EXAMPLES OF SECTIONS 7.4

Question 1. Find the general solution of the given systems.
(a)

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

(b)

$$\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}.$$

(c)

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \vec{x}.$$

(d)

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -8 & -5 & -5 \end{bmatrix} \vec{x}.$$

(e)

$$\vec{x}' = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} \vec{x}.$$

(f)

$$\vec{x}' = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \vec{x}.$$

SOLUTIONS.
Remark: To simplify the notation, we will not write an arrow on the top of the vectors.
1a. Start with the characteristic equation

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = -(3-\lambda)(2+\lambda)+4 = 0,$$

whose solutions are the eigenvalues

$$\lambda_1 = 2,\ \lambda_2 = -1.$$

Let us find the corresponding eigenvectors.
<u>$\lambda_1 = 2$</u>:

$$\begin{bmatrix} 3-\lambda_1 & -2 \\ 2 & -2-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} v_1 = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_1 = a \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

As we saw in class, we can drop the free variable a and write

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\underline{\lambda_2 = -1}$:

$$\begin{bmatrix} 3-\lambda_2 & -2 \\ 2 & -2-\lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_2 = a \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Again, we drop the free variable a, obtaining

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues and eigenvectors:

$$\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore the two linearly independent solutions are

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}, \text{ and } x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

1b. Proceeding as in the previous problem, we find

$$\lambda_1 = -2,\ \lambda_2 = -1.$$

and associated eigenvectors

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore the two linearly independent solutions are

$$x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}, \text{ and } x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

1c. Start with the characteristic equation

$$\det \begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)\Big((2-\lambda)(1-\lambda)-1\Big) - (1-\lambda-2) + 2\Big(1-2(2-\lambda)\Big) = 0.$$

Rearranging,

$$(2-\lambda)(1-\lambda)^2 - 1 + \lambda + 1 + \lambda - 6 + 4\lambda = (2-\lambda)(1-\lambda)^2 - 6(1-\lambda)$$
$$= (1-\lambda)\Big((2-\lambda)(1-\lambda) - 6\Big) = 0.$$

The eigenvalues are now easily found to be

$$\lambda_1 = 4,\ \lambda_2 = -1,\ \lambda_3 = 1.$$

Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 4}$:

$$\begin{bmatrix} 1-\lambda_1 & 1 & 2 \\ 1 & 2-\lambda_1 & 1 \\ 2 & 1 & 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix},$$

so we need to solve

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system and ignoring the free variable as before we obtain

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Repeating the process for $\lambda_2 = -1$, $\lambda_3 = 1$ we find, respectively

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues with corresponding eigenvectors

$$\lambda_1 = 4,\ \lambda_2 = -1,\ \lambda_3 = 1.$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The linearly independent solutions are

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t},\ x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t},\ x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t}.$$

1d. We proceed as in the previous problem, finding

$$\lambda_1 = -2,\ \lambda_2 = -1,\ \lambda_3 = 0.$$

with corresponding eigenvectors

$$v_1 = \begin{bmatrix} -2 \\ -1 \\ 7 \end{bmatrix},\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Hence

$$x_1 = \begin{bmatrix} -2 \\ -1 \\ 7 \end{bmatrix} e^{-2t},\ x_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} e^{-t},\ x_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Notice that $e^{\lambda t}$ does not appear in x_3 because the corresponding eigenvalue is zero, so that $e^{\lambda t} = e^{0t} = 1$.

1e. The characteristic equation is

$$\det \begin{bmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{bmatrix} = -9 + \lambda^2 + 18 = \lambda^2 + 9 = 0,$$

whose solutions are

$$\lambda_1 = 3i,\ \lambda_2 = -3i.$$

Recall that we saw in class that in the complex root case, the first root already gives two linearly independent solutions, so it is enough to consider $\lambda_1 = 3i$. We want to solve

$$\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Proceeding as before and ignoring the free variable we find

$$v = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix}.$$

This gives

$$x = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix} e^{3it}.$$

Next, we separate the real and imaginary parts,

$$\begin{aligned} x &= \begin{bmatrix} -2e^{3it} \\ (3+3i)e^{3it} \end{bmatrix} = \begin{bmatrix} -2\cos(3t) - 2i\sin(3t) \\ (3+3i)(\cos(3t) + i\sin(3t)) \end{bmatrix} \\ &= \begin{bmatrix} -2\cos(3t) - 2i\sin(3t) \\ 3\cos(3t) - \sin(3t) + i(3\cos(3t) + -3\sin(3t)) \end{bmatrix} \\ &= \begin{bmatrix} -2\cos(3t) \\ 3\cos(3t) - 3\sin(3t) \end{bmatrix} + i \begin{bmatrix} -2\sin(3t) \\ 3\sin(3t) + 3\cos(3t) \end{bmatrix}. \end{aligned}$$

Hence the two linearly independent solutions are

$$x_1 = \begin{bmatrix} -2\cos(3t) \\ 3\cos(3t) - 3\sin(3t) \end{bmatrix},$$

$$x_2 = \begin{bmatrix} -2\sin(3t) \\ 3\sin(3t) + 3\cos(3t) \end{bmatrix}.$$

1f. As before, we look for solutions of the characteristic equation

$$\det \begin{bmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{bmatrix} = 0.$$

The solutions are

$$\lambda_1 = 0,\ \lambda_2 = 2 \pm 3i \Rightarrow \lambda_1 = 0,\ \lambda_2 = 2 + 3i.$$

where as in the previous problem we can pick only one of the two complext roots. The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}.$$

The solution corresponding to $\lambda_1 = 0$ then becomes,

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

while the two solutions obtained from $\lambda_2 = 2 + 3i$ are

$$x_2 = \begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \\ 2\cos(3t) \end{bmatrix} e^{2t},$$

$$x_3 = \begin{bmatrix} \cos(3t) + \sin(3t) \\ -2\sin(3t) \\ 2\sin(3t) \end{bmatrix} e^{2t}.$$