§5.7 29 (Hint)

(a) (i) We look at the cofactor expansion of $p(\lambda) = \det(A - \lambda I)$ along the first row

$$p(\lambda) = (a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} + \underbrace{a_{12} \left| \cdots \right| + \cdots}_{\text{Note that this is a poly in } \lambda \text{ of degree } \le n-2}$$

By induction, we have

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + \underbrace{\cdots\cdots}_{\text{poly in } \lambda \text{ of degree } \le n-2}$$

So λ^n, λ^{n-1} are only contained in

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

The λ^{n-1} term in the above expression is

$$(-1)^{n-1}(a_{11} + \cdots + a_{nn})\lambda^{n-1},$$

indeed if we expand $(a_{11} - \lambda) \cdots (a_{nn} - \lambda)$, we can produce λ^{n-1} only if we choose $(n-1)$ $(-\lambda)$ from the (n) parentheses, and one constant a_{ii} from the remaining parenthesis. So all the possibilities are

$$(-\lambda)^{n-1} a_{11}, \quad (-\lambda)^{n-1} a_{22} \quad \cdots \quad (-\lambda)^{n-1} a_{nn}$$

$\square$

(ii) Set $\lambda = 0$. Then

$$p(\lambda)\big|_{\lambda=0} = (-1)^n \cdot 0 + b_1 \cdot 0 + \cdots + b_{n-1} \cdot 0 + b_n = b_n$$

On the other hand

$$p(\lambda)\big|_{\lambda=0} = \det(A - 0 \cdot I) = \det A$$

§5.7 33.

$\{V_1\}$ is a basis for E_1

$\{V_2, V_3\}$ is a basis for E_2

If

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0 \qquad (1)$$

then

$$c_1 A V_1 + c_2 A V_2 + c_3 A V_3 = 0$$

$$\Downarrow$$

$$c_1 \lambda_1 V_1 + c_2 \lambda_2 V_2 + c_3 \lambda_2 V_3 = 0 \qquad (2)$$

$(1) \times \lambda_1 - (2) \Rightarrow$

$$c_2(\lambda_1 - \lambda_2) V_2 + c_3(\lambda_1 - \lambda_2) V_3 = 0$$

V_2, V_3 are L. I. thus

$$\begin{cases} c_2(\lambda_1 - \lambda_2) = 0 \\ c_3(\lambda_1 - \lambda_2) = 0 \end{cases}$$

But $\lambda_1 \neq \lambda_2$. $\Rightarrow$ $c_2 = c_3 = 0$

Then $\lambda_1 = 0$