EXAMPLES OF SECTIONS 4.3

Question 1. Consider the set V of all triples (x, y, z) such that $x = 3$. Is V a vector space?

Question 2. Find all the subspaces of $\mathbb{R}^2$.

SOLUTIONS.

1. First notice that elements of V can be written as $(3, y, z)$. In order for V to be a vector space, there must exist a zero element, i.e., an element $q = (q_1, q_2, q_3)$ such that $q \in V$ and $q + u = u$ for every $u \in V$. But if $q \in V$ then it can be written as $q = (3, q_2, q_3)$, and it follows that

$$q + u = (3, q_2, q_3) + (3, u_2, u_3) = (6, q_2 + u_2, q_3 + u_3) \neq (3, u_2, u_3).$$

Therefore V it is not a vector space.

2. Assume that S is a subset of $\mathbb{R}^2$.

Case 1: First notice that $S = \{(0, 0)\}$ is a vector space.

Case 2: If there is a non-zero vector v in S, in order to be a subspace S should contain the straight line $\{tv : t \in \mathbb{R}\}$.

Subcase 2(a): If all vectors of $\mathbb{R}^2$ is, on the other hand, contained in $\{tv : t \in \mathbb{R}\}$, then $S = \{tv : t \in \mathbb{R}\}$ is a straight line through the origin.

Subcase 2(b): If there is another non-zero vector $u \in S$ with

$$u \notin \{tv : t \in \mathbb{R}\},$$

then u, v are not parallel. In this case, $\text{span}\{u, v\} = \mathbb{R}^2$. But by the closedness under vector addition and scalar multiplication, in order to be a subspace, S will contain $\text{span}\{u, v\} = \mathbb{R}^2$, thus $S = \mathbb{R}^2$.

To sum up, all the subspaces of $\mathbb{R}^2$ are $\{(0, 0)\}$, straight lines through the origin and $\mathbb{R}^2$ itself.

Remark 0.1. *Via a similar argument, we can prove that*

(i) *all the subspaces of* $\mathbb{R}$ *are* $\{(0, 0)\}$ *and* $\mathbb{R}$ *itself.*

(ii) *all the subspaces of* $\mathbb{R}^3$ *are* $\{(0, 0)\}$*, straight lines through the origin, planes through the origin and* $\mathbb{R}^3$ *itself.*