THE PROOFS FOR SOME FACT IN SECTION 5.6, 5.7

Question 1. Suppose that A is an $n \times n$ matrix, and the characteristic polynomial of A is

$$p(\lambda) = \pm(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_r)^{m_r},$$

where λ_i are different eigenvalues with multiplicity m_i.

The space of eigenvectors corresponding to the eigenvalue λ_i is called the eigenspace of λ_i, denoted by E_i.

Assume that $\dim E_i = \dim(\text{Null space of } (A - \lambda_i I_n)) = k_i$. Prove that

$$1 \leq k_i \leq m_i.$$

Proof. Since $\lambda_i I_n - A$ is singular, the fact that $1 \leq k_i$ is straightforward. We will prove $k_i \leq m_i$ by contradiction.

If assume, to the contrary, that $k_i > m_i$, then let $P = \{v_1, \cdots, v_{k_i}, v_{k_{i+1}}, \cdots, v_n\}$. Here $\{v_1, \cdots, v_{k_i}\}$ is a basis for $N(A - \lambda_i I_n)$, i.e, linearly independent eigenvectors with respect to λ_i, and $\{v_1, \cdots, v_n\}$ forms a basis for $\mathbb{R}^n$.

An easy computation shows that $AP = PD$, where D is of the form

$$\begin{bmatrix} \lambda_i e_1 & \cdots & \lambda_i e_{m_i} & B \end{bmatrix},$$

where $e_1, \cdots e_n$ are the standard basis of $\mathbb{R}^n$ and B is an $n \times (n - m_i)$ matrix. Therefore,

$$P^{-1}AP = D.$$

Using cofactor expansion, we can compute the characteristic polynomial of D. We can find that λ_i is a eigenvalue of D of multiplicity $k_i > m_i$. Now let us compute the set of eigenvalues for D.

$$|D - \lambda I_n| = |P^{-1}AP - \lambda I_n| = |P^{-1}AP - \lambda P^{-1} I_n P| = |P^{-1}||A - \lambda I_n||P| = |A - \lambda I_n|$$

So D has exactly the same characteristic polynomial as A, and thus has the same set of eigenvalues as A (even the multiplicities are the same). So $k_i > m_i$ is impossible and a contradiction. Therefore, $k_i \leq m_i$. □

Question 2. Assume that the conditions in **Question 1** still hold true. More precisely, $\lambda_1, \cdots, \lambda_r$ are different eigenvalues of A and $\{v_i, \cdots, v_{i,k_i}\}$ is the basis for E_i = Null space of $(A - \lambda_i I_n)$. Then

$$\cup_{i=1}^{r} \{v_i, \cdots, v_{i,k_i}\} \quad \text{are linearly independent.}$$

Proof. For simplicity, we assume that A has only two different eigenvalues λ_1, λ_2, and we will show that the eigenvectors v_1, v_2 with respect to λ_1, λ_2 are linearly independent.

If

$$a_1 v_1 + a_2 v_2 = 0, \tag{1}$$

multiplying both side by A

$$0 = a_1 A v_1 + a_2 A v_2 = a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2.$$

On the other hand, by multiplying both sides of (1) by λ_1, we have

$$0 = a_1 \lambda_1 v_1 + a_2 \lambda_1 v_2.$$

Substract the above two equalities.

$$0 = a_2(\lambda_1 - \lambda_2) v_2.$$

Since $\lambda_1 \neq \lambda_2$ and $v_2 \neq 0$, we infer that

$$a_2 = 0.$$

Similarly, we can conclude that

$$a_1 = 0.$$

Thus v_1, v_2 are linearly independent. □