1. What is the determinant of the following matrix?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 7 \\ 0 & 4 & -4 \\ 0 & -4 & -36 \end{vmatrix} = 160$$

A. 0
B. 8
C. 55
D. 120
E. 160

2. If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, then $\det \begin{bmatrix} a_1 & a_2 & 4a_3 - 2a_2 \\ b_1 & b_2 & 4b_3 - 2b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3 - c_2 \end{bmatrix} =$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} a_1 & a_2 & 4a_3 \\ b_1 & b_2 & 4b_3 \\ c_1 & c_2 & 4c_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & a_2 & 4a_3 \\ b_1 & b_2 & 4b_3 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a_1 & a_2 & 4a_3 - 2a_2 \\ b_1 & b_2 & 4b_3 - 2b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3 - c_2 \end{vmatrix}$$

A. 8
B. 6
C. 4
D. 2
E. 1

3. Let A be an $n \times n$ matrix. Which of the following statements must be true?

(i) If $A^T = -A$, then $\det A$ must be zero.
(ii) If $A^2 = A$, then A must be the identity matrix or the zero matrix.
(iii) If $\det A \neq 0$, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ only has the trivial solution.

A. None
B. (iii) only
C. (i) and (iii) only
D. (ii) and (iii) only
E. (i), (ii) and (iii)

(i) might not be true when n is even.
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is an counter example

(ii) Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

4. Which of the following vectors in R_3 is a linear combination of

$$v_1 = \begin{bmatrix} 4 & 2 & -3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}, v_3 = \begin{bmatrix} -2 & -1 & 4 \end{bmatrix}?$$

$$\left[\begin{array}{ccc|c} 4 & 2 & -2 & x \\ 2 & 1 & -1 & y \\ -3 & -2 & 4 & z \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & -2 & 4 & z \\ 2 & 1 & -1 & y \\ 4 & 2 & -2 & x \end{array}\right]$$

A. $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
B. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
C. $\begin{bmatrix} -2 & 2 & 3 \end{bmatrix}$
D. $\begin{bmatrix} 6 & 3 & 7 \end{bmatrix}$
E. None of the above.

$$\sim \left[\begin{array}{ccc|c} -1 & 0 & 2 & z+2y \\ 0 & 1 & -5 & -3y-2z \\ 0 & 0 & 0 & x-2y \end{array}\right] \Rightarrow x - 2y = 0$$

5. Which of the following sets of 2×2 matrices are vector spaces? (Here $\oplus$ and $\odot$ are the usual addition and scalar multiplication of matrices.)

(i) {all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = 3c - d$.}
(ii) {all 2×2 matrices A such that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.}
(iii) {all 2×2 matrices A such that $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is consistent.}

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$AX = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $BX = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

are consistent.

But $(A+B)X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is NOT

A. (i) and (ii)
B. (ii) only
C. (ii) and (iii)
D. (i) only
E. All of them are vector spaces.

(ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$AX = 0$, $BX = 0$ has nontrivial solutions. But $A + B = I_2$.

6. Let P_3 be the set of all polynomials of degree 3 or less. Which of the following subsets are subspaces of P_3? (Here $\oplus$ and $\odot$ are the standard addition and scalar multiplication.)

(i) {all polynomials $p(x)$ such that $p(1) \neq 0$}
(ii) {all polynomials $p(x)$ such that $p(x) = p(-x)$.}
(iii) {all polynomials $p(x)$ with $p(0) = p(1)$.}

A. (i) and (ii)
B. (ii) only
C. (ii) and (iii)
D. (i) and (iii)
E. All of them are vector spaces.

(i) $p(1) \neq 0$

$0 \cdot p(1) = 0$

7. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & t \\ 2 & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$. Choose a value of t so that the null space of A has dimension 0.

rank $A = 2$ for all t.

$\Rightarrow$ nullity of $A = 1$.

A. 0
B. 1
C. -1
D. 2
E. No such value of t exists.

8. Let A be a 3×5 matrix with rank 3. Which of the following statements is true?

A. A consistent linear system $A\mathbf{x} = \mathbf{b}$ must have a unique solution. / Can not be unique.
B. The null space of A has dimension 3. nullity of $A = 2$
C. The columns of A form a basis for the column space of A. columns are linearly dep.
D. The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^3$.
E. The rows of A form a linearly dependent set. rows of A are L.I.

9. Choose a value of a so that the vector $\begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to the vector $\begin{bmatrix} a \\ 1 \\ a^3 \end{bmatrix}$.

$-9a + a^3 = 0$

$a(a^2 - 9) = 0$

A. 1
B. 2
C. 3
D. 4
E. None of the above

10. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

A. is an eigenvector with eigenvalue 1.
B. is an eigenvector with eigenvalue -1.
C. is an eigenvector with eigenvalue i.
D. is an eigenvector with eigenvalue $-i$.
E. is not an eigenvector.

11. Consider the following differential equation:
$dx/dt = x + 5y$
$dy/dt = 5x + y$
Which of the following is a solution:

A. $x = 3e^{3t} + 2e^{-2t}$ and $y = 2e^{3t} - 2e^{-2t}$
B. $x = 3e^{4t} + 2e^{-4t}$ and $y = 2e^{6t} - 3e^{-t}$.
C. $x = 2e^{6t} + 2e^{-4t}$ and $y = 2e^{6t} - 2e^{-4t}$.
D. $x = 3e^{3t} + 2e^{2t}$ and $y = 2e^{3t} - 2e^{2t}$.
E. $x = 8e^{t} + 4e^{-3t}$ and $y = 9e^{t} - 4e^{-3t}$.

$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

$p(\lambda) = \begin{vmatrix} \lambda-1 & -5 \\ -5 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda - 24 = (\lambda-6)(\lambda+4)$

$\lambda_1 = 6$

$\lambda_1 I_2 - A = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -4$

$\lambda_2 I_2 - A \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$X(t) = b_1 \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix} + b_2 \begin{bmatrix} -e^{-4t} \\ e^{-4t} \end{bmatrix}$

Choose $b_1 = 2$
$b_2 = -2$

12. Suppose the vector $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Then which of the following statement is always correct:

A. $a = 0$, $b = c = -d$
B. $b = 0$, $a = b = -d$
C. $c = a = 0$, $a = b = -c$
D. $d = 0$, $a = b = c$
E. $a = b = 0$, $c = 2d + 1$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 2 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

13. Let $C[-\pi,\pi]$ be the real vector space of continuous functions defined on $[-\pi,\pi]$. Define an inner product on $C[-\pi,\pi]$ by

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt.$$

Then which of the following set of vectors are orthogonal?

A. $1, t, t^2$
B. $\sin^2 t, 1, \cos t$
C. $1, e^t, e^{2t}$
D. $1, t, t^2 - \frac{\pi^2}{3}$,
E. None of the above

14. Let A be an $n \times n$ matrix, which of the following statements is FALSE?

A. If A is a symmetric matrix, then A^T is also symmetric.
B. The product AA^T is always symmetric.
C. If A is skew symmetric, then A^3 is symmetric. $(A^3)^T = (A^T)^3 = (-A)^3 = -A^3$
D. If A is symmetric, then $A + A^2$ is symmetric. still skew-symmetric.
E. The sum $A + A^T$ is always symmetric.

15. A and B are $n \times n$ invertible matrices, which of the following statements is FALSE?

A. $(A^2)^{-1} = (A^{-1})^2$
B. $(A^{-1})^T = (A^T)^{-1}$.
C. $(AB^{-1})^{-1} = BA^{-1}$.
D. $(A + B^{-1})^{-1} = A^{-1} + B$.
E. $(aA)^{-1} = \frac{1}{a}A^{-1}$ for any nonzero real number a.

$(A + B^{-1})$ might not be invertible.

Counterexample: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

16. Consider the linear system

$$\begin{cases} x + 2y + 3z = a \\ 2x - y + z = b \\ 3x + y + 4z = c \end{cases}$$

Under which condition will this system be consistent?

A. $a = c - 2b$

B. $a = c - b$.

C. $b = a + c$.

D. $b = a - 2c$.

E. $c = 2a + b$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 2 & -1 & 1 & b \\ 3 & 1 & 4 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -5 & -5 & b-2a \\ 0 & -5 & -5 & c-3a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 1 & -\frac{1}{5}(b-2a \\ 0 & 0 & 0 & c-b-a \end{array}\right] \Rightarrow c - b - a = 0$$

17. Which of the following statements is FALSE?

A. The rows of any invertible $n \times n$ matrix span $\mathbb{R}^n$.

B. The rows of any orthogonal $n \times n$ matrix span $\mathbb{R}^n$.

C. The rows of any elementary $n \times n$ matrix span $\mathbb{R}^n$.

D. The rows of any symmetric $n \times n$ matrix span $\mathbb{R}^n$.

E. The rows of any nonsingular $n \times n$ matrix span $\mathbb{R}^n$.

orthogonal, elementary matrices are invertible, but sym matrices might NOT.

e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18. Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$. Which of the following collections of vectors is linearly independent?

A. The rows of the matrix A.

B. The rows of the matrix A^T.

C. The rows of the matrix $A \cdot A^T$.

D. The first three rows of A.

E. None of the above.

rank $A = 3$

$\Downarrow$

columns of A are $\mathcal{L}$ I.

C. AA^T is 4×4. But rank $AA^T \leq 3$

So. rows of AA^T are linearly dep.

D. $$\begin{vmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & -2 \\ 0 & -3 & -6 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$ $\Rightarrow$ first 3 rows of A are linearly dep.

19. In this problem, A is a 5×8 matrix. Find the TRUE statement.

A. If rref(A) has five leading 1's then the columns of A are linearly independent.

B. If the number of leading 1's of rref(A) equals three, then one can pick three columns of A that span the column space of A.

C. If columns 1,3 and 8 of rref(A) have no leading 1 then A must have five linearly independent rows.

D. If columns 1, 3, 4, 5 and 8 of rref(A) are exactly the ones that have no leading 1 then there are five linearly independent rows in A.

E. None of the above.

20. What are the eigenvalues of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$?

A. 1, 2.

B. 1, 2, 3.

C. $\pm 1, 2$.

D. $\pm 1, 2, 3$.

E. None of the above.

$$p(\lambda) = \begin{vmatrix} \lambda-1 & & & 0 \\ & \lambda-1 & -2 & \\ & -2 & \lambda-1 & \\ 0 & & & \lambda-2 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-2)\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda+1)$$

21. Which of the following matrices is NOT diagonalizable?

A. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

B. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

C. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

D. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $\lambda_1 = \lambda_2 = 2$ $\quad k = 2$ $\quad \lambda_1 I_2 - A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad m = 1 < k$

E. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

22. For an $n \times n$ real matrix A, which of the following statements are true?

i) If $A^T = A$, then all eigenvalues of A are real. A sym $\Rightarrow$ all eigenvalue of A are real.

ii) If A is an orthogonal matrix, then the columns of A form an orthonormal set.

iii) If all eigenvalues of A are equal to 1, then A is similar to the identity matrix I_n. Counterexample : $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

A. i) only.

B. ii) only.

C. i) and ii) only .

D. i), ii), iii).

E. None of the above.

23. Assume A and B are $n \times n$ symmetric matrices. Which of the following statements are always true?

(i) A^2 is symmetric. $(A^2)^T = A^T A^T = (A^T)^2 = A^2$

(ii) AB is symmetric.

(iii) ABA is symmetric. $(ABA)^T = A^T B^T A^T = ABA.$

A. none of these

B. (i) only

C. (ii) only

D. (i) and (iii) only.

E. (i), (ii) and (iii)

(ii) Counterexample : $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

24. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 5 \\ 0 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. Compute the $(2, 3)$-entry of $2A[B + A^T]$.

A. 4

B. 8

C. 26

D. 52

E. 104

25. Let W denote the vector space spanned by the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

and let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. Find the distance from $\mathbf{v}$ to W.

A. 0

B. 1

C. 2

D. $\sqrt{2}$

E. 4

v is orthogonal to u_1 & u_2

$\Rightarrow$ distance of v to $W = \|v\| = 2$