First order DE?

No →

① $y^{(n)} = f(x)$ integrate for n times

② 2nd order. Let $u = y'$

Yes ↓

(*) Write the DE in the form
$M(x,y)\,dx + N(x,y)\,dy$

This is always possible by simply multiplying both sides of the DE by dx

↓

Exact?

Yes →

Find out $\phi(x,y) = \int M(x,y)\,dx + h(y)$.
Then $\phi = c$ is the general solution

No ↓

(**) Rewrite the DE in the form
$\frac{dy}{dx} = f(x,y)$

This is always possible. Indeed, from (*) we have $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$.

↓

$\frac{dy}{dx} = p(x)q(y)$?

Yes →

separation of variables: $\int \frac{dy}{q(y)} = \int p(x)\,dx + C$

NO ↓

$\frac{dy}{dx} + p(x)y = q(x)$?

Yes →

$I(x) = e^{\int p(x)\,dx} \Rightarrow (I(x)y)' = I(x)q(x)$

No ↓

$\frac{dy}{dx} + p(x)y = q(x)y^n$?

Yes →

Divided by y^n. Let $u = y^{1-n}$. Then
$u' + (1-n)p(x)u = (1-n)q(x)$

NO ↓

No (Go back to (**))

$f(x,y) = f(tx, ty)$

Yes → Let $V = \frac{y}{x}$ $\Rightarrow$ $xV' + V = f(1, V)$

No (Go back to (*))

(i) If $\frac{\partial_y M - \partial_x N}{N}$ depends only on x, then

$$I(x) = e^{\int \frac{\partial_y M - \partial_x N}{N} dx}$$

is an integrating factor

(ii) If $\frac{\partial_y M - \partial_x N}{M}$ depends only on y, then

$$I(y) = e^{-\int \frac{\partial_y M - \partial_x N}{M} dy}$$

is an integrating factor

Exer: determine the type of the following DE.

(a) $(x^4 - 2t^3x)\,dt + (t^4 - 2tx^3)\,dx = 0$

(b) $\sec^2 t \tan x \, dt + \sec^2 x \tan t \, dx = 0$

(c) $x^2 + t^2 \frac{dx}{dt} = tx \frac{dx}{dt}$

(d) $\frac{dx}{dt} = -\frac{x}{t+x^3}$

(e) $x\,dy - 4y\,dx = x\sqrt{y}\,dx$

(f) $\left(\frac{1}{x}\sin\frac{t}{x} - \frac{x}{t^2}\cos\frac{x}{t} + 1\right)dt + \left(\frac{1}{t}\cos\frac{x}{t} - \frac{t}{x^2}\sin\frac{t}{x} + \frac{1}{x^2}\right)dx = 0$

(g) $dy + \frac{xy}{1+x^2}\,dx = \frac{1}{x(1+x^2)}\,dx$

Solution: (a) homogeneous
(b) Separable
(c) homogeneous
(d) exact equation
(e) Bernoulli equation
(f) exact equation
(g) first order linear equation.

Question: What is the type of
$$\frac{dx}{dt} = \frac{x}{t+x^3}$$