PROOF FOR MATRIX OF INNER PRODUCT

Suppose that $(\cdot,\cdot)$ is an inner product in $\mathbb{R}^n$.

We first compute the expression for the product of three matrices. Let $u = [u_i]$, $v = [v_j]$ be two n-column vectors and $C = [c_{ij}]$ be an $n \times n$ matrix. Then

$$\begin{aligned} u^T C v = (u^T C)v &= [\sum_{i=1}^n u_i C_{i1}, \sum_{i=1}^n u_i C_{i2}, \cdots, \sum_{i=1}^n u_i C_{in}] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i C_{ij} v_j. \end{aligned}$$

Now we take an arbitrary basis $S = \{s_1, \cdots, s_n\}$ for $\mathbb{R}^n$. Write any two vector u, v as linear combinations of S:

$$\begin{aligned} u &= u_1 s_1 + \cdots + u_n s_n \\ v &= v_1 s_1 + \cdots v_n s_n. \end{aligned}$$

Let $C = [c_{ij}]_{n\times n}$ with $c_{ij} = (a_i, a_j)$. Then

$$\begin{aligned} (u, v) = (u_1 s_1 + \cdots + u_n s_n, v) &= \sum_{i=1}^n u_i (s_i, v) \\ &= \sum_{i=1}^n u_i (s_i, v_1 s_1 + \cdots v_n s_n) \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i (s_i, s_j) v_j \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i c_{ij} v_j \\ &= [u_1, \cdots, u_n] C \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}. \end{aligned}$$

This matrix $C = [(v_i, v_j)]$ is called the matrix of inner product $(\cdot,\cdot)$ with respect to the basis $S = \{s_1, \cdots, s_n\}$.

Remark 0.1.

(i) *The matrix* $C = [(v_i, v_j)]$ *is symmetric. Moreover, it satisfies* $u^T C u \geq 0$ *for any vector* $u \in \mathbb{R}^n$*, and* $u^T C u = 0$ *only if* $u = 0$*. Such a matrix is called positive definite.*

(ii) *If* $S = \{s_1, \cdots, s_n\}$ *is an orthonormal basis, then* $C = I_n$*.*