§5.7 29 (Hint)

(a) (i) We look at the cofactor expansion of $p(\lambda) = \det(A - \lambda I)$ along the first row

$$p(\lambda) = (a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} + \underbrace{a_{12} \begin{vmatrix} \cdots \end{vmatrix} + \cdots}_{\text{Note that this is a poly in } \lambda \text{ of degree} \le n-2}$$

By induction, we have

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + \underbrace{\cdots\cdots}_{\text{poly in } \lambda \text{ of degree} \le n-2}$$

So $\lambda^n, \lambda^{n-1}$ are only contained in

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

The $\lambda^{n-1}$ term in the above expression is

$$(-1)^{n-1}(a_{11} + \cdots + a_{nn}) \lambda^{n-1},$$

indeed if we expand $(a_{11} - \lambda) \cdots (a_{nn} - \lambda)$, we can produce $\lambda^{n-1}$ only if we choose $(n-1)(-\lambda)$ from the $(n)$ parentheses, and one constant $a_{ii}$ from the remaining parenthesis. So all the possibilities are

$$(-\lambda)^{n-1} a_{11}, \quad (-\lambda)^{n-1} a_{22} \quad \cdots \quad (-\lambda)^{n-1} a_{nn}$$

□

(ii) Set $\lambda = 0$. Then

$$p(\lambda)\Big|_{\lambda=0} = (-1)^n \cdot 0 + b_1 \cdot 0 + \cdots + b_{n-1} \cdot 0 + b_n = b_n$$

On the other hand

$$p(\lambda)\Big|_{\lambda=0} = \det(A - 0 \cdot I) = \det A$$