1. The general solution of the following differential equation

$$2x^3e^{-y}dx = (x^4+2)dy$$

is

A. $y = C - \frac{1}{2}\ln(x^4+2)$

B. $x^4 = Ce^y - 2$

C. $\ln(\sqrt{x^4+2}) = Ce^y$

D. $\ln[(x^4+2e^y)^2] = C$

E. $\ln(\sqrt{x^4+2}) = e^y + C$

$$\frac{2x^3}{x^4+2}\,dx = e^y\,dy$$

$$\int \frac{2x^3}{x^4+2}\,dx = \int e^y\,dy \qquad u = x^4+2$$

$$\frac{1}{2}\int \frac{du}{u} = e^y$$

$$\frac{1}{2}\ln u + C = e^y$$

$$\ln\sqrt{x^4+2} = e^y + C$$

2. Suppose that $y = y(x)$ is the solution of

$$(4x + y)dx + (x + e^{-y} + 1)dy = 0, \quad y(0) = 0.$$

Then y satisfies

A. $5x^2 - xe^y + xy - x = 0$

B. $2x^2 + xy - e^{-y} + y = -1$

C. $2x^2 + xy - ye^{-y} + y = 0$

D. $(4x + y)^2 + (x + e^{-y} + 1)^2 = 1$

E. $(x + e^y)(4x + y) = 0$

$M = 4x + y \qquad N = x + e^{-y} + 1$

$\partial_y M = 1 = \partial_y N \qquad$ EXACT

$\Phi(x, y) = \int M(x, y)\, dx = 2x^2 + xy + h(y)$

$\partial_y \Phi = x + h'(y) = N = x + e^{-y} + 1$

$h'(y) = e^{-y} + 1$

$h(y) = -e^{-y} + y$

$\Phi = 2x^2 + xy + y - e^{-y} = C$

$\Phi(0, 0) = C = -1$

$\Phi = 2x^2 + xy - e^{-y} + y = -1$

3. A tank contains 200 liters of liquid. Initially, the tank contains pure water. At time $t = 0$, brine containing 3 g/L of salt begins to pour into the tank at a rate of 2 L/min, and the well stirred mixture is allowed to drain away at the same rate. How many minutes must elapse before there are 100 grams of salt in the tank?

A. $100 \ln \frac{6}{5}$

B. $600 - 600e^{-1}$

C. $600 - e^{-1}$

D. $600 + 600e$

E. $100 \ln \frac{7}{6}$

$r_1 = r_2 = 2 \qquad V_0 = 200 \qquad A_0 = 0 \qquad C_1 = 3$

$V = (r_1 - r_2)t + V_0 = 200$

$A' = r_1 C_1 - r_2 \frac{A}{V} = 6 - 2 \cdot \frac{A}{200} = 6 - \frac{A}{100}$

$A' + \frac{1}{100} A = 6 \qquad I(t) = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100} t}$

$\left(e^{\frac{1}{100} t} A \right)' = 6 \cdot e^{\frac{1}{100} t}$

$e^{\frac{1}{100} t} A = \int 6 e^{\frac{1}{100} t} dt = 600 e^{\frac{1}{100} t} + C$

$A = 600 + C e^{-\frac{1}{100} t}$

$A(0) = 600 + C = 0 \Rightarrow C = -600$

$A(t) = 600 - 600 e^{-\frac{1}{100} t} = 100$

$\frac{500}{600} = e^{-\frac{1}{100} t}$

$t = 100 \ln \frac{6}{5}$

4. The rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 5 & 1 & 5 & 4 \\ 7 & 0 & 8 & 5 \end{bmatrix}$$

is

A. 0

B. 1

C. 2

D. 3

E. 4

$$A \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -14 & 10 & -6 \\ 0 & -21 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -7 & 5 & -3 \\ 0 & -7 & 5 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -7 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, then the **sum** of the entries in the second row of A^{-1} is

A. -2

B. -1

C. 0

D. 1

E. 2

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 3 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right]$$

6. The general solution to $ty' - y = t^3e^{-t}$ is

A. $y = -t^2e^{-t} - te^{-t} + Ct$

B. $y = -te - t$

C. $y = -Cte^{-t} + t$

D. $y = -t^2e^{-t} + Ct$

E. $y = te^{-t} + Ct$

$$y' - \frac{1}{t}y = t^2e^{-t}$$

$$I(t) = e^{\int -\frac{1}{t}\,dt} = \frac{1}{t}$$

$$\left(\frac{1}{t}y\right)' = te^{-t}$$

$$\frac{1}{t}y = \int te^{-t}\,dt = -e^{-t}t + \int e^{-t}\,dt \quad IBP$$

$$= -te^{-t} - e^{-t} + C$$

$$y = t(-te^{-t} - e^{-t} + C)$$

$$= -t^2e^{-t} - te^{-t} + Ct$$

7. The determinant of $A = \begin{bmatrix} 1 & 7 & 3 & -5 \\ -3 & 2 & 6 & 0 \\ -1 & 3 & 11 & 4 \\ 2 & -6 & 5 & -9 \end{bmatrix}$ is

A. -2256

B. 3301

C. 4499

D. -3601

E. 0

$$\begin{vmatrix} 1 & 7 & 3 & -5 \\ -3 & 2 & 6 & 0 \\ -1 & 3 & 11 & 4 \\ 2 & -6 & 5 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 3 & -5 \\ 0 & 23 & 15 & -15 \\ 0 & 10 & 14 & -1 \\ 0 & -20 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 23 & 15 & -15 \\ 10 & 14 & -1 \\ -20 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 23 & 15 & -15 \\ 10 & 14 & -1 \\ 0 & 27 & -1 \end{vmatrix} = 23 \begin{vmatrix} 14 & -1 \\ 27 & -1 \end{vmatrix} - 10 \begin{vmatrix} 15 & -15 \\ 27 & -1 \end{vmatrix}$$

$$= -3601$$

8. A is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector. $A\mathbf{x} = \mathbf{b}$ has no solution. Consider the following statements:

(i) $m < n$

(ii) $n < m$

(iii) the rank of $A < n$

(iv) the rank of $A < m$

Which **must** be true?

A. only (ii)

B. only (i) and (iii)

C. only (iv)

D. only (iii)

E. None of the statements has to be true.

$Ax = b$ has no solution means

$$[A \mid b] \sim \left[\begin{array}{ccc|c} & * & & * \\ 0 & \cdots & 0 & * \end{array}\right]$$

non-zero.

So the number of non-zero rows in ref A = rank A $< m$ = the number of rows of A.

For (i), (ii), consider the system

$$0 \cdot x = 1.$$

For (iii), consider

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ 0 \cdot x_1 + 0 \cdot x_2 = 1 \end{cases}$$

9. For what value(s) of λ, is the following system of equations consistent?

$$\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 7x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$$

A. $\lambda = -2, 1$

B. $\lambda = 5, 6$

C. $\lambda = 1$

D. $\lambda = 0, 1$

E. $\lambda = 5$

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 7 & -4 & 11 & \lambda \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & -1 & 1 & 1 & 1 \\ 1 & 7 & -4 & 11 & \lambda \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -5 & 3 & -7 & -3 \\ 0 & 5 & -3 & 7 & \lambda-2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & 1 & -3/5 & -7/5 & 3/5 \\ 0 & 0 & 0 & 0 & \lambda-5 \end{array}\right]$$

when $\lambda \neq 5$. this system has no solution.

when $\lambda = 5$. this system always has infinitely many solutions.

10. If $y = y(x)$ is the solution to

$$y' = \frac{3y^2 + x^2}{2xy}, \quad y(1) = 1,$$

then $y(2) = ?$

A. $2\sqrt{3}$

B. 1

C. $2\sqrt{2}$

D. $\frac{\sqrt{3}}{2}$

E. 0

This is a homogeneous equation.

$y' = \frac{3}{2}\frac{y}{x} + \frac{1}{2}\frac{x}{y}$ Let $V = \frac{y}{x}$

$xV' + V = \frac{3}{2}V + \frac{1}{2}\frac{1}{V}$

$xV' - \frac{1}{2}V = \frac{1}{2}\frac{1}{V}$ $V' - \frac{1}{2x}V = \frac{1}{2x}\frac{1}{V}$

$2VV' - \frac{1}{x}V^2 = \frac{1}{x}$ Let $u = V^2$

$u' - \frac{1}{x}u = \frac{1}{x}$ $I(x) = \frac{1}{x}$

$\left(\frac{1}{x}u\right)' = \frac{1}{x^2}$ $u = x\left(-\frac{1}{x} + c\right) = -1 + Cx$

$u(1) = 1 = -1 + Cx$ $u = -1 + 2x$

$V = \sqrt{2x-1} = \frac{y}{x}$ $y = x\sqrt{2x-1}$

$y(2) = 2\sqrt{4-1} = 2\sqrt{3}$

11. Determine whether the following statement is true or false. If true, state the reason. If false, give a counterexample.

Suppose that A is a square matrix. If $A^2 = \mathbf{0}$, i.e., the zero matrix, then $A = \mathbf{0}$.

False

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$