EXAMPLES OF SECTION 7.1

Question 1. Find the eigenvalues and eigenvectors of the following matrices:
(a)

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Solutions.
a. Start with the characteristic polynomial

$$\det \begin{bmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{bmatrix} = (\lambda - 3)(2 + \lambda) + 4 = 0,$$

whose solutions are the eigenvalues

$$\lambda_1 = 2, \ \lambda_2 = -1.$$

Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 2}$:

$$\begin{bmatrix} \lambda_1 - 3 & 2 \\ -2 & \lambda_1 + 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} v_1 = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_1 = s \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

As we saw in class, we can drop the free variable s and write

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\underline{\lambda_2 = -1}$:

$$\begin{bmatrix} \lambda_2 - 3 & 2 \\ -2 & \lambda_2 + 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} v_2 = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_2 = s \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Again, we drop the free variable s, obtaining

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues and eigenvectors:

$$\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

b. Start with the characteristic polynomial

$$\det \begin{bmatrix} \lambda - 1 & -1 & -2 \\ -1 & \lambda - 2 & -1 \\ -2 & -1 & \lambda - 1 \end{bmatrix} = (\lambda - 1)\Big((2-\lambda)(1-\lambda) - 1\Big) + (1 - \lambda - 2) - 2\Big(1 - 2(2-\lambda)\Big) = 0.$$

Multiplying both sides by -1 and rearranging, we obtain

$$\begin{aligned} (2-\lambda)(1-\lambda)^2 - 1 + \lambda + 1 + \lambda - 6 + 4\lambda &= (2-\lambda)(1-\lambda)^2 - 6(1-\lambda) \\ &= (1-\lambda)\Big((2-\lambda)(1-\lambda) - 6\Big) = 0. \end{aligned}$$

The eigenvalues are now easily found to be

$$\lambda_1 = 4,\ \lambda_2 = -1,\ \lambda_3 = 1.$$

Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 4}$:

$$\begin{bmatrix} \lambda_1 - 1 & -1 & -2 \\ -1 & \lambda_1 - 2 & -1 \\ -2 & -1 & \lambda_1 - 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix},$$

so we need to solve

$$\begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system and ignoring the free variable as before we obtain

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Repeating the process for $\lambda_2 = -1$, $\lambda_3 = 1$ we find, respectively

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues with corresponding eigenvectors

$$\lambda_1 = 4,\ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \lambda_3 = 1,\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$