A PROOF FOR THE CRAMER'S RULE

Proof. Suppose that A is an $n \times n$ invertible matrix. We look at the linear system $AX = b$. Then this system has a unique solution

$$X = A^{-1}b = \frac{1}{|A|}\mathrm{adj}A \cdot b = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} b.$$

Here $\cdot$ denotes matrix multiplication. By the formula for matrix multiplication, the k-th unknown, x_k, can be written as

$$x_k = \frac{1}{|A|}\sum_{i=1}^{n} A_{ik}b_i = \frac{1}{|A|}\det\begin{bmatrix} c_1 & \cdots & c_{k-1} & b & c_{k+1} & \cdots & c_n \end{bmatrix}.$$

Here c_i denotes the i-th column of A. The second equality follows from the cofactor expansion of the matrix $\begin{bmatrix} c_1 & \cdots & c_{k-1} & b & c_{k+1} & \cdots & c_n \end{bmatrix}$ along the k-th column. □