EXAMPLES OF SECTIONS 4.9

In the problems below, let A be the matrix

$$A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}.$$

Question 1. Give the column and row spaces of A in terms of a basis.

Question 2. Find a basis for the null space of A.

Question 3. What is the nullity of A?

SOLUTIONS.

1. Applying Gauss-Jordan elimination we find

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two columns contain a leading one. Therefore the first two columns of A are linearly independent, and

$$\mathrm{rref}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

The non-zero rows of $rref(A)$ are the first and the second, therefore

$$\mathrm{rowspace}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 3 \end{bmatrix} \right\}.$$

Remark. It is important to remember that, after finding $\mathrm{rref}(A)$, the columns that form a basis of $\mathrm{colspace}(A)$ are the columns of the original matrix (i.e., A itself, as opposed to $\mathrm{rref}(A)$) which correspond to pivot columns, while a basis for $\mathrm{rowspace}(A)$ is given by the non-zero rows of $\mathrm{rref}(A)$ — and not of the original matrix A.

2. The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{cccc:c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore x_3 and x_4 are free variables. Denoting by $x_3 = s$, $x_4 = t$, we can then write

$$x_1 = -s - 5t,$$
$$x_2 = -s - 3t.$$

Therefore solutions $\vec{x} = (x_1, x_2, x_3, x_4)$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-5t \\ -s-3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} = s\vec{u} + t\vec{v},$$

where

$$\vec{u} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\vec{u}$ and $\vec{v}$ are a basis for the solution space of the system. In other words, *any* solution $\vec{x}$ of the system can be written as

$$\vec{x} = s\vec{u} + t\vec{v},$$

for some s, $t \in \mathbb{R}$.

3. Dimension of Null space of $A = 2$.