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HEGEL ON CALCULUS

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It is fair to say that Georg Wilhelm Friedrich Hegel's philosophy of mathematics and his interpretation of the calculus in particular have not been popular topics of conversation since the early part of the twentieth century. Changes in mathematics in the late nineteenth century, the new set-theoretical approach to understanding its foundations, and the rise of a sympathetic philosophical logic have all conspired to give prior philosophies of mathematics (including Hegel's) the untimely appearance of naïveté. The common view was expressed by Bertrand Russell:

> The great [mathematicians] of the seventeenth and eighteenth centuries were so much impressed by the *results* of their new methods that they did not trouble to examine their foundations. Although their arguments were fallacious, a special Providence saw to it that their conclusions were more or less true. Hegel fastened upon the obscurities in the foundations of mathematics, turned them into dialectical contradictions, and resolved them by nonsensical syntheses. . . .The resulting puzzles [of mathematics] were all cleared up during the nineteenth century, not by heroic philosophical doctrines such as that of Kant or that of Hegel, but by patient attention to detail (1956, 368–69).

Recently, however, interest in Hegel's discussion of calculus has been awakened by an unlikely source: Gilles Deleuze. In particular, work by Simon Duffy and Henry Somers-Hall has demonstrated how close Deleuze and Hegel are in their treatment of the calculus as compared with most other philosophers of mathematics. We also believe that Hegel's treatment of the calculus is worthy of serious examination, not least because the confusions he finds in early nineteenth-century analytical procedures are reproduced in virtually all of the current pedagogical presentations of calculus in the United States. Unlike Duffy and Somers-Hall, however, we believe that the value of Hegel's discussion turns primarily not on its relation to the methodological foundations of metaphysics but on his attempts to forge a connection between analytical techniques and the theory of numbers generally.

This connection does have a metaphysical payoff, but its beneficiary is mathematics rather than philosophy.

1. Objections to Hegel's Interpretation of Calculus

According to Deleuze, Hegel makes two mistakes in his interpretation of the calculus. First, though he sees that there is a true infinite in the notion of the infinitesimal, he ruins this insight by forcing his interpretation into a dialectical logic of contradiction that works on principles at odds with those of the calculus (Deleuze 1995, 310n9).[1] Second, Hegel thinks that mathematical procedures for working with differentials can never explain the relation between infinitesimals and reality; thus, he thinks that his own conceptual logic is required to show this relation. These two mistakes are related, of course, and, on Simon Duffy's view, they are further connected with an historical fact. In Leibniz, integration is characterized as both (a) the inverse transformation of differentiation and (b) the summation of rectangles under the curve (which rectangles have infinitesimal sections of the curve as one side). Hegel follows his own contemporaries in viewing (a) as more promising and (b) as untenable. But, subsequent to Hegel, Augustin-Louis Cauchy and Bernhard Riemann developed (b) with new mathematical techniques that are more powerful: sufficiently powerful, on Deleuze's view, to explain the reality of vanishing infinitesimals without recourse to the Hegelian dialectical logic of contradiction (Duffy 2009, 566–73).[2]

To this same general way of viewing Hegel's approach to calculus, Henry Somers-Hall adds the recognition that, in contrast to the infinitesimal interpretation of the calculus, Hegel takes Newton's conception of fluxions and the ultimate ratio of evanescent quantities to provide the best available interpretation at his time. However, Newton's presentation is afflicted with certain confusions, is thus insufficiently clear in its abstraction, and thus requires the dialectical logic of contradiction to demonstrate its import. Specifically, that logic tries to do so as a form of the unity of the finite and the infinite (Somers-Hall 2010, 561–62). But Hegel does not get beyond this opposition—he "only gets as far as the vanishing of the quantum [*dx*], and therefore leaves its status as vanished (from the realm of quanta at least) untouched" (Somers-Hall 2010, 567). For both Duffy and Somers-Hall, Deleuze then responds to this failure by building a metaphysics that depends on the distinctive and unique status of differentials.

In response to these objections and that of Russell, we advance the following interpretation: neither contradiction nor anything specific to Hegelian dialectical *method* is playing any fundamental role here; instead, the crucial notion is quantity itself. Before coming to his extended engagement with the calculus, Hegel had already advanced a conception of quantity very much in tune with later developments in the theory of real numbers. In particular, he advanced such a conception that embeds the notion of the limit of a series at the heart of quantitative determinacy.[3] Hegel's argument is more indirect than that of Deleuze: the calculus brings into greater relief problems that are inherent in any notion of quantitative determinacy, and so the same resources that must be brought to bear to make sense of the latter can be extended to make sense of the former.[4] According to Hegel one must look through the infelicities of analytical techniques to conceptualize what is actually being done, and focusing on the figure of the infinitesimal is a barrier to doing so. Once one does pierce the veil of these infelicities, however, Hegel thinks he can show the way in which quantity necessarily realizes itself as materiality. This is how the metaphysical payoff is to mathematics itself, since Hegel's interpretation seems to explain why mathematics has the powerful insight into the nature of reality that it does.

2. Hegel's Interpretation

A. Relation

The core of Hegel's analysis derives from the realization that derivatives are given by ratios (dy/dx) and, as such, they can be viewed as giving quality to the quantities that are the moments or relata of this relation (dy and dx). This depends on Hegel's conception of quality as the simplest form of a relation, that is, as the necessity of contrast for any determinate character. Mathematically speaking, it is legitimate to take the limit of these ratios, but only if one uses the composite (that is, one can take the limit of dy/dx but not dy or dx). This binds the relata to the ratio as mere moments (that is, dependent or subsidiary elements), which by themselves thus lose their nature as quanta (since quanta are supposed to be independent of comparison or variation). In Hegel's terminology, taking the limit (Grenze) of the ratio supersedes the qualitative character of the composite and transforms it back into a quantity:

> Quite generally: quantum is superseded quality; but quantum is infinite, it surpasses itself, is the negation of itself; this, its surpassing, is therefore *in itself* the negation of the negated quantity, the restoration of it; and what is posited is that the externality, which seemed to be a beyond, is determined as quantum's *own moment*. (WL 21.235–36; italics in original)[5]

This whole process is taken to be definitive of any quantum or number as such, and Hegel's description of the process is just the translation into his own jargon of an ordinary mathematical procedure central to calculus.

It is easy to read such jargon as *invoking as a premise* basic dialectical principles such as the purportedly inevitable negation of the negation, when, in fact, these are rather *exemplified by* the form of calculation that is being interpreted. The dialectical language provides a guiding thread rather than a driving force. Hegel's point is simply that symbols like *dx* should only be considered in ratios, and, as such, they are relational in nature, which ascribes a qualitative character to them:

> Now the conception of limit does indeed imply the stated true category of the *qualitatively* determined relation of variable magnitudes, for the forms of it which occur, *dx* and *dy*, are supposed to be taken simply as only moments of , and itself ought to be regarded as one single indivisible symbol. (WL 21.265)

The higher-order operations of taking the limit of the related terms is a way that the relation surpasses or transcends itself (*über sich hinausgeht*) and generates itself as quantity by dissolving and subsuming the independence of its constituents as specifiable quantities.

The consideration of ratios naturally leads Hegel back to simple ratios of numbers as an illustration. The difference between the ratios of numbers and the ratios of infinitesimals is that hidden in *dx* is a variable *x*. The variability of *x* is fundamental in the usual considerations, since, in order to take the limit, one lets *x* approach certain values, yet this variability is an *ad hoc* property.

It follows from all of this that infinitesimals are not, by themselves, *bona fide* quantities.[6] Indeed, it is very hard to say what a simple infinitesimal—usually symbolized by ε, i, or *dx*— is supposed to be.[7] As a stand-alone object, they are suspect and certainly not numbers (as George Berkeley correctly pointed out).

In Leibniz and Newton, there is a certain mysticism about this infinitesimal quantity, which is why calculus was at first just a way to compute (thus, the name), but not strictly well founded even within mathematics itself. Leibniz's version of the Fundamental Theorem of Calculus requires the difficult step of calculating the function of a curve from its law of tangency, for example, the curve $x=y^2$ with the law of tangency $x=y^3/3$:

$$\frac{dx}{dy} = \frac{(x+dx)-x}{dy} = \frac{(y+dy)^3 - y^3}{3 \cdot dy}$$
$$= \frac{y^3 + 3y^2 \cdot dy + 3y \cdot (dy)^2 + (dy)^3 - y^3}{3 \cdot dy}$$
$$= y^2 + y \cdot dy + \frac{(dy)^2}{3} = y^2$$

What is suspect in the calculation is the last equality. To obtain the desired result, one has to treat dy and hence dy^2 as zero. On the other hand, if one does this in the first or second line, one obtains problematic expressions. The question that has to be dealt with is the following: what is the reason to keep only the first term y^2 and drop the second two terms $y\mathrm{d}y$ and $\frac{dy^2}{3}$? Going back one equation, this becomes the question of why ultimately one only keeps the term $3y^2\mathrm{d}y$ of the numerator and not the terms with higher powers of $\mathrm{d}y$. There is a different rendering of the computations separating $\mathrm{d}x$ and $\mathrm{d}y$ that goes as follows, if $y=x^3$

$$y + dy = (x + dx)^3 = x^3 + 3x^2dx + 3xdx^2 + dx^3$$

Using that $y=x^3$, one arrives at $\mathrm{d}y=3x^2\mathrm{d}x$, but only if one drops the terms with higher powers of $\mathrm{d}x(3xdx^2 + dx^3)$ (and keeps only the linear term: that is, the term linear in $\mathrm{d}x(3x^2dx)$. This last step is the one that is under scrutiny, for, if $\mathrm{d}x$ were a number, then $\mathrm{d}x^2=0$ would imply that $\mathrm{d}x$ itself is zero. Thus, for Leibniz, this mysticism comes from the fact that an infinitesimal squared is treated as zero, but the infinitesimal itself is not (Laubenbacher and Pengelley 2000, 135). And this feature exposes early modern accounts to Berkeley's criticism, namely, that there is supposedly some quantity or quality that can be disregarded and that suddenly appears or disappears (Berkeley 1999, 73).

To legitimize this last step of the calculus, different arguments had been presented. For instance, in Newton, the arguments for limits involve the nebulous concept of fluxions, and Joseph-Louis Lagrange legitimizes the different treatment for the various terms in the sum by attributing physical meaning to each of them (such as speed, acceleration, and so on). Hegel decisively exposes the deficiency in both interpretations and, hence, the inadequacy of the arguments. What these approaches (and others like that of Lazare Carnot) have in common is that, in order to treat these objects sensibly, they are amalgamated with other properties to make their existence more credible. Hegel correctly points out that this amalgamation is an insufficient, confusing, and ad hoc response. This is summarized nicely in his criticism of Christian Wolff's presentation as employing the latter's "customary way of popularizing things—in effect, by polluting the concept and replacing it with false sense-representations" (WL 21.256).

Subsequently, Hegel offers his own argument for the selection of the linear term by appeal to the nature of the relation . But before going into the details of this argument, it will be convenient to recall the modern definition of a derivative and point out the relevant features for Hegel's philosophical analysis:

$$\frac{df}{dx}(x_0) = \lim_{\varepsilon\to 0}\frac{f(x_0+\varepsilon)-f(x_0)}{\epsilon}$$

The right-hand side contains two nested operations: first a ratio and then a limit. The left-hand side is indeed an unseparated ratio; it is often written as $\frac{dy}{dx}$, where one first sets $y = f(x)$ The realization that there is a second operation on the previously formed ratio that needs to be considered is central to Hegel's argument.

If we attend to the contrast between this modern way of proceeding and Newton's, we can bring the distinction between Newton and Hegel into relief. Newton does not yet have the concept of a limit (this comes from Cauchy), so Newton's version of the above formula is

$$dy = \frac{dy}{dx}dx$$

In this formula, the limit is already inherent in the quantities dx and dy, but only as implicit. The problem is that, in the calculation on the basis of this formula, higher-order terms are generated that have to vanish to generate the desired result (as we saw above with respect to Leibniz). Without the explicit concept of a limit, Leibniz and Newton are denied the possibility of explaining why this vanishing is legitimate. Furthermore, they have to explain why, in the determination of the ratio of vanishing, higher-order terms are neglected.

While Hegel embraces the fact that, in Newton's argument, a certain quality of the differential takes shape as the last value of the ratio (the *quacum evanescunt*), Hegel rejects Newton's reasoning for the negligibility of the higher-order terms legitimizing the calculation. In fact, he ponders what "could have brought Newton to deceive himself about such a proof" (WL 21.261). Hegel thinks that, had Newton only realized that the composite is a quality now—a ratio of the two quantities together that remains as the individual constituents vanish—then he would have had an adequate response to Berkeley and the other, more sympathetic critics who proposed alternative forms of analysis (WL 21.262–63). If the composite is essentially relational, then there is nothing strange in the idea that it might have a determinate character over and above whatever determinate character its relata have, and thus a determinate character that might remain *or even first become clear* in the relation between the changes between those relata undergo as they get smaller.

In fact, the way Hegel explains this is the way we currently take the limit of the ratio as it goes to zero in contemporary mathematics:

> Since at issue here is a *relation* and not a *sum*, the differential is completely given *by the first term*; and where there is the need of further terms, of differentials of higher orders, their determination does not involve the continuation of a series as *sum*, but the *repetition* rather of one and the same *relation*, the only one wanted and the one

which is, therefore, already *completely* determined in the *first term* (WL 21.264; italics in original).

What is important is not the series qua sequence or quantitative sum, but rather the basic form of the relation. That basic form is *revealed* by the operation of taking the limit of the ratio itself as the quantitative values of both numerator and denominator to zero. More concretely, Hegel argues that the linear term ($3x^2dx$) *constitutes* the relation between dx and dy, while the higher terms represent merely repeated applications and introduce no new quality or relation.

Hegel's solution holds that a *quality* is given by the ratio but then denies that this in any independent way survives when the quantities vanish by computing *modulo* higher-order terms (for example, as a Newtonian fluxion). Instead, the quality given by a ratio (Verhältnis) becomes the sought-after *quantity* when taking the limit. A surviving *independent* quality would be something with which the quantity was amalgamated (whether that is conceived as fluxion or something else). The key is that, for Hegel, the surviving quality is internal to the quantity generated by the limit-taking process rather than something that survives the destruction of the quantity in the infinitesimal. Thus (as Somers-Hall notes), Hegel does not follow Newton's argument in full, only that to which it points (the *ratio* with which they vanish); Newton just does not know what that means (Somers-Hall 2010, 562). Hegel explicitly says that he cannot endorse the fluxions or *quacum evanescent*—these are still the "something else" with which the differential is amalgamated to make it accessible to intuition.

Thus, two features distinguish Hegel's view from Newton's: first, this qualitative nature is something intrinsic to the mathematical phenomenon (WL 21.310–22), not something with which it is amalgamated in an ad hoc manner. Second, it is something essentially conceptual or logical, rather than intuitive or physical. But note that the qualitative nature is a conceptual or logical feature required to do justice to the mathematical phenomenon (that is, it is what is rendered as a limit in contemporary mathematics), not for idiosyncratically Hegelian methodological reasons (such as the purported inevitability of dialectical contradiction). There is, of course, a sense in which this conceptual or logical feature is added to the notion of quantity by means of a dialectical process. That process, however, works not by forcing the mathematical phenomenon into a preconceived pattern but rather by noticing that a widely recognized confusion about the phenomenon can be traced back to the tension between qualitative and quantitative aspects of that phenomenon. This is the sense in which contradiction functions here for Hegel as at most a heuristic thread to follow rather than a methodological premise, contra Deleuze.

We would do well to pause here and regard Hegel's analysis in the light of modern mathematical practice. Hegel realizes that nesting operations are necessary and that the ratio has to be understood as qualitative, and it is certainly true that we only get the derivative after taking the limit of the ratio as a whole. Before such an operation, we find only a quotient, and there is no way to take the limit just as a limit of the numerator divided by a limit of the denominator. This would yield the nonsensical and derided as Hegel and all critics have pointed out. This Gordian knot is cut by modern mathematics almost tautologically by defining the limit of a Cauchy sequence to be the class of the sequence itself, just like a decimal expansion defines a real number. In this connection, we should read Hegel as offering a philosophical concept (Begriff) to capture the nature of this class and an answer to the question of its justification. This answer is to insist on the *qualitative* nature of that composite, or its essential character of being a ratio. It then follows that, to take the limit, there must be brackets around the ratio, treating it as a whole. And, indeed, it is the case that the sequence of ratios is subjected to another ratio like equivalence.[8] Furthermore, the fact that a derivative is a becoming or nascent quantity is also reflected in the modern treatment, as the fact that a derivative can only be computed locally. One needs a neighborhood of a point to be able to define the derivative at that point; this neighborhood can be arbitrarily small, but it is necessary. Technically, this is called a germ, which suggests its nascent character.[9] Again, there are important philosophical and mathematical issues involved in conceptualizing the practice of calculus, but Hegel does manage to show that it can be done by extending the same logical resources already required to understand number as such—here, resources relating to ratios.

B. Variability

Resources related to ratios can be extended to eliminate the ad hoc character of the disregarding of all but the linear term, but the ad hoc nature of the variability at the heart of calculus remains. Philosophically, we would like to know why we have the two terms dy and dx in the ratio in the first place rather than determinate quantities. But *Hegel realizes that quanta are themselves inherently variable*, and, if it is in the basic nature of all quantitative determinacy to be variable, then the use of such variability in the methods of differential calculus is not an arbitrary and ungrounded novelty in mathematical practice but rather an extension of other mathematical uses of numbers that brings into relief one of their essential properties. Thus, Hegel gives a philosophical meaning to Newton's fluxions by reinterpreting it: on this interpretation, the *quacum evanescent* is not a vanishing but a creation; it is a becoming (Werden) that is inherent in quanta. But if that is true, then the logical form expressed so confusingly by infinitesimals should be amenable to treatment by the techniques that Hegel had already developed in his discussions of number. Here, unfortunately, we can no longer avoid one of the classic tropes of Hegelian metaphysics, namely, the distinction between true and false infinites; but we can also illuminate it from the modern mathematical perspective.

Hegel investigates the role of the infinite in both ratios and derivatives, frequently making use of his distinction between false and true infinities. The main examples for the former are series, whereas true infinities have to include a certain aspect of transcendence or change from mere quantity to quality and vice versa. The series do not have this property; they are merely truncated quantities. Here one would naturally be inclined to criticize Hegel for not realizing that the real numbers are about convergent sequences (to be precise, Cauchy sequences). However, one would miss an important point: the real numbers are not sequences, but *classes* of sequences, which is a *quality* of the sequences in Hegel's terms. And so it is quite right of Hegel to point out, for instance, that a decimal expansion, of, for example, one-third or one-seventh is not a true infinity, but that actually the ratio is much closer, since it contains a quality. In modern mathematical terminology, this is encoded by the fact that rational numbers are classes.[10] As Hegel explains it, quotients as rational numbers are constituted of three parts (the two relata and the value of their relation, which Hegel calls the exponent); this gives them a qualitative aspect. Their value, the class, is then again a quantity, but, for this, one needs an infinite process of taking classes of ratios. Hegel then simply applies this result from his number theory to infinitesimals by observing that one *must* add the variability of the constituent (here, the variability of x in dx). The variability is thus conceptually necessary rather than ad hoc. This insistence on variability resolves the nature of quantity of the constituents of the compound object given by the relation. The constituents are in these considerations just moments (WL 21.325). Only after the infinite process of taking classes of ratios of do we find a quantity.

In modern terms, it is true that, before taking the limit of the ratio , the ratio is that of functions and, in that sense, they are not finite quanta or real numbers; they become such only after evaluation. After this intermediate step of evaluation, one again gets a function. Unaccountably, Hegel tries to bypass this point. This is a shame because it would have helped him bring out some consequences of the necessity of that variability. Specifically, it is actually not true that dx contains a free variable x in any way; otherwise, if y is viewed as some other variable, there would be no relationship of dx to dy. Indeed $\frac{dy}{dx}$ is actually an archaic albeit widely used way of expressing $g\frac{df}{dx}$. The intermediate step left out by Hegel and other users of the $\frac{dy}{dx}$ notation is $y = f(x)$, which expresses y as a dependent variable. This only makes the depth of the relational tie between dy and dx (and thus their qualitative nature) clearer. Thus, the variability of quantity is deeply tied to its essentially relational nature.

C. Powers

So far, we have seen the application of the resources Hegel developed to comprehend ratios and variability to the case of the ratio $\frac{dy}{dx}$ and its varying numerator and denominator in particular. In another extension, Hegel takes up his understanding powers (WL21.197–203) and applies them to calculus. In fact, he claims that derivatives can be exhaustively analyzed into powers and thus treated algebraically.[11] There are two points of contact between derivatives and powers; one is mathematical, and the other is more philosophical.

The mathematical point is that differentiation picks out the information about the exponent. This is obvious in the equation that Hegel uses for his demonstration: $dx^n = nx^{n-1}dx$ (WL 21.273). As Hegel correctly points out, this is not as simple as it sounds, because it pertains to the relation of the variable x of the function and the power n that defines the function. Hegel reduces this special relation to binomials and then gives a clever argument assimilating it to his previous discussion of powers. A payoff for Hegel is that, in this line of reasoning, he can construct his own interpretation of the fundamental step of picking out the linear term and, moreover, its coefficient. The argument proceeds in several steps.

(1) He first observes that, in a variable quantity, it is inherently always possible to consider adding to them and, hence, one inherently is led to reflect upon the function of a sum as in $(x + a)^n$. This expression can be expanded by considering the binomial expansion $(x + a)^n = x^n + nx^{n-1}a + \cdots$. This is crucially inherent in quantity itself as quantities can be taken to their nth power, as argued previously.

(2) The linear term $nx^{n-1}a$ can now be taken to be a relation between a and x. Hegel now asks: Why should we consider this linear term to be fundamental? The first answer is that this is the fundamental relation between powers and the leading and following terms are all iterates of this fundamental relation (this step is the more developed version of the argument from ratios we considered in § IIA). Indeed, as he points out, formally this follows from the product rule.

(3) Hegel's second answer is that it is not the position of the terms in the expansion that is fundamental, but rather the relation that each

term and its successor stand. This relationship is fully contained in the power being considered: the power n for the first two terms, the power n-1 for the second and third term, and so on. Here, there is a role reversal in which each term is essential for its successor.

(4) In calculus, usually a is replaced by an infinitesimal dx and expanded only to the first order, but Hegel posits that one may as well set a equal to 1, or the unit. This is, in a sense, more natural, since now one is augmenting by units (Einheiten). This circumvents the introduction of infinitesimals and directly yields the expression $x^n + nx^{n-1}a + \cdots$, which only involves different powers of x and thus eliminates the need for an infinitesimal.

(5) The coefficients are now the relations between the different powers of x and in particular the coefficient n of the linear term, which gives the relation between the nth power and the n-1st power. This is why we are justified in picking out the linear term and discarding the rest: which relation one picks depends on the power one is considering, but, for the function at hand, that power is n, which, therefore, constitutes the primate. The first successor is then fundamental and the further successors higher-derived quantities. This is the meaning of his identification of "the *reduction* of the magnitude to the next lowest power [die *Herabsetzung* der Größe auf die nächst niedrigere Potenz]" (WL 21.282.12–13) as the main driver of the analytical technique.

It would be good at this point to connect this positive argument with Hegel's earlier criticism of Newton and others for their amalgamation of sensible or otherwise nonmathematical qualities with the derivative that remains as the ratio goes to zero. The argument Hegel provides replaces this nonmathematical quality with a mathematical one: namely, the relation between a power and the next-lowest power. The argument from ratios tells you why the series qua series is not that important (that is, it tells you why what is important is the relation at which the series points), and the argument from powers tells you that that relation is to be found in the linear term (that is, in the relation between the nth and n-1th powers). So one can consider Hegel to have an overarching a two-step argument. The first step tells you the point of the mathematical formalism, and the second tells you the part of the formalism to isolate to determinate its content.

This connects with the earlier point that the progression of the ratio to 0/0 is the mathematical form of the divergence of measures (for example, in phase transitions), which is itself the way that Hegel introduces a nonmathematical concept, namely, essence (Wesen). What Hegel might have seen given his own analysis of powers is that the taking of the limit is itself a way to generate the concrete conception of the relation between dramatically different scales that is built into the notion of an essence. This would be a different route to a far more Deleuzian metaphysics, in which strictly mathematical tools do more work than the more traditionally metaphysical concepts Hegel employs in the Doctrine of Essence. But, instead of using the divergence of measures, he turns to a different mathematical resource developed earlier in his number theory: the relation between unit and amount.

This leads us to the philosophical point of contact between derivatives and powers, which concerns the coincidence of the two conceptual aspects of number (Einheit and Anzahl, or unit and amount) (WL 21.275). This is given by noticing that, in a multiplication one factor is the unit) and the other is the amount. When these two coincide, then their difference in quality is resolved, and this transforms multiplication to power and to a quantity. Thus, in computing squares, there is a similar transcendence as in ratios. In this treatment, power (namely, squares) and, with it, higher powers now become an inherent property of quanta (by their variability and their relation to units), namely, that they can be units, thus giving rise to powers, and that they can be augmented by units, thus giving rise to the binomial expansion and the arguments above. Here the mechanism that produces a quantity from the ratio becomes the underlying principle for the definition of powers. Thus, calculus becomes all about powers, which are inherent in the variability of quanta, which was inherent in quanta to start with. Nothing in the way of fundamental principles has to be added that is specific to the calculus itself. The power analysis thus works together with the ratio analysis to show that calculus can be handled by extension of the same basic techniques required for defining real numbers.

But, in the same way that no additional mathematical principles have to be added, it is important to Hegel that neither do any geometrical or physical ideas have to be added. This was also important to Lagrange, as one can see from the subtitle of his "Théorie des Fonctions Analytiques: Principes du Calcul différentielle, dégagés de toute considération d'infiniment petits, d'évanouissantes, de limite et de fuxions, réduits à l'analyse algébrique des quantités finies" (Theory of Analytic Functions: The Principles of Differential Calculus, Set apart from All Considerations of Infinitesimals, Effervescents, Limits and Fluxtions and Reduced to an Algebraic Analysis of Finite Algebraic Quantities). Hegel also rids calculus of these superfluous terms. Furthermore, he establishes algebra's primacy over geometry, which is also the approach of modern mathematics.[12] As mentioned before, the algebraic approach both Hegel and Lagrange pursue is only valid for analytic functions.

Hegel, however, differentiates himself from Lagrange by claiming that the introduction of power series is unnecessary and misleading; only individual powers, and not power *series*, are required. Hegel is right on this point, which deserves some untangling. The power series are only necessary to make taking a derivative an algebraic operation, not to define derivatives. They are key to extend differential to the wider class of analytic functions, which contains exponentials sine and cosine for instance, in contrast to polynomials. In this setting, one additionally needs theorems to say that one can interchange summation and differentiation, effectively separating the two concepts. These are developments in analysis, as even the modern definition of a function, which were not quite worked out to a modern level or rigor yet at Hegel and Lagrange's time. Correctly, these theorems deal with two types of infinities that Hegel bemoans as being confounded. The modern mathematical analysis actually does not confound them, but rather realizes in a precise way that they are two different limits whose interchangeability, as order of operations, is not always guaranteed. Thus, Hegel is right that the series are not the reason for the negligibility of higher orders. Their interplay is, however, tantamount to establishing the algebraic nature of differentiation for a suitable class of functions. Furthermore, Lagrange does make the mistake that Hegel could have foreseen of defining the functions as power series instead of using classes of these. The fact that different power series can give different functions and not all functions are represented by power series was the reason the Cauchy wrote his textbook and refuted Lagrange's view.

D. Integration

Hegel completes his interpretation of calculus with a treatment of integration. This serves a two-fold purpose of demonstrating the applicability of his reasoning that the "reduction" of powers is the fundamental concept (which is inverse to the coincidence of amount and unit in the forming of powers) while also showing the primacy of algebraic thought over geometry. This inversion of the reduction and the coincidence of unit and amount ties into the property of the integral being an antiderivative. Using the method of Bonaventura Cavalieri,[13] he discusses integration as a sum of infinitesimals,[14] and he observes that, in these manipulations, the dx in $\int f dx$ naturally has different flavors depending on whether one is using them to compute area, length, or volume. In Cavalieri's principle, it is not so much that one is multiplying lines by lines to get an area, but that one is again reflecting upon the ratio of two areas (usually to say that they are equal). This ties back into the application of Hegel's primate of the next leading power, which is most apparent in the calculation of area as the integral over the function giving the boundary. Here one imagines vertical lines making up the area; these are then "summed" together. The integral then is the transition of a sum of lengths to an area, which is exactly going up one power from length to area=(length)^2.

From a modern perspective, there are two correct results here. When defining integrals, one is actually defining areas or lengths, again a point missed in most explanations and textbooks. The area of a square or volume of a higher dimensional version is defined by an integral, and Hegel holds this identification to be a real insight: "The true merit of mathematical acumen is that, from results already known elsewhere, it has found that certain sides of a mathematical object stand to each other in the relationship of original and derived function, and it has found which these sides are" (WL 21.295). The way to obtain the results "known" from geometry is that one fixes the normalization that the length of the interval from zero to one is indeed one. The other integrals then can be viewed as products and transformations of this integral and, hence, as ratios. Again, the power analysis, thus, works together with the ratio analysis to show that calculus can be handled by extension of the same basic techniques required for defining real numbers.

The second correct result of Hegel's construal of Cavalieri's method is one that Hegel himself misses (but would have been accessible for him at this point): the resolution of Cavalieri's paradox by means of the chain rule. The paradox pertains to the area of the two subtriangles of a triangle created by an altitude of its largest angle. The lines parallel to the height are in 1–1 correspondence, and, under this correspondence, line segments of equal height are paired. So, one could argue, the areas of the two subtriangles should be the same.[15] But this is only the case for isosceles triangles, and not triangles in which the base is otherwise cut into two unequal parts, $\overline{ab}$ and $\overline{bc}$. This is illustrated in Figure 1.

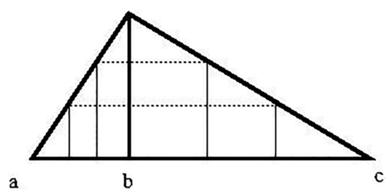

Figure 1

This can be resolved using the chain rule. If one segment $\overline{ab}$ is taken as the unit, the other $\overline{bc}$ is some multiple of it, say k, and hence if dx is the infinitesimal (or measure) on the first and dy is the infinitesimal on the other, then the chain rule says that $dx=kdy$ and hence $\int_a^b dx = \int_b^c kdy$ is the correct identity.

This resolution makes use of several of the features that are important to Hegel. First, k represents the difference in normalization and, hence, a ratio of areas. Second, in the chain rule, it becomes apparent that dx and dy are actually bound together by their ratio; it is this ratio, which is k, that is meant by $\frac{dy}{dx}$. Third, they should not exist alone but are bound by the integral sign, which lends them sense. Finally, as in the case of powers, one can see that the "infinite sum" represented by the integral can have three interpretations: (1) the sum giving the unit and the lines giving the different amounts, as a product (as such this is an area, albeit normalized differently in the two triangles); or (2) regarding this as a sum, we would obtain a line composed of all the pieces, which is not meant, neither in algebra nor in geometry; but (3), finally, the integral sign gives the magnitude of this line, which is a number. Again, there is a normalization (in Hegel's terminology, this is the degree [Grad]).

3. Conclusion

One of the great difficulties in interpreting Hegel's understanding of calculus and of mathematics generally is that he wrote right on the cusp of some of the great technical and conceptual breakthroughs in mathematics in the mid-nineteenth century, breakthroughs that Russell rightly credits with having resolved difficulties that had lingered throughout the history of mathematics and in sharpened form through the early modern period. Chief among these fundamental problems was the problem of understanding mathematical continuity. Although we have many intuitive presentations of continuity (for example, in geometrical figures), it had always seemed difficult to understand such continuity in terms of the discrete numbers that seem, equally intuitively, to be paradigmatic (for example, the natural numbers). For example, the attempt to build up a continuous line out of extensionless points seems incoherent. Hegel thought that, on this point and on the attempts to demonstrate the reality of infinitesimals by means of the amalgamated properties such as fluxions, mathematics foundered on the contradictions of its own practice and it fell to philosophy to explain the objective reality of mathematical concepts (WL 21.237, 252, 256–57, and 272). As strong as Hegel's analysis is, nonetheless it is at this point perhaps at its weakest, since he denies mathematics even the basic capability to treat these topics adequately. Of course, as Russell rightly points out, this has been proven wrong by subsequent developments. Modern mathematics has solved the problems of continuity by using limits and an appropriate definition of the real numbers, yet Hegel's position on calculus still provides constructive insights into the subject that remain relevant. This hinges on the fact that the modern mathematical treatment is at heart very technical and in a sense tautological—a fact that is glossed over in the usual education where recourse is taken to older forms of justifications of calculus—and such a technical and tautological account is in no position to explain the relation of calculus to reality.

The fact that the solutions developed to the problem of the continuum are tautological and thus not intrinsically informative is glossed over in the standard way most of us learned calculus, at least in English-speaking countries. To exemplify this, consider from a Hegelian perspective what is now used as the standard high school and early college definition of a derivative of a function as the slope of the tangent line to graph. This tangent is considered as a limit of secant lines. This is a plausible geometric construction, but how is this legitimized? The key question is what this limit is. In modern mathematical terms, one chooses a local parameterization, that is, a representation in terms of functions and then takes the derivative of these functions. The derivative itself is a limit. If this limit exists, then it gives the direction of the tangent line. The exact mathematical background is actually not that important for the discussion at hand; we only need that the calculation uses a nongeometric representation and a limit. At this point, one should recall that historically there was a struggle to define the real numbers, and one of the two definitions of real numbers that come to be used is Cauchy sequences modulo null sequences. The limit is then a real number by virtue of representing such a sequence, which is the tautological part of the definition mentioned above. Bypassing these modern achievements, we still are used to handling the real number line as a basically pre-existing geometric object. The derivation used today for derivative in calculus instruction either uses spontaneous velocity (already debunked by Zeno) or the characteristic triangle of which Hegel rightfully says does not hold up to rigorous scrutiny.

These, then, also enter into both common and intellectual debate. More than that: in the guise of infinitesimal variations, as in Jean Le Rond d'Alembert's principle, they are alive and well in physics. One of the struggles of mathematical physics has been to make sense of these variations and integrals in infinite dimensional spaces as needed for a rigorous definition of path integrals. These older forms of justification are susceptible to valid criticisms, which have been voiced from Zeno to Berkeley and in a very detailed way by Hegel. But Hegel does more than criticize: he draws attention to the key issue, namely, how infinitesimals as limits are employed, and offers a fundamental philosophical treatment that has a metaphysical payoff for mathematics itself.

Though Hegel is clearly opposed to the amalgamation of conceptual structures of mathematics with physical or dynamic notions intended to make those structures more intuitive, this very opposition has a materialist motivation. So long as such physical or dynamic notions are taken as given and then merely combined with the conceptual structures, Hegel thinks that the problem of showing the reality of such structures is insoluble; the question has simply been begged. In contrast, Hegel at several points provides an argument that what we mean by physical or material objects has largely to do with the integrity and continuity that such conceptual structures describe. This is clearest in his discussion of measure, where he argues that to be a physical object just is for something to define its own scale and metric (WL 21.345–46). But Hegel pursues a similar argument in his discussions of calculus (WL 21.255 and 279), which date from the same late revision to the *Science of Logic*. There is no space here to pursue the issue, but it is clear that Hegel means to provide his own argument for the objectivity of mathematics that circumvents the Kantian reference via intuition to space and time. Space and time provide the privileged contexts in which mathematical relations can be observed and tested with greatest clarity, but the objectivity of those relations is secured more directly, by the way that they constitute the very core of our conception of a material object as such.

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Keywords: Hegel, calculus, Newton, mathematics, Deleuze

NOTES

1. Though more sympathetic, similar interpretations of Hegel's view as driven by extraquantitative methodological concerns are found in Lacroix (2000) and Moretto (2013).

2. Or, as Henry Somers-Hall puts it, which are sufficient to "finesse the paradoxes resulting from [the infinitesimal interpretation of the calculus]" (2010, 560).

3. See Kaufmann and Yeomans (2017). Deleuze takes the view that "the limit must be conceived not as the limit of a function but as a genuine cut" (1995, 172), and Hegel appears to agree (see WL 21.265).

4. Compare Somers-Hall: "For [Hegel], the difficulty of differentials appearing in the resultant formulae is resolved . . .through recognizing that the status of the nascent ratio differs from that of normal numbers" (2010, 570–71).

5. Citations to Hegel's *Science of Logic* (WL) are to Hegel 1968, vol. 21; English translations are modified from Hegel (2010).

6. Of course, there is nonstandard analysis, which gives a mathematical home to infinitesimals, but only in a very construed or particular way. There is also an algebraic way to counter the problem of the square of an object being zero, with the object itself not being zero, in the ring of so-called dual numbers.

The latter is again a local technique that does not undermine the general point that *dx* is not a number.

7. As *dx*, there is another mathematical interpretation, namely, that of a one form as used in integrals. Hegel treats this interpretation in the third remark added to the discussion of the infinite (WL 21.299–309).

8. The equivalence in mathematics goes through the definition of function, which is given by evaluation. The differentiation is defined pointwise. Here Hegel did not foreshadow this development but discards it (WL 21.259).

9. This is the extra structure of a topology on the real numbers that is needed to capture continuity. See Kaufmann and Yeomans (2017).

10. Hegel's argument for this as a conceptual truth is one of the high points of his philosophy of mathematics. See Kaufmann and Yeomans (2017)

11. This is not quite true, though it arguably applies to so-called analytic functions.

12. Its independence and yet relevance for physics is an entirely different matter, but the current view within mathematics is one that fits well with Hegel's general analytical framework.

13. Cavalieri's principle in two dimensions states that two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas. An example is two parallelograms with the same height and base.

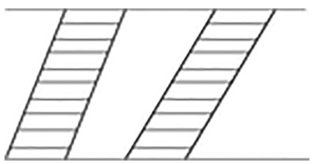

Figure 2

14. The integral sign invented by Leibniz as an elongated S stands, indeed, for summa.

15. This conclusion is prohibited by Cavalieri's principle, since he assumes that the line segments have to be cut by the same line.

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