MA 265 FINAL EXAM FALL 2011

Name ______________________________

PUID# ______________________________

Lecturer ______________________________

Section#__________ Class Time __________

INSTRUCTIONS

1. Make sure you have a complete test. There are **11** different test pages, including this cover page.
2. Your PUID# is your student identification number. DO NOT list your social security number.
3. **Using a #2 pencil**, fill in each of the following items on your answer sheet:
   (a) On the top left side, print your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION NUMBER, write in your 4 digit section number (for example 0012 or 0003) and fill in the little circles. If you do not know your section number, ask your lecturer.
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student I.D. number (NOT your social security number) and fill in the little circles.
   (d) On the bottom right, print your instructor's name and the course number.
4. Do any necessary work for each problem in the space provided or on the back of the pages of this test. No partial credit is given but your work may be considered if your grade is on the borderline. Circle your answers in this test.
5. Each problem is worth 10 points. The maximum possible score is 200 points.
6. **Using a #2 pencil**, put your answers to questions 1–20 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect.
7. NO books, notes or calculators are allowed on this exam. Turn off all electronic devices.
8. After you have finished the exam, hand in your answer sheet and this test to your lecturer.

1. Suppose A and B are $n \times n$ matrices. Which of the following statements are always TRUE?

   (i) $(-A)^T = -(A^T)$

   (ii) $(A+B)^T = A^T + B^T$

   (iii) $(A^T B)^T = AB^T$ $\quad (A^T B)^T = B^T (A^T)^T = B^T A$

   A. (i) only

   B. (ii) only

   C. (i) and (ii) only

   D. (ii) and (iii) only

   E. (i), (ii), and (iii)

B^T, A in general don't commute

Example

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

2. Let A, B, C be arbitrary square $n \times n$ matrices. What is the transpose of

$$(A^T + 2011B)C^{-1}$$

   A. $(A + 2011B^T)(C^T)^{-1}$

   B. $(C^{-1})^T A + (2011C^T)^{-1} B^T$

   C. $(C^{-1})^T A + 2011B^T(C^T)^{-1}$

   D. $(C^{-1})^T A + (2011C^{-1})^T B^T$

   E. $(A(C^{-1})^T)^T + (2011C^{-1})^T B^T$

$$\left[(A^T + 2011 B) C^{-1} \right]^T$$

$$= (C^{-1})^T (A^T + 2011 B)^T$$

$$= (C^{-1})^T \left[(A^T)^T + 2011 B^T \right]$$

$$= (C^{-1})^T A + 2011 (C^{-1})^T B^T$$

$$= (C^{-1})^T A + (2011 C^{-1})^T B^T$$

3. Let A be a nonsingular matrix with inverse

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Which of the following statements is FALSE?

A. For arbitrary 2×2 matrices B, C, if $BA = BC$ then $A = C$.

B. A^T is invertible.

C. For arbitrary 2×2 matrices B, C, if $AB = AC$ then $B = C$.

D. $AA^{-1} = A^{-1}A$.

E. A is a symmetric matrix.

A is true if B is invertible.

Counterexample $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

4. Suppose the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & a & 1 \end{bmatrix}$$

is invertible. What is the $(2, 1)$-entry of the inverse of A?

A. $\frac{2}{4a-1}$

B. $\frac{-2}{4a-1}$

C. $\frac{2-3a}{4a+1}$

D. $\frac{-2+3a}{4a+1}$

E. $\frac{2-3a}{-4a-1}$

$$(A^{-1})_{21} = A_{12} / |A|$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & a-4 & -5 \end{vmatrix} = 4a-1$$

5. Given constants a and b, consider the linear system

$$\begin{array}{rcrcl} 9x & + & 2y & = & a \\ 4x & + & y & = & b \end{array}$$

Expressing the solutions with determinants, one gets...

A. $x = \begin{vmatrix} a & b \\ 4 & 1 \end{vmatrix}$ and $y = \begin{vmatrix} 9 & 2 \\ a & b \end{vmatrix}$

B. $x = \begin{vmatrix} 9 & 2 \\ a & b \end{vmatrix}$ and $y = \begin{vmatrix} a & b \\ 4 & 1 \end{vmatrix}$

C. $x = \begin{vmatrix} a & 2 \\ 4 & b \end{vmatrix}$ and $y = \begin{vmatrix} 9 & a \\ b & 1 \end{vmatrix}$

D. $x = \begin{vmatrix} a & 2 \\ b & 1 \end{vmatrix}$ and $y = \begin{vmatrix} 9 & a \\ 4 & b \end{vmatrix}$

E. $x = \begin{vmatrix} 9 & a \\ 4 & b \end{vmatrix}$ and $y = \begin{vmatrix} a & 2 \\ b & 1 \end{vmatrix}$

Use Cramer's rule

$$X = \frac{\begin{vmatrix} 9 & 2 \\ b & 1 \end{vmatrix}}{\begin{vmatrix} 9 & 2 \\ 4 & 1 \end{vmatrix}} = \begin{vmatrix} 9 & 2 \\ b & 1 \end{vmatrix}$$

$$y = \frac{\begin{vmatrix} 9 & a \\ 4 & b \end{vmatrix}}{\begin{vmatrix} 9 & 2 \\ 4 & 1 \end{vmatrix}} = \begin{vmatrix} 9 & a \\ 4 & b \end{vmatrix}$$

6. Let V be the set of positive real numbers. Define the operations $\mathbf{u} \oplus \mathbf{v} = e^{\mathbf{u}} e^{\mathbf{v}}$ for every $\mathbf{u}$ and $\mathbf{v}$ in V, and $c \odot \mathbf{u} = e^{c}\mathbf{u}$ for every real number c and every $\mathbf{u}$ in V.

Which of the following statements are TRUE?

(i) V is closed under $\oplus$.

(ii) V is closed under $\odot$.

(iii) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for every $\mathbf{u}$ and $\mathbf{v}$ in V.

(iv) $1 \odot \mathbf{u} = \mathbf{u}$ for every $\mathbf{u}$ in V. $\longrightarrow$ $1 \odot u = e^1 u \neq u$ for $u \neq 0$

A. (i) and (iii) but not (ii)

B. (ii) and (iv)

C. (i) and (ii) but not (iii)

D. (iii) and (iv)

E. (i) and (ii) and (iii)

7. Let A be an $m \times n$ matrix. Which set listed below is NOT a subspace of $\mathbb{R}^n$?

A. All the solutions of the system $A\mathbf{x} = \mathbf{0}$.

B. All the vectors that can be written as a linear combination of the columns of A.

C. The set spanned by the columns of A and the zero vector in $\mathbb{R}^n$.

D. The vectors $\mathbf{v}$ in $\mathbb{R}^n$ with $A\mathbf{v} = 0$ whose first and last components are equal to each other.

E. All the solutions of the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is the unit vector $(1, 0, \ldots, 0)$.

A is $N(A)$

B is colspace(A)

C = B

$D = \left\{ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : \begin{cases} AX = 0 \\ x_1 - x_n = 0 \end{cases} \right\}$ is still the solution space to a homogeneous linear system.

E is the solutions to an inhomogeneous linear system, and thus not a vector space.

8. Which of the following sets of vectors spans $\mathbb{R}_2$?

A. [1 1] → Not enough vectors

B. [0 0], [1 1]

C. [1 1], [2 1], [0,0]

D. [0 0], [1 1], [2 2]

E. [1 2], [-2 -4] → colinear

B. $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ has rank 1

D $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ rank 1

C $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ has rank 2.

9. Let $A = \begin{bmatrix} 2 & 0 & 4 & 6 & 1 \\ 1 & 1 & 4 & 2 & -1 \\ 3 & 1 & 8 & 8 & 0 \\ 1 & -1 & 0 & 4 & 3 \\ 6 & 6 & 24 & 12 & 10 \\ 7 & 5 & 24 & 16 & 13 \end{bmatrix}$.

A row echelon form of A is $U = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is

rank $A = 3$ $\Rightarrow$ nullity of $A = 5 - 3 = 2$

A. 1

B. 2

C. 3

D. 4

E. 5

10. Suppose the matrix A has eigenvalues $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = [0\ 5\ 3]^T$, $\mathbf{v}_2 = [2\ 0\ 1]^T$, $\mathbf{v}_3 = [1\ -1\ 0]^T$.
If you diagonalize A as $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 2 & 2 & 0 \\ p_{21} & p_{22} & 2 \\ p_{31} & p_{32} & p_{33} \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$2V_3 \quad V_2 \quad \frac{2}{5}V_1 \qquad\qquad \lambda_3 \quad \lambda_2 \quad \lambda_1$

then

$$\Rightarrow P = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & 1 & 6/5 \end{bmatrix}$$

A. $p_{31} = 1$

B. $p_{32} = 1$

C. $p_{33} = 1$

D. The given conditions are not sufficient to determine P

E. It is not possible to find P in the given shape.

11. Consider the following system of equations:

$$\begin{cases} x + 2y + 3z &= a \\ x + y + z &= b \\ 5x + 7y + 9z &= c \end{cases}$$

Under which condition will this system be consistent?

A. $a = c - 5b$

B. $a = 2b - 5c$

C. $c = 3b + 2a$

D. $c = 3b - 2a$

E. $c = 7b - 2a$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 1 & 1 & b \\ 5 & 7 & 9 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 1 & 2 & 3 & a \\ 5 & 7 & 9 & c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & a-b \\ 0 & 2 & 4 & c-5b \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & a-b \\ 0 & 0 & 0 & c-5b-2a+2b \end{array}\right]$$

$$\Rightarrow \quad c - 5b - 2a + 2b = -2a - 3b + c = 0$$

12. The best line fit $y = ax + b$ to the data

x	-3	0	1	2
y	1	0	1	2

is given by

A. $y = x/4 + 2$

B. $y = x/4 + 1/14$

C. $y = 14x + 4$

D. $y = 1$

E. $y = x/7 + 1$

We need to find the least square solution $\hat{x}$ to $AX = B$, where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} b \\ a \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix} \quad A^T B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^T A \hat{x} = A^T B$$

$$\Rightarrow \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 1/7 \end{bmatrix}$$

13. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation satisfying $L(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}1\\2\end{bmatrix}$, and $L(\begin{bmatrix}-1\\1\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$. Then $L(\begin{bmatrix}0\\1\end{bmatrix}) =$

A. $\begin{bmatrix}1\\1\end{bmatrix}$

B. $\begin{bmatrix}-1\\1\end{bmatrix}$

C. $\begin{bmatrix}1\\-1\end{bmatrix}$

D. $\begin{bmatrix}-1\\-1\end{bmatrix}$

E. $\begin{bmatrix}0\\1\end{bmatrix}$

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\Rightarrow L(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\end{bmatrix}$$

14. Let $\mathbf{w}_1 = \begin{bmatrix}0\\-24\\0\end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix}7\\0\\6\end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix}-6\\0\\7\end{bmatrix}$ be an orthogonal set that spans $\mathbb{R}^3$.

Write the vector $\mathbf{v} = \begin{bmatrix}2\\2\\-1\end{bmatrix}$ as $a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3$. What is a_1?

A. $a_1 = -1/12$

B. $a_1 = 0$

C. $a_1 = 1/24$

D. $a_1 = 1/12$

E. $a_1 = 2\sqrt{24}$

Since w_1 w_2 w_3 are orthogonal basis dot product both sides of

$v = a_1 w_1 + a_2 w_2 + a_3 w_3$

by w_1 $\Rightarrow$

$(v, w_1) = a_1 (w_1, w_1)$

$$\Rightarrow a_1 = \frac{(v, w_1)}{\|w_1\|^2} = \frac{-48}{24^2} = -\frac{1}{12}$$

15. The determinant of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is

A. 0

B. 2

C. -2

D. 8

E. -8

$$|A| = (-1)^{1+6} \begin{vmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{vmatrix} = (-1)(-1)^{1+5}\, 2 \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix}$$

$$= (-1)(-1)^{3+1}\, 2 \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = (-1)(-1)^{3+1}\, 4 \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 8$$

16. Let A be a real 2 by 2 matrix with inverse A^{-1}. If you know that $A^2 = 8A^{-1}$ then the determinant of A is

A. 1

B. 2

C. 4

D. 8

E. None of the above

$$|A^2| = |A|^2 = |8A^{-1}|$$

$$= 8^2 |A^{-1}| = 8^2 \frac{1}{|A|}$$

$$\Rightarrow |A|^3 = 64 \quad \Rightarrow |A| = 4$$

17. Given that

$$\det \overbrace{\begin{pmatrix} a & b & c \\ 1 & 1 & 1 \\ d & e & f \end{pmatrix}}^{A} = 3,$$

the determinant of

$$B = \begin{pmatrix} 2 & 2 & 2 \\ a+1-d & b+1-e & c+1-f \\ d & e & f \end{pmatrix}$$

is

$$|B| = -\begin{vmatrix} a+1-d & b+1-e & c+1-f \\ 2 & 2 & 2 \\ d & e & f \end{vmatrix}$$

$$= -2\begin{vmatrix} a+1 & b+1 & c+1 \\ 1 & 1 & 1 \\ d & e & f \end{vmatrix}$$

$$= -2|A| = -6$$

A. 6

B. -6

C. 2

D. 3

E. -3

18. Let $x_1(t), x_2(t)$ be solutions to the system of linear differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with the initial conditions $x_1(0) = 1$ and $x_2(0) = 2$. Then $x_1(1)$ equals

A. $2e - e^2$

B. $2e - e^3$

C. $3e^2 - 2e^3$

D. $4e^2 - 3e^3$

E. $-2e + 3e^3$

$$p(\lambda) = \begin{vmatrix} \lambda-1 & 1 \\ -2 & \lambda-4 \end{vmatrix} = \lambda^2 - 5\lambda + 6$$

$$= (\lambda-2)(\lambda-3)$$

$\lambda_1 = 2$ $\quad \lambda_1 I_2 - A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

$\lambda_2 I_2 - A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$X(t) = b_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + b_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{3t}$ $\quad \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow b_1 = -4 \quad b_2 = -3$

19. Apply the Gram–Schmidt algorithm with the standard inner product to the vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ to get an orthonormal set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. What is $\mathbf{w}_3$?

$$V_1 = U_1$$

$$V_2 = U_2 - \frac{(U_2, V_1)}{\|V_1\|^2} V_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$V_3 = U_3 - \frac{(U_3, V_1)}{\|V_1\|^2} V_1 - \frac{(U_3, V_2)}{\|V_2\|^2} V_2$$

$$= \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$W_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A. $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

D. $1/\sqrt{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

20. For which real number(s) λ are the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ \lambda \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ \lambda \\ 1 \end{bmatrix}$ orthogonal?

A. Exactly if $\lambda = 2$.

B. Exactly if $\lambda = -2$.

C. Exactly if $\lambda = 1$.

D. Exactly if $\lambda = -1$.

E. For both $\lambda = 2$ and $\lambda = -2$.

$$-4 + \lambda^2 = 0$$

$$\lambda = \pm 2$$