra map $f : E_*^\wedge E \to \mathrm{Hom}_{cts}(\mathbb{G}_n, E_*)$. Such a map is adjoint to a continuous map

$$\mathbb{G}_n \to \mathrm{Hom}_{\mathsf{Alg}^\wedge_{E_*}}(E_*^\wedge E, E_*).$$

Let $(\mathbb{G}^u, 1, \alpha^u)$ be the universal deformation over E_*. The E_*-algebra structure map $E_* \to E_*$ is just the identity map, which classifies this deformation. By Proposition 5.1.4,

$$\mathrm{Hom}_{\mathsf{Alg}^\wedge_{E_*}}(E_*^\wedge E, E_*) \cong \{(j : k \to k, \gamma : \Gamma \xrightarrow{\sim} \Gamma \otimes^j k)\}.$$

Since k is a finite field, any map $k \to k$ is an isomorphism. So the right-hand side is exactly $\mathbb{G}_n$, defining the desired map.

For simplicity's sake, we now restrict everything to degree zero. To show that the map $f : E_0^\wedge E \to \mathrm{Hom}_{cts}(\mathbb{G}_n, E_0)$ is an isomorphism, it suffices, since both sides are flat and complete E_0-algebras, that it induces an isomorphism mod I_n. Now, I_n is an invariant ideal in BP, so

$$E_0^\wedge E/I_n = k \otimes_{BPP_0} BPP_0BPP \otimes_{BPP_0} k.$$

A map from this into a ring R is the same as a pair of maps $i, j : k \to R$ and an isomorphism $\gamma : \Gamma \otimes_k^i R \to \Gamma \otimes_k^j R$ of formal groups over R. Now, if Γ is the Honda formal group over $k \supseteq \mathbb{F}_{p^n}$, we have a coordinate x for Γ with $[p]_\Gamma(x) = x^{p^n}$, and this must commute in the obvious way with any isomorphism γ. It follows that the coefficients of γ, viewed as a power series in x, are fixed by the nth power of Frobenius. Since R is an $\mathbb{F}_{p^n}$-algebra via i, the subring of elements of R fixed by the nth power of Frobenius is a product of copies of $\mathbb{F}_{p^n}$. By Theorem 2.2.8, the isomorphism γ is defined over $\mathbb{F}_{p^n}$.

Thus, the data (i, j, γ) is always base changed from data of the form

$$(1 : k \to k, j : k \to k, \gamma : \Gamma \xrightarrow{\sim} j^*\Gamma).$$

Since k is finite, j is an isomorphism. This is precisely an element of the Morava stabilizer group $\mathbb{G}_n = \mathrm{Aut}(k, \Gamma)$ (cf. Definition 2.2.5). Thus, the map $E_0^\wedge E/I_n \to \mathrm{Hom}_{cts}(\mathbb{G}_n, k)$ is an isomorphism. $\square$

5.2. $E_{n-1,*}^\wedge E_n$

In this section, we let k be a finite field containing $\mathbb{F}_p$ and $\mathbb{F}_{p^{n-1}}$. We let E be the E-theory associated to a height n formal group Γ over k, and F the E-theory associated to the height $n-1$ Honda formal group Γ_{n-1} over k. As before, $LE = L_{K(n-1)}E$, Λ is the ring $Wk((u_{n-1}))_p^\wedge \subseteq LE_0$, and $\mathbb{H}$ is the base change of the formal group of E_0 to $k((u_{n-1}))$. As we showed (Theorem 4.2.2), a continuous map from LE_0 to a complete local ring R represents a deformation of $\mathbb{H}$ together with an augmentation lifting its map $\bar{i} : k((u_{n-1})) \to R/\mathfrak{m}$ to a p-adically continuous map $i : \Lambda \to R$. We will study the ring $F_*^\wedge E = \pi_* L_{K(n-1)}(F \wedge E)$.

Proposition 5.2.1. *$F^\wedge_* E = F^\wedge_* LE$.*

Proof. The map $E \to LE$ is a $K(n-1)$-local equivalence, so remains so after smashing with F. $\square$

Proposition 5.2.2. *$F^\wedge_* E$ is even periodic and flat over F_*.*

Proof. As in the proof of Lemma 5.1.2, F and LE are both even periodic and Landweber exact, so

$$F_* LE = F_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} LE_*,$$

which is even periodic and flat over F_*, since $F \wedge E$ is L_{n-1}-local. The $K(n-1)$-localization satisfies

$$F^\wedge_* LE = \pi_* \operatorname{holim}_I (F \wedge LE/I),$$

where I ranges over a cofinal set of ideals of the form $(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$. The objects in the limit diagram are

$$F_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} LE_*/I,$$

and the maps in the diagram are surjective. Therefore, $F^\wedge_* E$ is also even periodic.

It remains to show that $F^\wedge_* E$ is F_*-flat. The above implies that

$$F^\wedge_* LE = (F_* LE)^\wedge_{I_{n-1}},$$

(the degreewise completion), and therefore that

$$F^\wedge_* LE/I = (F_* LE)/I = F_*/I \otimes_{BP_*} BP_* BP \otimes_{BP_*} LE_*/I,$$

where I is a power of I_{n-1}. Since each of these is a flat F_*/I-module and the maps in the limit diagram computing the completion are surjective, [**Stacks**, Tag 0912] implies that $F_*^\wedge LE$ is a flat $(F_*)^\wedge I = F_*$-module. $\square$

For $x \in BP_*$, write x for $\eta_L(x)$ and $\overline{x}$ for $\eta_R(x)$, elements of BP_*BP. The map $BP_* \to F_*$ sends

$$v_i \mapsto u^{p^i-1}u_i, \quad i \leq n-2,$$

$$v_{n-1} \mapsto u^{p^{n-1}-1},$$

$$v_i \mapsto 0, \quad i \geq n.$$

Likewise, the map $BP_* \to LE_*$ sends

$$\overline{v_i} \mapsto u^{p^i-1}u_i, \quad i \leq n-1,$$

$$\overline{v_n} \mapsto u^{p^n-1},$$

$$\overline{v_i} \mapsto 0, \quad i \geq n+1$$

We can thus write

$$F_*^\wedge E = \left(\frac{Wk[[u_1, \ldots, u_{n-2}]][u^{\pm 1}][t_1, t_2, \ldots] \otimes Wk((u_{n-1}))[[u_1^E, \ldots, u_{n-2}^E]][(u^E)^{\pm 1}]}{(\overline{v_1} - (u^E)^{p^i-1}u_i^E, \ldots, \overline{v_n} - (u^E)^{p^n-1}, \overline{v_{n+1}}, \ldots)} \right)^\wedge_{I_{n-1}}.$$

In degree zero, let $s_i = t_i(u^E)^{1-p^i}$ and $w = u(u^E)^{-1}$. Note that the ideal I_{n-1} contains p and all u_i and u_i^E (and thus all v_i and $\overline{v_i}$) for $1 \leq i \leq n-2$. Therefore,

$$F_0^\wedge E/I_{n-1} = k((u_{n-1}))[s_1, s_2, \ldots, w^{\pm 1}]/((u^E)^{1-p^{n-1}}\overline{v_{n-1}} - u_{n-1}, (u^E)^{1-p^n}\overline{v_n} - 1, \overline{v_{n+1}}, \ldots).$$

Now, by [**Rav04**, 4.3.1],

$$\overline{v_{n-1+i}} \equiv v_{n-1+i} + v_{n-1} t_i^{p^{n-1}} - v_{n-1}^{p^i} t_i \pmod{(p, v_1, \ldots, v_{n-2}, t_1, \ldots, t_{i-1})}. \tag{5.1}$$

Scaling to degree zero by multiplying by appropriate powers of $\overline{u}$, we get relations in $F_0^\wedge E/I_{n-1}$:

$$\begin{aligned} u_{n-1} &= w^{p^{n-1}-1}, \\ 1 &= w^{p^{n-1}-1} s_1^{p^{n-1}} - w^{p(p^{n-1}-1)} s_1, \\ 0 &\equiv w^{p^{n-1}-1} s_i^{p^{n-1}} - w^{p^i(p^{n-1}-1)} s_i + f_i(s_1, \ldots, s_{i-1}) \text{ for } i \geq 2. \end{aligned} \tag{5.2}$$

The first relation gives an embedding of $k((w))$ into $F_0^\wedge E/I_{n-1}$, as a tamely ramified extension of $k((u_{n-1}))$ (recall that k contains $\mathbb{F}_{p^{n-1}}$). The remaining relations inductively define s_i as solutions to higher Artin-Schreier equations over the ring generated by $k((w))$ and $s_1, \ldots, s_{i-1}$. In particular, $F_0^\wedge E/I_{n-1}$ is ind-étale over $k((u_{n-1}))$.

We now describe the functor represented by $F_0^\wedge E$.

Lemma 5.2.3. *For any complete local ring R, pre-composition with the completion map $F_0 E \to F_0^\wedge E$ induces an isomorphism*

$$\mathrm{Hom}_{cts}(F_0^\wedge E, R) \cong \mathrm{Hom}_{F_0, cts}(F_0 E, R_*).$$

Moreover, for any continuous map $F_0^\wedge E \to F_0$, the composition

$$LE_0 \xrightarrow{\eta_R} F_0 E \to R$$

is also continuous.

Proof. This is just as in Lemma 5.1.3. Since I_{n-1} is an invariant ideal in BP_*, any complete local ring R with a map $f : F_0E \to R$ such that the restriction to F_0 is continuous must be complete with respect to $I_{n-1} \cdot F_0E$, so that f factors uniquely through the completion $F_0^\wedge E$. Moreover, f also sends $\eta_R(I_{n-1} \cdot E_0)$ into $I_{n-1}R$, so that the map $E_0 \to R$ coming from the right unit is also continuous. $\square$

Theorem 5.2.4. *Let R be a complete local F_0-algebra. There is a natural isomorphism between continuous F_0-algebra maps $F_0^\wedge E \to R$ and pairs (j, γ), where $j : \Lambda \to R$ is a p-adically continuous map and γ is an isomorphism of formal groups over R, $\gamma : \Gamma_{n-1} \otimes_k^i R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes_{k((u_{n-1}))}^{\bar{j}} R/\mathfrak{m}$.*

Proof. As before, we have

$$F_0LE = \pi_0(F_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} LE_*) = F_0 \otimes_{BPP_0} BPP_0BPP \otimes_{BPP_0} LE_0.$$

An F_0-algebra map $F_0LE \to R$ is equivalent to a map $LE_0 \to R$ and an isomorphism over R between the base changes of the formal groups of F_0 and LE_0.

If R is complete local, then the previous lemma tells us that $\mathrm{Hom}_{F_0,cts}(F_0E, R) = \mathrm{Hom}_{F_0,cts}(F_0^\wedge E, R)$, and that the map $LE_0 \to R$ is continuous. Hence, the map $LE_0 \to R$ represents an object of $\mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}(R)$. Likewise, the structure map $F_0 \to R$ represents an object of $\mathsf{Def}_{\Gamma_{n-1}}(R)$, say $(\mathbb{G}, i, \alpha)$. Thus, we have a pullback of groupoids:

$$\begin{tikzcd}
\mathrm{Hom}_{F_0,cts}(F_0^\wedge E, R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Def}_{\mathbb{H}}^{\mathrm{aug}}(R) \arrow[d] \\
\{*\} \arrow[r, "\mathbb{G}"'] & \mathcal{M}_{\mathrm{fg}}(R).
\end{tikzcd}$$

In other words, a map $f : F_0E \to R$ corresponds to the data:

$$(\mathbb{G}', j : \Lambda \to R, \beta : \mathbb{H} \otimes^{\bar{j}}_{k((u_{n-1}))} R/\mathfrak{m} \xrightarrow{\sim} \mathbb{G}' \otimes_R R/\mathfrak{m}) \in \mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}}(R);\ \phi : \mathbb{G} \xrightarrow{\sim} \mathbb{G}'.$$

There is a unique isomorphism in the pullback groupoid which restricts to $\phi^{-1} : \mathbb{G}' \to \mathbb{G}$ on formal groups. Composing with this isomorphism, one gets a unique object in the pullback groupoid isomorphic to f whose underlying formal group is $\mathbb{G}$ and whose underlying isomorphism of formal groups is the identity. It follows that the groupoid is locally contractible, as expected. The rest of the data is given by j and β, or equivalently by j and

$$\gamma = \beta^{-1}\alpha : \Gamma_{n-1} \otimes^i_k R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes^{\bar{j}}_{k((u_{n-1}))} R/\mathfrak{m}.$$

□

Proposition 5.2.5. *F_0E/I_{n-1} is of the form* $\mathrm{Hom}(\mathrm{Gal}(k/\mathbb{F}_p), L)$*, where L is a field. Therefore, $F_0^\wedge E$ is a finite product of complete local rings.*

Proof. Armed with the above result, this is essentially a reinterpretation of a result of Torii, [**To11**, Theorem 2.7], which in turn reinterprets a result from [**Gr79**]. For R a complete local $k = F_0/I_{n-1}$-algebra,

$$\mathrm{Hom}_k(F_0E/I_{n-1}, R) = \{(\bar{j} : k((u_{n-1})) \to R, \gamma : \Gamma_{n-1} \otimes^i_k R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes^{\bar{j}}_{k((u_{n-1}))} R/\mathfrak{m})\}.$$

The étaleness of isomorphisms, Theorem 2.2.8, says that we can equivalently define γ as an isomorphism between Γ_{n-1} and $\mathbb{H}$ over R. There is a smallest extension L of $k((u_{n-1}))$ over which Γ_{n-1} and $\mathbb{H}$ become isomorphic, given by adjoining the coefficients of an isomorphism between any choice of formal group laws for Γ_{n-1} and $\mathbb{H}$. Torii proves that $L/k((u_{n-1}))$ is Galois with Galois group $\mathrm{Aut}_k(\Gamma_{n-1})$. On the other hand, having chosen $\bar{j} : k((u_{n-1})) \to R$ such that an isomorphism $\Gamma_{n-1} \otimes R \to \mathbb{H} \otimes R$ exists, the set of such isomorphisms is clearly a torsor for this group. It follows that a k-algebra map $F_0E/I_{n-1} \to R$ is precisely a map $L \to R$, which is not necessarily a k-algebra map. In other words, $F_0E/I_{n-1} = \mathrm{Hom}(\mathrm{Gal}(k/\mathbb{F}_p), L)$.

Since k is finite, this is a finite product of fields. The corresponding splitting for $F_0^\wedge E$ itself follows from Hensel's lemma. □

Remark 5.2.6. Of course, the coefficients of the universal isomorphism of formal groups defined over L are precisely the s_i, and the algebraic equations they satisfy over $k((u_{n-1}))$ were partially computed above in (5.2). What is not clear is that the equation defining s_i is irreducible over the field defined by w and $s_1, \ldots, s_{i-1}$.

One should compare this result with the computation of $E_0^\wedge E$. In both cases, the object calculated is a flat extension of a Lubin-Tate ring, it represents an isomorphism of deformations of formal groups, and its reduction mod I_{n-1} carries the action of a Morava stabilizer group. In the case of $E_0^\wedge E$, this action splits, and in fact $E_0^\wedge E$ is a profinite group algebra for the Morava stabilizer group. In the case of $F_0^\wedge E$, the action is in a certain sense as complicated as possible, so that the only part of the action that splits is the Galois group. This is one way of stating the idea that the formal group of LE_0 is as complicated a height $n-1$ formal group as possible.

5.3. Co-operations for localized E-theory

We conclude this chapter by studying $K(n-1)$-localized co-operations, by much the same argument as before.

Theorem 5.3.1. *Let R be a complete local LE_0-algebra, with $LE_0 \to R$ classifying $(\mathbb{G}, i, \alpha) \in \mathsf{Def}^{\mathrm{aug}}_{\mathbb{H}}(R)$. There is a natural isomorphism between continuous LE_0-algebra maps $LE_0^\wedge LE \to R$ and pairs (j, γ), where $j : \Lambda \to R$ is a p-adically continuous map and γ is an isomorphism of formal groups over $R/\mathfrak{m}$, $\gamma : \mathbb{H} \otimes^{\bar{i}}_{k((u_{n-1}))} R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes^{\bar{j}}_{k((u_{n-1}))} R/\mathfrak{m}$.*

This proceeds by the same essential steps as in the previous section:

- LE_0 is Landweber exact, so that $LE_0^\wedge LE$ is even periodic and flat over LE_0.
- If R is a complete LE_0-algebra, continuous LE_0-algebra maps $LE_0^\wedge LE \to R$ are equivalent to LE_0-algebra maps $LE_0 LE \to R$, and induce continuous maps $LE_0 \to R$ through the right unit.
- The representability results imply that a continuous map $LE_0 LE \to R$ represents a pair of augmented deformations with an isomorphism between their underlying formal groups. This implies the statement of the theorem.

Remark 5.3.2. Unlike the case of $\pi_0 L_{K(n)}(E \wedge E)$, $LE_0^\wedge LE$ is not formally étale over LE_0 (nor is $F_0^\wedge E$ formally étale over F_0). However, in the data (j, γ), the isomorphism γ is insensitive to infinitesimal thickenings. Constructing a lift in a diagram of the form

$$\begin{tikzcd} LE_0 \arrow[r] \arrow[d] & R \arrow[d] \\ LE_0^\wedge LE \arrow[r] \arrow[ur, dashed] & R/I \end{tikzcd}$$

where R is complete local and I is a square-zero ideal reduces to constructing a lift in the diagram

$$\begin{tikzcd} Wk \arrow[r] \arrow[d] & R \arrow[d] \\ \Lambda = Wk((u_{n-1}))^{\wedge}_p \arrow[r] \arrow[ur, dashed] & R/I. \end{tikzcd}$$

In other words, the complete cotangent complex $\mathbb{L}_{LE_0^{\wedge}LE/LE_0}$ is a base change of $\mathbb{L}_{\Lambda/Wk}$. As a consequence of Proposition 3.2.12, the map $Wk \to \Lambda$ is formally smooth if k is perfect, meaning that its completed cotangent complex is Λ-free and concentrated in degree zero. Note that

$$\operatorname{rank}_{\Lambda} \mathbb{L}_{\Lambda/Wk} = \operatorname{rank}_{k((u_{n-1}))} \mathbb{L}_{k((u_{n-1}))/k} = \infty$$

as $k((u_{n-1}))$ has infinite transcendence degree over k. (If k is finite, this last fact follows from a cardinality argument: $k((u_{n-1}))$ is uncountable, while any field of finite transcendence degree over k is countable.)

CHAPTER 6

Power operations and E_∞ structures

6.1. Power operations: topology

If A is an E_∞ ring spectrum, the d-fold multiplication map $m : A^{\wedge d} \to A$ is commutative up to coherent homotopy, and factors through the A has multiplication maps $m : A^{\wedge d}_{h\Sigma_d} \to A$ which give rise to power operations in the homotopy of A. To be precise, a class $x \in \pi_0 A$ is represented by a map

$$x : S \to A;$$

the composition

$$\Sigma^\infty_+ B\Sigma_d \simeq S^{\wedge d}_{h\Sigma_d} \xrightarrow{x^{\wedge d}} A^{\wedge d}_{h\Sigma_d} \xrightarrow{m} A$$

then gives a class in $A^0(B\Sigma_d)$, called the **total dth power operation** on x, $P_d(x)$. This operation satisfies the following properties ([**BMMS**], [**Re09**, 3.9]):

- $P_d(xy) = P_d(x)P_d(y)$.
- $P_d(x+y) \equiv P_d(x) + P_d(y)$ modulo the sums of the images of the transfer maps $A^0(B\Sigma_i \times B\Sigma_{d-i}) \to A^0(B\Sigma_d)$ for $1 \le i \le d-1$.
- Composing with the map $A^0(B\Sigma_d) \to A^0(*) = \pi_0 A$ given by the inclusion of a basepoint to $B\Sigma_d$ sends $P_d(x)$ to x^d.

In nonzero degrees, power operations can still be defined but land in the cohomology of an extended power of a sphere, or equivalently a Thom spectrum of a virtual bundle over $B\Sigma_d$; see [**Re09**, 3.11].

In good cases, one can identify $A^0(B\Sigma_d)$ and decompose the total power operation into more manageable pieces. Most famously, the mod p cohomology of $B\Sigma_p$ for $p > 2$ is the tensor product of an exterior algebra on a generator y in degree $2p-1$ and a polynomial algebra on a generator z in degree $2p-2$. This has at most a one-dimensional $\mathbb{F}_p$-vector space in each degree. If A is an E_∞ algebra over $H\mathbb{F}_p$, one can write

$$A^*(B\Sigma_p) = \pi_{-*}A \otimes_{\mathbb{F}_p} H^*(B\Sigma_p; \mathbb{F}_p) = \pi_{-*}A \otimes \mathbb{F}_p[y,z]/(y^2),$$

and take the coefficients of $P_d(x)$ with respect to monomials in y and z to produce further elements of A_*, called the **Dyer-Lashof operations** on x. These operations are a specific algebraic structure that act on the homotopy of any E_∞-$H\mathbb{F}_p$-algebra. In particular, if A is an arbitrary E_∞-algebra, then $H\mathbb{F}_p \wedge A$ is an E_∞-$H\mathbb{F}_p$-algebra, so we get Dyer-Lashof operations on the homology of A; if X is a space, then $F(\Sigma^\infty_+ X, H\mathbb{F}_p)$ is an E_∞-$H\mathbb{F}_p$-algebra, so we get operations on the cohomology of X (which are just the Steenrod operations).

Likewise, let A be an E_∞-algebra over an E-theory E. Besides examples of the above form, a key example here is $A = L_{K(n)}(E \wedge X)$, where X is an arbitrary E_∞-algebra. We can write $A^0(B\Sigma_d) = A_0 \otimes_{E_0} E^0(B\Sigma_d)$, using work of Strickland [**Str98**] which implies that $E^0(B\Sigma_d)$ is always free over E_0. Strickland further proves that the quotient of $E^0(B\Sigma_{p^k})$ by transfers represents deformations of a formal group together with a degree p^k subgroup.

This suggests that Dyer-Lashof-type operations on algebras over E-theory should have something to do with subgroups of formal groups. We now turn to this relationship.

6.2. Power operations: algebra

We now let E be the E-theory associated to a height n Honda formal group over a finite field k containing $\mathbb{F}_{p^n}$. If X is a spectrum, then the zeroth completed E-homology of X receives the structure of an $E_0^\wedge E$-comodule. Put differently, $E_0^\wedge X$ is a quasicoherent sheaf on the formal stack represented by $(E_0, E_0^\wedge E)$. As a functor on CLN, this stack represents deformations $(\mathbb{G}, i, \alpha)$ of the base formal group Γ, together with isomorphisms of their underlying formal groups. In other words, $E_0^\wedge X$ gives us the following information:

- For every $R \in \mathsf{CLN}$ and for every $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$, an R-module $M_\mathbb{G}$, namely the tensor product $R \otimes_{E_0} E_0^\wedge X$ along the map classifying $(\mathbb{G}, i, \alpha)$.
- For every pair of deformations $(\mathbb{G}_1, i_1, \alpha_1)$ and $(\mathbb{G}_2, i_2, \alpha_2)$ over R and isomorphism $\phi : \mathbb{G}_1 \to \mathbb{G}_2$, an isomorphism of R-modules $\theta_\phi : M_{\mathbb{G}_2} \to M_{\mathbb{G}_1}$. In fact, étaleness of the moduli of isomorphisms (Theorem 2.2.8) means that ϕ is induced by an element g of the Morava stabilizer group, which lifts to a commutative diagram

$$
\begin{tikzcd}
E_0 \arrow[r, "g^*", "\cong"'] \arrow[d, "\mathbb{G}_2"'] & E_0 \arrow[d, "\mathbb{G}_1"] \\
R \arrow[r, equal] & R.
\end{tikzcd}
$$

  This induces the desired isomorphism $R \otimes_{E_0}^{\mathbb{G}_2} E_0^\wedge X \to R \otimes_{E_0}^{\mathbb{G}_1} E_0^\wedge X$.
- Moreover, these R-modules and isomorphisms are natural in R. A map $f : R \to R'$ induces isomorphisms $M_\mathbb{G} \otimes_R R' \cong M_{f^*\mathbb{G}}$ carrying each θ_ϕ to $\theta_{f^*\phi}$; given

a composition $R \to R' \to R''$, the corresponding triangle of isomorphisms of modules commutes.

Now suppose that X is an E_∞ ring spectrum. Then $E_0^\wedge X$ has more structure, in the form of power operations. This corresponds to an extension of the above structure by an action of isogenies of formal groups, as was worked out by Rezk [**Re09**] following work of Ando-Hopkins-Strickland [**AHS04**]. (See Equation (2.1) for discussion of the Frobenius map of a formal group.)

Definition 6.2.1. Let Γ be a height n formal group over a perfect field k and let R be a complete local ring. Let σ be the Frobenius automorphism of k. The **deformations of Frobenius category** $\mathsf{DefFrob}_\Gamma(R)$ is defined as follows. The objects of $\mathsf{DefFrob}_\Gamma(R)$ are just deformations $(\mathbb{G}, i, \alpha)$ of Γ over R. The morphisms are naturally graded by $r \geq 0$; a morphism $\phi : (\mathbb{G}_1, i_1, \alpha_1) \to (\mathbb{G}_2, i_2, \alpha_2)$ of degree r is the requirement that $i_1 \circ \sigma^r = i_2$, together with a formal group homomorphism $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ over R such that the square

$$\begin{array}{ccc} \Gamma \otimes_k^{i_1} R/\mathfrak{m} & \xrightarrow{\mathrm{Frob}^r} & (\Gamma \otimes_k^{i_1} R/\mathfrak{m})^{(p^r)} = \Gamma \otimes_k^{i_2} R/\mathfrak{m} \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ \mathbb{G}_1 \otimes_R R/\mathfrak{m} & \xrightarrow[\phi]{} & \mathbb{G}_2 \otimes_R R/\mathfrak{m} \end{array}$$

commutes. In particular, the morphisms of degree zero are precisely those of the groupoid $\mathsf{Def}_\Gamma(R)$.

One can think of $\mathsf{DefFrob}_\Gamma$ as a sort of stack, but valued in categories rather than groupoids. The following definition then is then a definition of a quasicoherent sheaf on this stack.

Definition 6.2.2. An **isogeny module** (resp. **isogeny algebra**) for (k,Γ) is the following information:

- For every $R \in \mathsf{CLN}$, a functor M_R from $\mathsf{DefFrob}_\Gamma^{op}$ to the category of complete R-modules (resp. complete R-algebras) and continuous maps. We will often write $M_R(\mathbb{G})$, or just $M(\mathbb{G})$, for $M_R(\mathbb{G}, i, \alpha)$.
- For every map $f : R \to R'$, a continuous natural isomorphism $M_R(\mathbb{G}) \otimes_R R' \cong M_{R'}(f^*\mathbb{G})$, such that for each composition $R \to R' \to R''$, the corresponding triangle of isomorphisms commutes.

An **isogeny algebra** for (k,Γ) is an isogeny module M together with a lift of M_R to a functor from $\mathsf{DefFrob}_\Gamma^{op}$ to Alg_R, such that the base change transformations $R' \otimes_{R'} M_R \to M_{R'}$ are isomorphisms of algebras. We write $\mathsf{IsogMod}_\Gamma$ and $\mathsf{IsogAlg}_\Gamma$ for the categories of isogeny modules and algebras.

Remark 6.2.3. An isogeny module has an underlying E_0-module M, given by evaluation at the universal deformation $(\mathbb{G}^u, 1, \alpha^u) \in \mathsf{Def}_\Gamma(E_0)$. For any R and $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$, $M(\mathbb{G})$ is the base change $M \otimes_{E_0} R$ along the map $E_0 \to R$ classifying $(\mathbb{G}, i, \alpha)$. We will often write M for the whole isogeny module, where this does not cause confusion.

Remark 6.2.4. An isogeny module can be thought of as a comodule, as follows. There is a moduli space $\mathrm{Isog}_{\Gamma,r}$ parametrizing pairs of a deformation $\mathbb{G}$ of Γ and a deformation of Frob^r out of $\mathbb{G}$. By a theorem of Strickland [**Str97, Str98**], this is an affine formal scheme, represented by a ring $A_r = E^0 B\Sigma_{p^r}/I$, where I is the transfer ideal, which is finite free over E_0. The pair $(E_0, \prod_r A_r)$ then has the structure of a (graded) Hopf algebroid: maps $\eta_L, \eta_R : E_0 \to A_r$ representing source and target, $A_r \to A_s \otimes_{E_0} A_{r-s}$ representing composition, and so on, with the exception that $\prod_r A_r$ has no conjugation automorphism (as the morphisms in the deformations of Frobenius category are not invertible). An isogeny module is then a (complete) comodule for this object, in the usual sense.

Definition 6.2.5. An isogeny algebra A for (k, Γ) **satisfies the Frobenius congruence** if, for each $R \in \mathsf{CLN}$ of characteristic p and each $(\mathbb{G}, i, \alpha) \in \mathsf{DefFrob}_\Gamma(R)$, the following diagram commutes:

$$\begin{tikzcd} \sigma_R^*(A(\mathbb{G})) \arrow[r, "\cong"] \arrow[dr, "\mathrm{Frob}_{A(\mathbb{G})}"'] & A(\sigma^*\mathbb{G}) = A(\mathbb{G}^{(p)}) \arrow[d, "\theta_{\mathrm{Frob}_{\mathbb{G}}}"] \\ & A(\mathbb{G}). \end{tikzcd}$$

Here $\mathrm{Frob}_A : \sigma_R^* A \to A$ is the Frobenius relative to R, defined for any R-algebra A; see Equation (2.1).

We are now in a position to state Rezk's main theorem. Note that our $\mathbb{T}$-algebras correspond to his analytically complete $\mathbb{T}$-algebras; these are also algebras over a monad by the main theorem of [**BF15**].

Theorem 6.2.6. *Let X be a $K(n)$-local E_∞-algebra over E, such that $\pi_* X$ is concentrated in even degrees. Then $\pi_0 X$ is naturally an algebra for a monad $\mathbb{T}$ on E-modules, with the following properties.*

(1) If $\pi_* X$ is a finite free E_*-module, then there is a map from $\mathbb{T}(\pi_* X)$ to the homotopy groups of the free $K(n)$-local E_∞-algebra over X, which becomes an isomorphism after I_n-completion of $\mathbb{T}(\pi_* X)$.

(2) The free algebra functor $\mathsf{Mod}_{E_*} \to \mathsf{Alg}_{\mathbb{T}}$ is the left Kan extension of its restriction to finite free E_*-modules.

(3) There is a functor $\mathsf{Alg}_{\mathbb{T}} \to \mathsf{IsogAlg}_{\Gamma}$ *which restricts to an equivalence between the subcategories of p-torsion-free* $\mathbb{T}$*-algebras and p-torsion-free isogeny algebras satisfying the Frobenius congruence.*

In other words, torsion-free $\mathbb{T}$-algebras reduce to isogeny algebras satisfying a certain property. By virtue of the Kan extension property, $\mathbb{T}$-algebras which are not necessarily torsion-free are isogeny algebras with an extra piece of structure: an operation which appears in the torsion-free case as a witness to the Frobenius congruence. This operation is not necessarily additive and multiplicative, but at least does comprehensible things to addition and multiplication, because of the Frobenius congruence.

Example 6.2.7. When $n = 1$, $\mathbb{T}$-algebras are elsewhere known as θ-algebras. The height 1 Lubin-Tate ring is $\mathbb{Z}_p$: the multiplicative formal group over $\mathbb{F}_p$ has a unique deformation over any p-complete ring R, namely the multiplicative formal group over R. For any r, there is a unique deformation of Frob^r to $\widehat{\mathbb{G}_m} \otimes R$, namely the multiplication by p^r map

$$[p^r] : \widehat{\mathbb{G}_m} \otimes R \to \widehat{\mathbb{G}_m} \otimes R.$$

Thus, an isogeny algebra A for $\widehat{\mathbb{G}_m} \otimes \mathbb{F}_p$ is precisely a complete $\mathbb{Z}_p$-algebra with a single ring operation ψ^p. The Frobenius congruence is the condition that

$$\psi^p(x) \equiv x^p \pmod{p}.$$

If A is an isogeny algebra satisfying the Frobenius congruence, we can therefore define an operation

$$\theta(x) = \frac{1}{p}(\psi^p(x) - x^p).$$

The fact that ψ^p preserves addition and multiplication then implies certain relations on θ, namely

$$\theta(x+y) = \theta(x) + \theta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i},$$

$$\theta(xy) = \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y).$$

A $\mathbb{T}$-algebra is then a complete $\mathbb{Z}_p$-algebra with an operation θ (which is automatically continuous) satisfying these identities. Note that θ determines ψ^p, but ψ^p generally only determines θ in torsion-free θ-algebras.

For higher values of n, $\mathbb{T}$-algebras are much more complicated, but like θ-algebras in certain essential respects. In general, there is a finite set of pth power operations corresponding to the $(p^{n-1})/(p-1)$ order p subgroups of the universal deformation of Γ. These do not necessarily commute with each other, but satisfy Adem relations coming from order p^2 subgroups of the universal deformation. These generators and relations completely describe the algebra of additive operations on $\mathbb{T}$-algebras, a reflection of the fact that a deformation of Frob^r decomposes as an r-fold composition of deformations of Frob^1. Finally, a single operation, like ψ^p, will satisfy the Frobenius congruence, producing a single non-additive θ-like operation on $\mathbb{T}$-algebras, which is uniquely determined in the torsion-free case. At height 2 and small primes, these structures have been calculated explicitly by [**Re08**] and [**Zhu14**]. At all heights and primes, the operation satisfying the Frobenius congruence is identified by [**Sta16**].

This is all a variation on a by now familiar theme: topological information about E-theory is expressible in terms of algebraic information about deformations of formal groups. Here, we are interested in using Rezk's power operations to analyze E_∞ structures on LE. This relies on the following result, which to the author's knowledge is not proved in the literature, and is here stated as a conjecture.

Conjecture 6.2.8. Let A be an $\mathbb{T}$-algebra. Then there are successively defined obstructions to realizing A as $\pi_0 X$, where X is an even periodic E_∞-algebra over E, in the $\mathbb{T}$-algebra André-Quillen cohomology groups

$$D^{s+2}_{\mathsf{Alg}_{\mathbb{T}}}(\pi_0 X, \Omega^s \pi_0 X),\ \ s \geq 1.$$

There are successively defined obstructions to the uniqueness of this realization in

$$D^{s+1}_{\mathsf{Alg}_{\mathbb{T}}}(\pi_0 X, \Omega^s \pi_0 X),\ \ s \geq 1.$$

The height 1 version is proved in [**GH05**], and a similar result is claimed in [**Re13**]. For the definition of André-Quillen cohomology, see Section 6.3.3. We indicate why the conjecture is plausible. The obstruction theory machinery of [**GH05**] proceeds by building certain kinds of free resolutions of E_∞-E-algebras. Given an E_∞-E-algebra X, one can construct a simplicial resolution $P_\bullet \to X$ such that each P_n is a free E_∞-E-algebra on a free E-module, and such that the underlying degeneracy diagram of $P_\bullet$ is free on a diagram of these modules, in the sense of [**GH05**, 1.1.9]. By the first statement of Theorem 6.2.6, $\pi_* P_\bullet$ is a similar diagram of completions of free $\mathbb{T}$-algebras. Conversely, given a $\mathbb{T}$-algebra A one wants to realize, one can attempt to do so by realizing a resolution of A by a simplicial $\mathbb{T}$-algebra of this form. To make this work, one needs compatible model structures on simplicial E_∞-E-algebras and simplicial $\mathbb{T}$-algebras, for which one needs $\mathbb{T}$ to be homotopically adapted to E in the sense of [**GH05**, 1.4.16]. However, this is more or less the content of Theorem 6.2.6, and the fact that $\mathbb{T}$ is a monad rather than an operad is not an essential obstacle.

6.3. Constructions with isogeny algebras and $\mathbb{T}$-algebras

6.3.1. The functor of points

Acomplete E_0-algebra A defines functors, which we write

$$\mathrm{Spf}\, A = \mathrm{Spf}\, A_R : \mathsf{Def}_\Gamma(R) \to \mathsf{Sets}$$

sending each $E_0 \to R$, classifying $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$, to the set of continuous ring maps

$$(\mathrm{Spf}\, A_R)(\mathbb{G}, i, \alpha) = \mathrm{Hom}_{R,cts}(A_R(\mathbb{G}), R) = \mathrm{Hom}_{E_0,cts}(A, R).$$

A map $f : R \to R'$ induces a natural isomorphism $f_* \,\mathrm{Spf}\, A_R \cong \mathrm{Spf}\, A_{R'}$, and a composition $R \to R' \to R''$ induces a commutative triangle of natural isomorphisms.

The structure of an isogeny algebra on A then induces an action of each $\mathsf{DefFrob}_\Gamma(R)$ on $\mathrm{Spf}\, A_R$: for each $\phi : (\mathbb{G}_1, i_1, \alpha_1) \to (\mathbb{G}_2, i_2, \alpha_2)$, there is a corresponding (covariant!) map $\phi_* : (\mathrm{Spf}\, A)(\mathbb{G}_1) \to (\mathrm{Spf}\, A)(\mathbb{G}_2)$. These maps are moreover compatible with base change in R. As it is by now too late to stop, we call this structure an **affine isogeny scheme**.

The Frobenius congruence is then the following statement: if R has characteristic p and $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$, then the map

$$\mathrm{Frob}_{\mathbb{G},*} : (\mathrm{Spf}\, A)(\mathbb{G}) = \mathrm{Hom}_{E_0/p}(A/p, R) \to \mathrm{Hom}_{E_0/p}(A/p, R) = (\mathrm{Spf}\, A)(\mathbb{G}^{(p)})$$

is given by precomposing with the Frobenius of A/p.

In settings like ours, where it is easier to talk about rings in terms of the functors they represent, it is useful to know the following:

Proposition 6.3.1. *If A is a finite product of complete local E_0-algebras, then it any isogeny algebra structure on A is uniquely given by its affine isogeny scheme.*

Proof. The isogeny algebra structure maps are given via Remark 6.2.4 by maps

$$\lambda_r : A \to A \otimes_{E_0} E^0 B\Sigma_{p^r}/I.$$

By Strickland's theorem [**Str97**], $E^0 B\Sigma_{p^r}/I$ is finite free over E_0, and thus a finite product of complete local E_0-algebras. Thus, the same is true for $A \otimes_{E_0} E^0 B\Sigma_{p^r}/I$. It follows that the structure map λ_r is given by a finite product of E_0-linear maps from A to complete local E_0-algebras, all of which can be identified with points of $\operatorname{Spf} A$. By the Yoneda lemma and the fact that $E^0 B\Sigma_{p^r}/I$ carries the universal deformation of Frob^{p^r} [**Re09**, 11.9], the maps λ_r define an isogeny algebra structure on A if and only if they define an affine isogeny scheme structure on $\operatorname{Spf} A$. $\square$

6.3.2. The isogeny algebra of E-theory

One example of an E_∞-algebra over E is E itself. The corresponding functor of points is just the constant functor: $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R) \mapsto *$. There is a unique possible structure of an affine isogeny scheme on this, and thus a unique possible structure of an affine isogeny algebra on E.

6.3.3. André-Quillen cohomology

The obstruction groups occurring in Conjecture 6.2.8 are André-Quillen cohomology groups of $\mathbb{T}$-algebras. These are the derived functors of derivations from a $\mathbb{T}$-algebra into a module over a $\mathbb{T}$-algebra. Let's define all the terms in this sentence.

Definition 6.3.2. Let A be a $\mathbb{T}$-algebra. The category of A**-modules**, Mod_A, is the category of abelian group objects in the slice category $(\mathsf{Alg}_{\mathbb{T}})_{/A}$.

If B is an A-module, the zero of B is a $\mathbb{T}$-algebra map $A \to B$ splitting the slice map $B \to A$. The module itself is best thought of as the kernel of $B \to A$, and B as a split square-zero extension $A \oplus M$. Replacing $\mathbb{T}$-algebras with rings for a moment, all this is very exact. An abelian group object in $(\mathsf{Rings})_{/A}$ is precisely a ring of the form $A \oplus M$, where M is an A-module and the multiplication is given by

$$(a, m)(b, n) = (ab, an + bm).$$

Moreover, a map of abelian group objects $A \oplus M \to A \oplus N$ is precisely a map of the form $1 \oplus f$ where $f : M \to N$ is an A-module map.

A module over a $\mathbb{T}$-algebra A therefore has an underlying ring of the form $A \oplus M$, and the $\mathbb{T}$-algebra operations restrict to operations on $A \oplus M$. We will typically call M the 'A-module', and $A \oplus M$ the 'split square-zero extension'; however, the two notions are obviously equivalent.

Example 6.3.3 ([**GH05**])**.** Suppose that $n = 1$, so that $\mathbb{T}$-algebras are θ-algebras. Let A be a θ-algebra. A split square-zero extension $A \oplus M$ has an operation

$$\psi^p : A \oplus M \to A \oplus M$$

with

$$\psi^p(a, m) = (\psi^p(a), \psi^p_M(a, m))$$

for some other operation $\psi^p_M : A \oplus M \to M$. We can write

$$\begin{aligned} \psi^p(a, m) &= (a, m)^p + p\theta(a, m) \\ &= (a^p, pa^{p-1}m) + (p\theta(a), p\theta_M(a, m)), \end{aligned}$$

which means that

$$\psi^p_M(a, m) = pa^{p-1}m + p\theta_M(a, m).$$

Restricting to $M = \ker(A \oplus M \to A)$, we have

$$\psi^p(0, m) = p\theta_M(0, m).$$

Additionally,

$$\psi^p(0, am) = \psi^p((a, 0)(0, m)) = (\psi^p(a), 0)(0, \psi^p_M(0, m)) = (0, \psi^p(a)(\psi^p_M(0, m))),$$

which implies that

$$\theta_M(0, am) = \psi^p(a)\theta_M(0, m).$$

Thus, an A-module M is precisely a complete module M over the complete $\mathbb{Z}_p$-algebra A, together with an operation θ such that $\theta(m+n) = \theta(m) + \theta(n)$ and $\theta(am) = \psi^p(a)\theta(m)$.

Definition 6.3.4. A **derivation** from a $\mathbb{T}$-algebra A into an A-module M is a map $A \to A \oplus M$ in $(\mathsf{Alg}_{\mathbb{T}})_{/A}$. We write $\mathrm{Der}_{\mathbb{T}}(A, M)$ for the set of derivations from A into M. This is an abelian group, using the abelian group structure of $A \oplus M$. We emphasize that $\mathbb{T}$-algebra derivations are required to be continuous.

Example 6.3.5. Again, consider the case of θ-algebras. A derivation $A \to A \oplus M$ is a map $a \mapsto (a, \delta(a))$ satisfying

$$(ab, \delta(ab)) = (a, \delta(a))(b, \delta(b))$$

and

$$(\theta(a), \delta\theta(a)) = \theta(a, \delta(a)).$$

This is equivalent to

$$\delta(ab) = a\delta(b) + b\delta(a)$$

and

$$\delta\theta(a) = \theta\delta(a) + pa^{p-1}\delta(a).$$

Finally, the following is a special case of [**GH05**, Proposition 2.3.1].

Proposition 6.3.6. *There is a model structure on the category of simplicial* $\mathbb{T}$*-algebras in which a morphism* $A_\bullet \to B_\bullet$ *is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) in the Quillen model structure on simplicial commutative rings.*

Definition 6.3.7. The **André-Quillen cohomology** of a $\mathbb{T}$-algebra A with coefficients in an A-module M,

$$D^s_{\mathbb{T}}(A, M),$$

is the sth left derived functor of $D^s_{\mathbb{T}}(A, \cdot)$ applied to M.

6.3.4. Comodules

For the purposes of obstruction theory, we care more about $K(n)$-local E_∞-algebras in general than those which are algebras over E. If X is a $K(n)$-local E_∞-algebra, then $L_{K(n)}(E \wedge X)$ is a $K(n)$-local E_∞-E-algebra. We can recover X as the limit of a $K(n)$-local Adams resolution:

$$X = \operatorname{holim}\left(L_{K(n)}(E \wedge X) \rightleftarrows L_{K(n)}(E \wedge E \wedge X) \cdots \right)$$

On taking homotopy groups, the maps in this diagram are all determined by $E^\wedge_* X$ as an E_*-comodule algebra, or equivalently, as an E_*-algebra with a $\mathbb{G}_n$-action (at least if E is associated to the Honda formal group over a finite field, by Theorem 5.1.1).

Thus, we should think of $E^\wedge_0 X$ as a $\mathbb{T}$-algebra in comodules, or equivalently, as a $\mathbb{G}_n$-equivariant $\mathbb{T}$-algebra. Such a thing can be defined in terms of isogeny algebras. First, note that $\operatorname{Aut}(k, \Gamma)$ acts on the groupoid $\mathsf{DefFrob}_\Gamma(R)$. On objects, this is precisely the action of Remark 2.3.8:

$$(\tau, g)(\mathbb{G}, i, \alpha) = (\mathbb{G}, i\tau, \alpha g^{-1} : \Gamma \otimes^{i\tau}_k R/\mathfrak{m} \xrightarrow{g^{-1}} \Gamma \otimes^i_k R/\mathfrak{m} \xrightarrow{\alpha} \mathbb{G} \otimes_R R/\mathfrak{m}).$$

Given $\phi : (\mathbb{G}_1, i, \alpha_1) \to (\mathbb{G}_2, i\sigma^r, \alpha_2)$ deforming Frob^r, (τ, g) sends this to the same underlying formal group homomorphism $\phi : \mathbb{G}_1 \to \mathbb{G}_2$, a map between the objects $(\mathbb{G}_1, i\tau, \alpha_1 g^{-1}) \to (\mathbb{G}_2, i\sigma^r\tau, \alpha_2 g^{-1})$. This makes sense because the following diagram commutes.

$$\begin{tikzcd}
\Gamma \otimes^{i\tau} R/\mathfrak{m} \arrow[r, "g^{-1}"] \arrow[d, "\mathrm{Frob}^r"'] & \Gamma \otimes_k^{i} R/\mathfrak{m} \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{Frob}^r"'] & \mathbb{G}_1 \otimes R/\mathfrak{m} \arrow[d, "\phi"] \\
\Gamma \otimes^{i\tau\sigma^r} R/\mathfrak{m} \arrow[r, "(\sigma^r)^* g^{-1}"'] \arrow[d, equal] & \Gamma \otimes_k^{i\sigma^r} R/\mathfrak{m} \arrow[r, "\alpha_2"'] \arrow[d, equal] & \mathbb{G}_2 \otimes R/\mathfrak{m} \\
\Gamma \otimes_k^{i\sigma^r\tau} \arrow[r, "g^{-1}"'] & \Gamma \otimes_k^{i\sigma^r} R/\mathfrak{m} &
\end{tikzcd}$$

(Since k is a finite field, $\mathrm{Gal}_{k/\mathbb{F}_p}$ is abelian and contains the Frobenius σ.)

Definition 6.3.8. An **isogeny comodule algebra** is an isogeny algebra A with the following additional structure. For every $(\mathbb{G}, i, \alpha) \in \mathsf{Def}_\Gamma(R)$ and every $(\tau, g) \in \mathrm{Aut}(k, \Gamma)$, there is an isomorphism of R-algebras

$$(\tau, g)^* : A(\mathbb{G}, i\tau, \alpha g^{-1}) \to A(\mathbb{G}, i, \alpha),$$

such that $(\tau_1, g_1)^*(\tau_2, g_2)^* = (\tau_2\tau_1, \tau_2^*(g_1)g_2)^*$ (cf. Definition 2.2.5 and Remark 2.3.8), and making the following diagrams commute.

$$\begin{tikzcd}
A(\mathbb{G}_2, i\sigma^r\tau, \alpha_2 g^{-1}) \arrow[r, "(\tau,g)^*"] \arrow[d, "\phi^*"'] & A(\mathbb{G}_2, i\sigma^r, \alpha_2) \arrow[d, "\phi^*"] \\
A(\mathbb{G}_1, i\tau, \alpha_1 g^{-1}) \arrow[r, "(\tau,g)^*"] & A(\mathbb{G}_1, i, \alpha_1).
\end{tikzcd} \tag{6.1}$$

Again, this can be interpreted in terms of the affine isogeny scheme. The affine isogeny scheme of an isogeny comodule algebra is a collection of functors, compatible under base change in R,

$$\mathrm{Spf}\, A_R : \mathsf{DefFrob}_\Gamma(R) \to \mathsf{Sets},$$

together with bijections

$$(\tau, g)_* : (\mathrm{Spf}\, A_R)(\mathbb{G}, i, \alpha) \to (\mathrm{Spf}\, A_R)(\mathbb{G}, i\tau, \alpha g^{-1})$$

compatible under composition in $\mathbb{G}_n$ and commuting with morphisms in $\mathsf{DefFrob}$ via the diagram (6.1).

6.4. Power operations on $E^\wedge_{n-1,0}E$

We now specialize to the case where F and E are E-theories corresponding to the Honda formal groups, respectively of heights $n-1$ and n, over a finite field k which contains both $\mathbb{F}_{p^{n-1}}$ and $\mathbb{F}_{p^n}$. As we saw in Section 5.2, the ring $F^\wedge_0 E$ represents a fairly simple functor on complete local rings, and is moreover a finite product of complete local rings.

Theorem 6.4.1. *There exists a $\mathbb{T}$-algebra structure on $F^\wedge_0 E$ for each lift of the Frobenius of $k((u_{n-1}))$ to Λ. These structures are all compatible with the action of the Morava stabilizer group for F, and thus define $\mathbb{T}$-algebras in comodules.*

Proof. As $F^\wedge_0 E$ is a finite product of complete local rings (Proposition 5.2.5), it suffices by Proposition 6.3.1 to define the corresponding affine isogeny scheme. If R is a complete local ring, then

$$\mathrm{Spf}\, F_0^\wedge E : \mathsf{Def}_\Gamma(R) \to \mathsf{Sets}$$

satisfies

$$(\mathrm{Spf}\, F_0^\wedge E)(\mathbb{G}, i, \alpha) = \{(j : \Lambda \to R, \gamma : \Gamma_{n-1} \otimes_k^i R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes_{k((u_{n-1}))}^{\bar{j}} R/\mathfrak{m})\}.$$

Let Σ be a Frobenius lift on Λ, and write σ for the Frobenius maps of k and of $k((u_{n-1}))$. We define an isogeny algebra structure as follows. For any morphism $\phi : (\mathbb{G}_1, i_1, \alpha_1) \to (\mathbb{G}_2, i_2, \alpha_2)$ which deforms Frob^r over R, and for each (j_1, γ_1) as above, let $\phi_*(j_1, \gamma_1) = (j_2, \gamma_2)$, where $j_2 = j_1 \circ \Sigma^r$, and where γ_2 is the unique map of formal groups making the following diagram commute:

(6.2)
$$\begin{tikzcd}
\Gamma \otimes^{i_1} R/\mathfrak{m} \arrow[r, "\gamma_1"] \arrow[d, "\mathrm{Frob}^r"'] & \mathbb{H} \otimes^{\overline{j_1}} R/\mathfrak{m} \arrow[d, "\mathrm{Frob}^r"] \\
\Gamma \otimes^{i_1\sigma^r} R/\mathfrak{m} \arrow[r, dashed, "\gamma_2"'] & \mathbb{H} \otimes^{\overline{j_1}\sigma^r} R/\mathfrak{m} = \mathbb{H} \otimes^{\overline{j_1\Sigma^r}} R/\mathfrak{m}.
\end{tikzcd}$$

Such a γ_2 exists and is unique because the maps $\Gamma \otimes^i R/\mathfrak{m} \to \Gamma \otimes^{i\sigma} R/\mathfrak{m}$ and $\Gamma \otimes^i R/\mathfrak{m} \to \mathbb{H} \otimes^{\bar{j}\sigma} R/\mathfrak{m}$ have the same kernel.

This operation is clearly an action of $\mathsf{DefFrob}_\Gamma(R)$ on $\mathrm{Spf}\, F_0^\wedge E$, and compatible with base change. Furthermore, the Morava stabilizer group for F, $\mathbb{G}_{n-1} = \mathrm{Aut}(k, \Gamma_{n-1})$, acts on points (j, γ) by precomposition with γ: (τ, g) sends (j, γ) to

$$(\tau, g)(j, \gamma) = (j, \gamma g^{-1} : \Gamma \otimes_k^{i\tau} R/\mathfrak{m} \xrightarrow{g^{-1}} \Gamma \otimes_k^i R/\mathfrak{m} \xrightarrow{\gamma} \mathbb{H} \otimes R/\mathfrak{m}).$$

Given a morphism ϕ deforming Frob^r, checking the commutativity of Equation (6.1) amounts to observing that the following diagram commutes:

$$
\begin{tikzcd}
\Gamma \otimes^{i_1\tau} R/\mathfrak{m} \arrow[r, "g^{-1}"] \arrow[d, "\mathrm{Frob}_r"'] & \Gamma \otimes^{i_1} R/\mathfrak{m} \arrow[r, "\gamma_1"] \arrow[d, "\mathrm{Frob}^r"'] & \mathbb{H} \otimes^{\overline{j_1}} R/\mathfrak{m} \arrow[d, "\mathrm{Frob}^r"] \\
\Gamma \otimes^{i\tau\sigma^r} R/\mathfrak{m} \arrow[r, "(\sigma^r)^* g^{-1}"'] \arrow[d, equal] & \Gamma \otimes_k^{i\sigma^r} R/\mathfrak{m} \arrow[r, "\gamma_2"'] \arrow[d, equal] & \mathbb{H}_2 \otimes R/\mathfrak{m} \\
\Gamma \otimes_k^{i\sigma^r\tau} \arrow[r, "g^{-1}"'] & \Gamma \otimes_k^{i\sigma^r} R/\mathfrak{m} &
\end{tikzcd}
\tag{6.3}
$$

We now show that it satisfies the Frobenius congruence. We need to use the fact that, as a complete F_0-algebra, $F_0^\wedge E$ is generated by u_{n-1} and the coefficients w and s_i of the universal isomorphism from Γ_{n-1} to $\mathbb{H}$. Let R be a complete local F_0-algebra of characteristic p, with the map $F_0 \to R$ classifying $(\mathbb{G}, i, \alpha)$. Suppose given an F_0-algebra map $f : F_0^\wedge E/p \to R$, classifying a pair (j, γ). We want to show that $\mathrm{Frob}_{\mathbb{G},*}(j, \gamma)$ is classified by the composition

$$F_0^\wedge E/p \xrightarrow{\sigma} F_0^\wedge E/p \to R.$$

It suffices to show that, in the map $\mathrm{Frob}_{\mathbb{G},*} f : F_0^\wedge E/p \to R$ classifying $\mathrm{Frob}_{\mathbb{G},*}(j, \gamma)$, we have $(\mathrm{Frob}_{\mathbb{G},*} f)(x) = f(x)^p$, where $x = u_{n-1}$, w, or s_i. If R has characteristic p, then the map $j : \Lambda \to R$ factors through $\overline{j} : k((u_{n-1})) \to R$. Then $j \circ \Sigma$ factors through $\overline{j} \circ \sigma_{k((u_{n-1}))}$. This map j is nothing but the restriction of the map $F_0^\wedge E \to R$ to $\Lambda \subseteq LE_0 \to F_0^\wedge E$, and reducing mod p gives the statement for u_{n-1}.

Finally, choose formal group laws for Γ_{n-1} and $\mathbb{H}$. Then w and s_i are the coefficients of the universal isomorphism $x \mapsto wx + \sum s_i x^{p^{i+1}}$ between Γ_{n-1} and $\mathbb{H}$. In particular, $f(s_i)$ are the coefficients of the isomorphism γ. In formulas, the square (6.2) is

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & wx + \sum s_i x^{p^{i+1}} \arrow[d, mapsto] \\ x^p \arrow[r, mapsto, dashed] & w^p x^p + \sum s_i^p x^{p^{i+2}}. \end{tikzcd}$$

Clearly, the new isomorphism $\mathrm{Frob}_{\mathbb{G},*}(\gamma)$ has coefficients w^p and s_i^p, as desired. $\square$

6.5. The height 2 case

We specialize to the case $n = 2$ and $n - 1 = 1$. We write $K = E_1$ and $E = E_2$. In this case, a height 1 $\mathbb{T}$-algebra is a θ-algebra, and a height 1 $\mathbb{T}$-comodule algebra is a θ-algebra together with an action of $\mathbb{Z}_p^\times$ that commutes with θ. By [**GH05**], Conjecture 6.2.8 is true and has a corresponding version for comodules:

Theorem 6.5.1. *Let A be an p-complete θ-comodule algebra with a commuting $\mathbb{Z}_p^\times$ action. Then there are successively defined obstructions to realizing A as $K_0^\wedge X$, where X is an E_∞ algebra such that $K_0^\wedge X$ is concentrated in even degrees, in the André-Quillen cohomology groups*

$$D^{s+2}_{\mathsf{ComodAlg}_\theta}(A, \Omega^s A), \ s \geq 1.$$

There are successively defined obstructions to the uniqueness of this realization in

$$D^{s+1}_{\mathsf{ComodAlg}_\theta}(A, \Omega^s A), \ s \geq 1.$$

In the previous section, we constructed θ-comodule algebra structures on $K_0^\wedge E$ corresponding to Frobenius lifts on $\Lambda = Wk((u_1))_p^\wedge$ – that is, to θ-algebra structures on LE_0. There is a simple reason for this. For any $K(1)$-local E_∞-algebra X, the action of θ on $K_*^\wedge X$ commutes with the $\mathbb{Z}_p^\times$-action, and restricts to an action on

$$\pi_*(L_{K(1)}(K \wedge X))^{h\mathbb{Z}_p^\times} = \pi_* L_{K(1)} X.$$

This situation is special to height 1, and has to do with the fact that there is a unique order p subgroup of Γ_1, which is therefore preserved by the Morava stabilizer group.

For $X = E$, LE_* is still even periodic, and the homotopy fixed point spectral sequence collapses to $LE_* = (K_*^\wedge E)^{\mathbb{Z}_p^\times}$. That is, the θ-algebra LE_0 is just a subring of $K_0^\wedge E$, and this is precisely Λ with its chosen θ-algebra structure.

We now prove that any θ-algebra structure on Λ is induced by an E_∞-algebra structure on LE, using the argument of [**GH05**, Section 2.4.3].

Theorem 6.5.2. *For any p-complete θ-algebra A with underlying ring equal to $K_0^\wedge E$, there is an even periodic E_∞-algebra X with $K_0^\wedge X = A$ as θ-algebras, and with $X \simeq L_{K(1)} E_2$ as homotopy commutative ring spectra.*

Proof. We want to show that the obstruction groups

$$D^{s+2}_{\mathsf{ComodAlg}_\theta}(A, \Omega^s A)$$

vanish for $s \geq 1$. Since $\Omega^s A \cong K_0^\wedge \Omega^s E$ is an induced $\mathbb{Z}_p^\times$-module, these reduce [**GH05**, Proposition 2.4.7] to André-Quillen cohomology of θ-algebras without $\mathbb{Z}_p^\times$-action:

$$D^{s+2}_{\mathsf{ComodAlg}_\theta}(A, \Omega^s A) \cong D^{s+2}_{\mathsf{Alg}_\theta}(A, \Omega^s LE_0).$$

The complete cotangent complex of A over Wk is a θ-module, and there is a composite functor spectral sequence

$$\mathrm{Ext}^p_{\mathsf{Mod}_{\theta,A/}}(\pi_q \mathbb{L}_{A/Wk}, \Omega^s LE_0) \Rightarrow D^{p+q}(A, \Omega^s LE_0).$$

But $Wk = K_0 \to K_0^\wedge E = A$ is formally smooth by Theorem 5.2.4 and Remark 5.3.2. Thus, $\mathbb{L}_{A/Wk}$ is just the Kähler differentials $\Omega_{A/Wk}$ concentrated in degree zero, and these are a p-completion of a free module. Finally, there is a resolution $\Omega_{A/Wk}$ by free θ-modules over A,

$$0 \to A[\theta] \otimes_A \Omega_{A/Wk} \xrightarrow{\theta} A[\theta] \otimes_A \Omega_{A/Wk} \to \Omega_{A/Wk}.$$

For any complete θ-module M over A,

$$\mathrm{Ext}^*_{\mathsf{Mod}_{\theta,A/}}(A[\theta] \otimes_A \Omega_{A/Wk}, M) = \mathrm{Ext}^*_{\mathsf{Mod}_A}(\Omega_{A/Wk}, M),$$

which is concentrated in degree zero because A is pro-free. Thus, the André-Quillen cohomology groups are concentrated in cohomological degrees 0 and 1, and in particular, those that can contain obstructions vanish.

This produces a $K(1)$-local E_∞-algebra X with $K_0^\wedge X = A$. Since $K_0^\wedge X \cong K_0^\wedge E$ as $\mathbb{Z}_p^\times$-modules, we also have $K_t^\wedge X \cong K_t^\wedge E$ as $\mathbb{Z}_p^\times$-modules for all t. Thus, the homotopy fixed points spectral sequence

$$H^s_{cts}(\mathbb{Z}_p^\times, K_t^\wedge X) \Rightarrow \pi_{t-s}X$$

has the same E_2 page as that for $L_{K(1)}E$. This is concentrated in cohomological degree zero and collapses to give $\pi_* X = \pi_* L_{K(1)}E$. So X is even periodic, and in particular, complex orientable. Finally, the Landweber exact functor theorem furnishes a (non-unique) equivalence of homotopy commutative ring spectra $X \simeq L_{K(1)}E$. $\square$

We conclude by showing that this construction actually does furnish non-equivalent E_∞ algebras. Let X and X' be two of the E_∞-algebras constructed by the above theorem. An equivalence $X \to X'$ induces a p-adically continuous isomorphism of θ-algebras $\pi_0 X \to \pi_0 X'$. Thus, the question of how many non-equivalent E_∞-algebra structures there are on $L_{K(1)}E_2$ reduces to the purely algebraic question of how many non-isomorphic θ-algebra structures there are on $Wk((u_1))_p^\wedge$.

This appears to be a thorny question. To take the reader into the brambles a little, here is a failed line of argument. There is, of course, a standard E_∞ structure on LE, namely that by $K(1)$-localizing the canonical one on E. The operation ψ^p on $\pi_0 LE$ is induced by a quotient of the formal group of LE by a certain subgroup, namely the kernel of multiplication by p. One can try to extend this subgroup to a subgroup K of the formal group of E. The corresponding quotient operation by K should then induce a height 2 power operation on $\pi_0 E$. In particular, it follows that the operation ψ^p preserves the subring $\pi_0 E = Wk[[u_1]]$ of $\pi_0 LE$. One can easily define θ-algebra structures on $\pi_0 LE$ that do not have this property, thus giving exotic E_∞ structures on LE.

The problem with this argument is that the additive height 2 power operations are really coefficients of a total power operation

$$P_p : E_0 \to E^0 B\Sigma_p/I,$$

where I is the transfer ideal. The ring $E^0 B\Sigma_p/I$ represents [**Str98**] deformations of the formal group of E together with a cyclic order p subgroup. At height 2, this is a rank $p+1$ free module over E_0, and mapping to various rank 1 factors gives power operations of the form $E_0 \to E_0$. However, the height 1 power operation ψ^p is induced from $K(1)$-localizing the total power operation, not any of its coefficients. (As a slogan, the order p subgroup of Γ_2 that lands in the formal group of its $K(1)$-localization varies nontrivially around $\pi_0 L_{K(1)} E_2$.) That is, there is a commutative square

$$\begin{tikzcd}
E_0 \arrow[r, "P_p"] \arrow[d] & E^0 B\Sigma_p/I \arrow[d] & \\
LE_0(LE^0 B\Sigma_p)/I \arrow[r, "\psi^p"'] & {} \arrow[r, equal] & LE_0.
\end{tikzcd}$$

The operation ψ^p sends E_0 to the image of $E^0 B\Sigma_p/I$ in LE_0, and the right-hand vertical map is somewhat elusive. An example is produced in [**Zhu14**, Section 4]: his formula (at $p = 3$) is

$$\psi^3(u_1) = u_1^3 - 27u_1^2 + 183u_1 - 180 + 186u_1^{-1} + (\text{lower order terms in } u_1).$$

Instead of studying this power operation, we simply produce non-isomorphic θ-algebra structures on LE_0. We start with some results and definitions about the structure of θ-algebras.

Recall that

$$\theta(fg) = f^p\theta(g) + \theta(f)g^p + p\theta(f)\theta(g),$$

$$\theta(f+g) = \theta(f) + \theta(g) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} f^i g^{p-i},$$

$$\theta(n) = 0 \text{ for } n \in \mathbb{Z}_p.$$

Lemma 6.5.3. *The operation θ descends to a map $\theta : k((u_1)) \to k((u_1))$.*

Proof.

$$\theta(f+pg) = \theta(f) + p^p\theta(g) + \sum \frac{1}{p}\binom{p}{i} p^{p-i} f^i g^{p-i},$$

which is congruent to f mod p. Thus, the reduction of $\theta(f)$ mod p only depends on the class of f mod p. $\square$

We will show that there are two θ-algebra structures, θ_0 and θ, such that there is no p-adically continuous automorphism of $Wk((u_1))^\wedge_p$ making the diagram

$$\begin{tikzcd}
Wk((u_1))^\wedge_p \arrow[r, "\theta_0"] \arrow[d, "f"'] & Wk((u_1))^\wedge_p \arrow[d, "f"'] \\
Wk((u_1))^\wedge_p \arrow[r, "\theta"'] & Wk((u_1))^\wedge_p
\end{tikzcd}$$

commute. For both θ-algebra structures, ψ^p will be pipe-continuous. In other words, for each n, it will induce a continuous map $W_nk((u_1)) \to W_nk((u_1))$ for the topology in which

$\{u_1^r W_n k[[u_1]]\}$ is a basis of neighborhoods of zero. Such a map is uniquely determined by the image of u_1. However, there is *a priori* a possibility of a non-pipe-continuous isomorphism f intertwining the two pipe-continuous θ-algebra structures. We first deal with this possibility.

Proposition 6.5.4. *Every automorphism of $Wk((u_1))_p^\wedge$, with k a finite field, is pipe-continuous.*

Proof. Such an automorphism is automatically p-adically continuous. The statement follows from Lemma 4.3.9, as soon as we can prove that the induced automorphism of $k((u_1))$ is continuous. Suppose that $k = \mathbb{F}_q$. Let S be the set of power series $1+a_1u_1+\cdots$. Then S is multiplicatively closed, and any $f \in S$ has a $(q-1)$th root $g \in S$, by the binomial theorem. On the other hand, an $f \notin S$ either has $v_{u_1}(f) = 0$ but constant term not equal to 1, in which case it does not have a $(q-1)$th root at all, or $v_{u_1}(f) \neq 0$, in which case it has at most a $(q-1)^m$th root for some maximal m.

It follows that S is exactly the set of elements of $k((u_1))$ which have a $(q-1)^m$th root for all m. Thus, any automorphism of $k((u_1))$ preserves S. Subtracting 1, we see that any automorphism of $k((u_1))$ preserves the set $u_1k[[u_1]]$, and thus that it preserves $u_1^r k[[u_1]]$ for every r. Thus, any automorphism is continuous. □

As an aside, we note that similar arguments apply to θ-algebras. This defeats one possible attempt to construct non-isomorphic θ-algebra structures: defining a discontinuous and a continuous θ-algebra structure, and using Proposition 6.5.4 to show that they are not isomorphic.

Proposition 6.5.5. *Any Frobenius lift on $Wk((u_1))^\wedge_p$, with k a finite field, is pipe-continuous.*

Proof. Again, any Frobenius lift is p-adically continuous, so this follows immediately from Lemma 4.3.9 and the fact that the Frobenius map is continuous on $k((u_1))$. $\square$

Corollary 6.5.6. *If ψ^p is a Frobenius lift of $Wk((u_1))^\wedge_p$, then the mod p reduction of the associated θ is a continuous self-map of $k((u_1))$.*

Proof. The reduction of θ mod p can be recovered from reduction of ψ^p mod p^2 by the formula $\psi^p(f) = f^p + p\theta(f)$ inside $W_2k((u_1))$. In other words, $\theta(f) = \frac{1}{p}(\psi^p(f) - p\theta(f))$, where division by p is the obvious isomorphism from the p-torsion of $W_2k((u_1))$ to $k((u_1))$. Since $\psi^p(f)$ and $f \mapsto f^p$ are both continuous on $W_2k((u_1))$, it follows that θ is also continuous. $\square$

Proposition 6.5.7. *There are two non-isomorphic θ-algebra structures on $Wk((u_1))^\wedge_p$.*

Proof. Let ψ_0^p and ψ^p be the unique pipe-continuous endomorphisms satisfying $\psi_0^p(u_1) = u_1^p$ and $\psi^p(u_1) = u_1^p + p$. Thus, $\theta_0(u_1) = 0$ and $\theta(u_1) = 1$. Suppose that f is an automorphism of $Wk((u_1))^\wedge_p$ such that $f\theta_0 = \theta f$. By Lemma 6.5.3, this equation makes sense mod p, giving a commutative diagram

$$\begin{CD}
k((u_1)) @>{\theta}>{0}> k((u_1))\\
@V{f}VV @V{f}VV\\
k((u_1)) @>>{\theta}> k((u_1))
\end{CD}$$

where f is, by Proposition 6.5.4, a continuous automorphism of $k((u_1))$.

We now work over $k((u_1))$, meaning that all equations are mod p unless it is stated otherwise. We have $\theta(f(u_1)) = 0$, which property is stable under multiplication by elements of k (since $\theta(af) = a^p\theta(f)$). Thus, without loss of generality, we can take

$$f(u_1) = u_1 + g(u_1)$$

where u_1^2 divides g. Then

$$\theta(u_1 + g) = 1 + \theta(g) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} u_1^i g^{p-i}. \tag{6.4}$$

Let v denote the u_1-adic valuation on $k((u_1))$. This satisfies

$$\begin{aligned} v(fg) &= v(f) + v(g), \text{ and} \\ v(f+g) &\geq \min(v(f), v(g)), \text{ with equality if } v(f) \neq v(g). \end{aligned}$$

In particular, for $\theta(u_1 + g)$ to be equal to zero, the two elements of lowest valuation in (6.4) must have equal valuation. We have

$$\begin{aligned} v(1) &= 0, \text{ and} \\ v(u_1^i g^{p-i}) &= i + (p-i)v(g) \geq p + 1. \end{aligned}$$

Thus, we must have $v(\theta(g)) = 0$.

However, this never happens. Still working mod p,

$$\theta(u_1^n) = u_1^p \theta(u_1^{n-1}) + u_1^{p(n-1)}\theta(u_1) = n u_1^{p(n-1)}.$$

Thus, a monomial g with $v(g) \geq 2$ has $v(\theta(g)) \geq 2$ as well. By induction and using the theta sum formula, the same is true for polynomials. By continuity of θ (Corollary 6.5.6), the same is true for power series. $\square$

Corollary 6.5.8. *There are non-isomorphic E_∞ structures on $L_{K(1)}E_2$.*

Proof. This follows by combining Proposition 6.5.7 with Theorem 6.5.2 and the fact that an equivalence of E_∞ structures on $L_{K(1)}E_2$ induces an equivalence of $\mathbb{G}_1$-equivariant θ-algebras on $(E_1)^\wedge_0 E_2$. $\square$

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