EXAMPLES OF SECTION 7.6

Example 1. *Find a general solution to the following differential equation.*

$$\begin{cases} X' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} X + \begin{bmatrix} 12e^{3t} \\ 18e^{2t} \end{bmatrix} \\ X(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \end{cases}$$

Solution. 1. 1a. Start with the characteristic equation

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (\lambda - 5)(\lambda + 1) = 0$$

whose solutions are the eigenvalues

2. Let us find the corresponding eigenvectors. When $\lambda_1 = 5$,

$$A - 5I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}.$$

This gives an eigenvector $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. When $\lambda_2 = -1$,

$$A + I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

This gives an eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

3. Therefore, the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix},$$

We will look for solutions of the form

$$X(t) = \Phi(t) \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Then

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \Phi^{-1}(t) \begin{bmatrix} 12e^{3t} \\ 18e^{2t} \end{bmatrix}$$

By Cramer's rule, we have

$$u_1'(t) = \frac{\det \begin{bmatrix} 12e^{3t} & -e^{-t} \\ 18e^{2t} & e^{-t} \end{bmatrix}}{\det \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix}} = 6e^{-3t} + 4e^{-2t},$$

and

$$u_1'(t) = \frac{\det \begin{bmatrix} e^{5t} & 12e^{3t} \\ 2e^{5t} & 18e^{2t} \end{bmatrix}}{\det \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix}} = -8e^{4t} + 6e^{3t},$$

Taking antiderivative yields

$$u_1 = -2e^{-3t} - 2e^{-2t} + c_1, \quad u_2 = -2e^{4t} + 2e^{3t} + c_2.$$

Then the general solution to the inhomogeneous equation is

$$X(t) = \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix} \begin{bmatrix} -2e^{-3t} - 2e^{-2t} \\ -2e^{4t} + 2e^{3t} \end{bmatrix}.$$

4. We plug the initial condition to find out c_1 and c_2. Set $t = 0$ in the above expression for the general solutions, we have

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

So we infer that

$$c_1 = 5, \quad c_2 = -2.$$

To sum up, the solution to the initial value problem is

$$X(t) = \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix} \begin{bmatrix} 5 - 2e^{-3t} - 2e^{-2t} \\ -2 - 2e^{4t} + 2e^{3t} \end{bmatrix}.$$

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Example 2. *Find the general solution to the following differential equation.*

$$X' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X + \begin{bmatrix} e^{2t} \\ 2e^{t} \end{bmatrix}$$

Solution. Following Step **1**, we find out that the eigenvalues of the coefficient matrix is $\lambda_{1,2} = \pm i$ and the corresponding eigenvectors are $v_{1,2} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$. We compute

$$e^{\lambda_1 t} v_1 = e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Therefore, the general solutions to the corresponding homogeneous system is

$$X(t) = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Following Step **3**, we infer that

$$u_1'(t) = \frac{\det \begin{bmatrix} e^{2t} & \cos t \\ 2e^t & \sin t \end{bmatrix}}{\det \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix}} = -e^{2t} \sin t + 2e^t \cos t.$$

By applying integration by parts twice, we get

$$\begin{aligned} u_1(t) &= \int (-e^{2t} \sin t + 2e^t \cos t)\, dt \\ &= \frac{1}{5} e^{2t} \cos t - \frac{2}{5} e^{2t} \sin t + e^t \sin t + e^t \cos t + c_1. \end{aligned}$$

Similarly, we have

$$u_2(t) = -\frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t + \frac{1}{5} e^{2t} \sin t + \frac{2}{5} e^{2t} \cos t + c_2.$$

Thus the general solution is given by

$$X(t) = \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} \frac{1}{5} e^{2t} \cos t - \frac{2}{5} e^{2t} \sin t + e^t \sin t + e^t \cos t + c_1 \\ -\frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t + \frac{1}{5} e^{2t} \sin t + \frac{2}{5} e^{2t} \cos t + c_2 \end{bmatrix}.$$

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