EXAMPLES OF SECTIONS 5.1

Question 1. T is a linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$, $\mathbb{P}_2$ is the space of all polynomials of degree no more than 2, and

$$T(x^2-1) = x^2+x-3, \quad T(2x) = 4x, \quad T(3x+2) = 2x+6.$$

Find $T(1)$, $T(x)$, and $T(x^2)$.

Solutions.

1. We identify T as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ by the map

$$ax^2+bx+c \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

By the given conditions, we have

$$T(1,0,-1) = (1,1,-3), \quad T(0,2,0) = (0,4,0), \quad T(0,3,2) = (0,2,6).$$

We immediately have

$$T(0,1,0) = \frac{1}{2}T(0,2,0) = (0,2,0).$$

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}.$$

Solving

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we get $\vec{x} = \begin{bmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$. Therefore,

$$T(0,0,1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 2 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}.$$

Finally,

$$T(1,0,0) = T(1,0,-1) + T(0,0,1) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Now we restore this result back to the space $\mathbb{P}_2$ and obtain

$$T(1) = -2x+3, \quad T(X) = 2x, \quad T(x^2) = x^2 - x.$$