EXAMPLES OF SECTIONS 2.3

Question 1. Find the inverse of

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Question 2. Solve the system

$$A\vec{x} = \vec{b},$$

where A is the matrix of question 1 and

$$\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

by using the inverse of A.

Question 3. For what value(s) of k does

$$\begin{cases} x + y + 3z = 0 \\ 2x + 4y + 5z = 0 \\ x - y + k^2 z = 0 \end{cases}$$

have infinitely many solutions?

SOLUTIONS.

1. Write

$$\begin{bmatrix} 3 & 5 & 6 & \vdots & 1 & 0 & 0 \\ 2 & 4 & 3 & \vdots & 0 & 1 & 0 \\ 2 & 3 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

and apply Gauss-Jordan elimination.

$$\begin{bmatrix} 3 & 5 & 6 & \vdots & 1 & 0 & 0 \\ 2 & 4 & 3 & \vdots & 0 & 1 & 0 \\ 2 & 3 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{A_{21}(-1)} \begin{bmatrix} 1 & 1 & 3 & \vdots & 1 & -1 & 0 \\ 2 & 4 & 3 & \vdots & 0 & 1 & 0 \\ 2 & 3 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{A_{32}(-1)}$$

$$\begin{bmatrix} 1 & 1 & 3 & \vdots & 1 & -1 & 0 \\ 0 & 1 & -2 & \vdots & 0 & 1 & -1 \\ 2 & 3 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{A_{13}(-2)} \begin{bmatrix} 1 & 1 & 3 & \vdots & 1 & -1 & 0 \\ 0 & 1 & -2 & \vdots & 0 & 1 & -1 \\ 0 & 1 & -1 & \vdots & -2 & 2 & 1 \end{bmatrix}$$

$$\underset{\sim}{A_{23}(-1)} \begin{bmatrix} 1 & 1 & 3 & \vdots & 1 & -1 & 0 \\ 0 & 1 & -2 & \vdots & 0 & 1 & -1 \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{bmatrix} \underset{\sim}{\begin{matrix} A_{32}(2) \\ A_{31}(-3) \end{matrix}}$$

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 7 & -4 & -6 \\ 0 & 1 & 0 & \vdots & -4 & 3 & 3 \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{bmatrix} \underset{\sim}{A_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & \vdots & 11 & -7 & -9 \\ 0 & 1 & 0 & \vdots & -4 & 3 & 3 \\ 0 & 0 & 1 & \vdots & -2 & 1 & 2 \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix}.$$

2.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -14 \\ 6 \\ 2 \end{bmatrix}.$$

3. The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 3 & \vdots & 0 \\ 2 & 4 & 5 & \vdots & 0 \\ 1 & -1 & k^2 & \vdots & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 1 & 3 & \vdots & 0 \\ 2 & 4 & 5 & \vdots & 0 \\ 1 & -1 & k^2 & \vdots & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & \vdots & 0 \\ 0 & 2 & -1 & \vdots & 0 \\ 0 & -2 & k^2-3 & \vdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & \vdots & 0 \\ 0 & 2 & -1 & \vdots & 0 \\ 0 & 0 & k^2-4 & \vdots & 0 \end{bmatrix}.$$

If $k^2 - 4 = 0$, then A is singular. There are infinitely many solutions in this case. If $k^2 - 4 \neq 0$, then $A \sim I_3$. In this case, we only have the trivial solution.

Therefore, this homogeneous linear system has infinitely many solutions if and only if $k = 2$ or $k = -2$.

Remark 0.1. *We can also use rank to analyze this problem. When* $k^2 - 4 = 0$*, rank*$(A) < 3$*. Then there are infinitely many solutions. When* $k^2 - 4 \neq 0$*, rank*$(A) = 3$*. In this case, we only have the trivial solution.*