THE PROOFS FOR SOME FACT IN SECTION 7.3

Question 1. Suppose that A is an $n \times n$ symmetric matrix, and λ_1, λ_2 are two distinct eigenvalues of A. Assume that v_1 and v_2 are the corresponding eigenvectors of λ_1 and λ_2, respectively. Then $(v_1, v_2) = 0$.

Proof.

$$\begin{aligned} \lambda_1(v_1, v_2) &= (\lambda_1 v_1, v_2) = (Av_1, v_2) \\ &= (Av_1)^T v_2 = v_1^T A^T v_2. \end{aligned}$$

Since A symmetric, we have $A^T = A$. So the last equality becomes

$$v_1^T A v_2 = v_1^T (Av_2) = (v_1, Av_2) = (v_1, \lambda_2 v_2) = \lambda_2 (v_1, v_2).$$

Combing these two equalities, we infer that

$$(\lambda_1 - \lambda_2)(v_1, v_2) = 0$$

But λ_1, λ_2 are two distinct eigenvalues, so $\lambda_1 - \lambda_2 \neq 0$, which implies

$$(v_1, v_2) = 0.$$

□