EXAMPLES OF SECTION 6.7

Example 1. *Find a general solution to the following differential equation.*

$$2y'' + 18y = 18\tan(3t).$$

Solution. Diving both sides by 2, the differential equation that we will actually be solving is

$$y'' + 9y = 9\tan(3t).$$

The general solution to the corresponding homogeneous equation is

$$y_c(t) = C_1\cos(3t) + C_2\sin(3t).$$

So, we have

$$y_1 = \cos(3t), \quad y_2 = \sin(3t).$$

Now we seek a solution of the form

$$y = u_1 y_1 + u_2 y_2.$$

The Wronskian of these two functions is

$$W(y_1, y_2)(t) = \det\begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix} = 3.$$

We find out u_1 and u_2:

$$\begin{aligned} u_1 &= -\int \frac{9\sin(3t)\tan(3t)}{3}\,dt = -3\int \frac{\sin^2(3t)}{\cos(3t)}\,dt = -3\int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt \\ &= 3\int [\cos(3t) - \sec(3t)]\,dt = \sin(3t) - \ln|\sec(3t) + \tan(3t)|. \end{aligned}$$

Similarly, we find

$$u_2 = -\cos(3t).$$

Therefore, A particular solution is given by

$$\begin{aligned} y_p &= \cos(3t)[\sin(3t) - \ln|\sec(3t) + \tan(3t)|] - \sin(3t)\cos(3t) \\ &= -\cos(3t)\ln|\sec(3t) + \tan(3t)|. \end{aligned}$$

The general solution is

$$y = C_1\cos(3t) + C_2\sin(3t) - \cos(3t)\ln|\sec(3t) + \tan(3t)|.$$

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Example 2. *Find a general solution to the following differential equation.*

$$y'' - 2y' + y = \frac{e^t}{t^2+1}.$$

Solution. The general solution to the corresponding homogeneous equation is

$$y_c(t) = C_1 e^t + C_2 t e^t.$$

So, we have

$$y_1 = e^t, \quad y_2 = te^t.$$

Now we seek a solution of the form

$$y = u_1 y_1 + u_2 y_2.$$

The Wronskian of these two functions is

$$W(y_1, y_2)(t) = \det \begin{bmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{bmatrix} = e^{2t}.$$

Let's find the Green function:

$$K(t,s) = e^{-2t} \det \begin{bmatrix} e^s & e^t \\ se^s & te^t \end{bmatrix} = (t-s)e^{t-s}.$$

A particular solution is now given by

$$y_p = \int K(t,s)F(s)\,ds = \int (t-s)e^{t-s}\frac{e^s}{s^2+1}\,ds = e^t \int \frac{t-s}{s^2+1}\,ds = te^t \tan^{-1} t - \frac{1}{2}e^t \ln(1+t^2).$$

The general solution is

$$y = C_1 e^t + C_2 t e^t + te^t \tan^{-1} t - \frac{1}{2} e^t \ln(1+t^2).$$

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