CHAPTER 4 REVIEW

1. Finite dimensional vector spaces

Any finite dimensional vector space can be identified as a Euclidean space.

Example 1.1. *$M_{m\times n}(\mathbb{R}) = M_{mn}(\mathbb{R})$, the space of all real valued $m \times n$ matrix, can be identified as $\mathbb{R}^{mn}$. Every matrix*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

is mapped to the column vector

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & a_{21} & \cdots & a_{2n} & \cdots & a_{m1} & \cdots & a_{mn} \end{bmatrix}^T.$$

Question: How to find a basis for $M_{m\times n}(\mathbb{R})$?
Answer: $\{M_{ij} : 1 \le i \le m, 1 \le j \le n\}$ forms a basis for $M_{m\times n}(\mathbb{R})$, where M_{ij} is the $m \times n$ matrix with 1 in the (i, j)-entry and 0 elsewhere.

Example 1.2. *$S_n(\mathbb{R})$, the space of all real valued symmetric $n \times n$ matrix, can be identified as $\mathbb{R}^{\frac{n(n+1)}{2}}$, thus has dimension $\frac{n(n+1)}{2}$ (Consider why?). Consider how to find a basis for $S_n(\mathbb{R})$. Recall any symmetric matrix is symmetric with respect to the main diagonal.*

Example 1.3. *$\mathbb{P}_n$, the space of all polynomials of degree no more than n, can be identified as $\mathbb{R}^{n+1}$. Every polynomial*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$$

is mapped to the column vector

$$\begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix}^T.$$

Question: How to find a basis for $\mathbb{P}_n$?
Answer: $\{t^n, t^{n-1}, \cdots, t, 1\}$ forms a basis for $\mathbb{P}_n$.

Example 1.4. *More generally, any n-dimensional vector space V can be identified as $\mathbb{R}^n$. Since V is n-dimensional, we can find a a basis $\{v_1, \cdots, v_n\}$. For every v in V, we can find a unique linear combination*

$$v = c_1 v_1 + \cdots + c_n v_n.$$

(Consider why this linear combination is unique). Then the vector v is mapped to the column vector

$$\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T.$$

This column vector is called the coordinates of v with respect to the basis $\{v_1, \cdots, v_n\}$.

2. Infinitely dimensional vector spaces

There does exist infinitely dimensional vector space. A vector space is of infinite dimension if it has a basis containing infinitely many vectors.

Example 2.1. *$P :=$ the set of all polynomials is an infinite dimensional vector space. $\{1, x, x^2, \cdots\}$ is a basis of P. This space can be recognized as $\mathbb{R}^\infty$.*

Example 2.2. *Let I be an interval or the real line $\mathbb{R}$.*

$$C^n(I) = \{f : I \to \mathbb{R} : f \ n \text{ times differentiable},$$
$$f, f', \cdots, f^{(n)} \text{ are all continuous.}\}$$

is an infinite dimensional vector space. Indeed, there is a basis of $C^n(I)$ containing $\{1, x, x^2, \cdots\}$, and thus has infinitely many elements.

3. Subspaces

A subspace S is a subset of a vector space V, which is a vector space itself if equipped with the vector addition and scalar multiplication of V.

Example 3.1. *State all the subspaces of $\mathbb{R}^3$.*

Solution. Subspaces of $\mathbb{R}^3$ are $\mathbb{R}^3$ itself and all the planes and lines passing through the origin. ◀

Remark 3.2. *A subset S of a vector space V is a subspace if and only if S is closed under the* **same** *vector addition and scalar multiplication.*

Remark 3.3. *If 0_V is not in S, then S is not a subspace.*

Example 3.4. *The null space of an $m \times n$ matrix A, that is, the set of solutions to the homogeneous linear system*

$$Ax = 0, \tag{1}$$

is a subspace of $\mathbb{R}^n$. The dimension of this subspace is $n - \operatorname{rank}(A)$.

Example 3.5. *The null space of an $m \times n$ matrix A, that is, the set of solutions to the nonhomogeneous linear system*

$$Ax = b \neq 0, \tag{2}$$

is never a subspace of $\mathbb{R}^n$.

Example 3.6. *The set of all the polynomial ax^2+bx+c satisfying $a+b=c$ forms a subspace of $\mathbb{P}_2$. The dimension of this subspace is 2.(Consider why?)*

4. Linear dependence and independence

Proposition 4.1. *A set of vectors $S = \{v_1, \cdots, v_n\}$ is linearly dependent if and only if some v_k can be expressed as a linear combination of the* **other** *vectors in S.*

Summary: How to determine whether a given set $\{v_1, \cdots, v_n\}$ in $\mathbb{R}^m$ is linearly independent or not?

- If $m < n$, not linearly independent. (Consider why?)
- If $m \geq n$, let $A = [v_1 \quad \cdots \quad v_n]$
- If $\text{rank}(A) = n$, that is, the number of columns of A, then $\{v_1, \cdots, v_n\}$ is linearly independent. Otherwise, not.

Remark 4.2. *When $m = n$, A is a square matrix. In this case, the last step can be replaced by computing the determinant of A. More precisely, if $\det(A) \neq 0$, then $\{v_1, \cdots, v_n\}$ is linearly independent. Otherwise, not.*

Summary: How to find a linearly independent subset out of a given subset $\{v_1, \cdots, v_n\}$ in $\mathbb{R}^m$?

- Let $A = [v_1 \quad \cdots \quad v_n]$
- A linearly independent subset of $\{v_1, \cdots, v_n\}$ consists of the columns containing the leading 1's in $\text{ref}(A)$.

5. Spanning set

Summary: How to determine whether a given set $\{v_1, \cdots, v_n\}$ is a spanning set of $\mathbb{R}^m$?

- If $m > n$, not a spanning set. (Consider why?)

- Let
$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.$$
- If $\text{rank}(A)$ = the number of rows of A, or equivalently, there is no bottom zero row in $\text{ref}(A)$, then $\{v_1, \cdots, v_n\}$ is a spanning set. Otherwise, not.

Remark 5.1. *When $m = n$, A is a square matrix. In this case, the last step can be replaced by computing the determinant of A. More precisely, if $\det(A) \neq 0$, then $\{v_1, \cdots, v_n\}$ is spanning set. Otherwise, not.*

Example 5.2. *Find a spanning set for the plane*
$$x + 2y - 3z = 0$$
in $\mathbb{R}^3$.

Solution. This plane actually gives a homogeneous linear system
$$x + 2y - 3z = 0.$$
Solving it, we obtain the general expression for the solutions
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$
Then $\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\}$ is a spanning set of the plane $x + 2y - 3z = 0$. ◀

Remark 5.3. *In the above example, the set $\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\}$ is a spanning set of the plane $x + 2y - 3z = 0$. But the matrix*
$$A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
has rank 2, which is smaller than 3. This seems to be a contradiction. However, in this example, the vector space is not the whole $\mathbb{R}^3$, but just a subspace $x + 2y - 3z = 0$. Recall the argument leading to the conclusion
$$\{v_1, \cdots, v_n\} \text{ spans } \mathbb{R}^m \Longleftrightarrow \text{rank}(A) = m.$$
First, we look at the linear system $AX = b$ for arbitrary $b \in \mathbb{R}^m$. If rank$(A) < m$, then $\text{ref}(A)$ has at least one bottom zero row. Since b is arbitrary, we can always find a proper b such that rank$(A) <$ rank$(A|b)$.

But in the above example, b is not arbitrary, since b always belongs to the subspace $x + 2y - 3z = 0$.

Theorem 5.4 (rank nullity theorem)**.** *Given an $m \times n$ matrix A, or equivalently a homogeneous linear system $Ax = 0$, then*

$$\mathrm{rank}(A) + \dim(\textit{Null space of } A) = n = \textit{the number of unknowns.}$$

Summary: How to find a spanning set of a **subspace** in $\mathbb{R}^n$? (NOT $\mathbb{R}^n$!)

- A subspace of $\mathbb{R}^n$ is usually given by a homogeneous linear system $Ax = 0$, where A is a $m \times n$ matrix.
- Solving this linear system, the solutions can be expressed as

  $$\vec{x} = c_1 v_1 + \cdots c_k v_k,$$

  where $c_1, \cdots, c_k$ are free parameters, and $v_1, \cdots, v_k$ are fixed vectors in $\mathbb{R}^n$. Here

  $$k = n - \mathrm{rank}(A)$$

  by Theorem 5.4.
- Then $\{v_1, \cdots, v_k\}$ is a spanning set of the subspace.

Remark 5.5. $\{v_1, \cdots, v_k\}$ *is indeed a basis of this subspace!!*

Theorem 5.6. *If a vector space V is of dimension n and $\{v_1 \cdots, v_n\}$ is a subset of V, then the following statements are equivalent.*

1. $\{v_1 \cdots, v_n\}$ *is linearly independent.*
2. $\{v_1 \cdots, v_n\}$ *is a spanning set.*
3. $\{v_1 \cdots, v_n\}$ *is a basis.*

Example 5.7. *Is*

$$S = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} \}$$

a spanning set for the plane

$$x + 2y - 3z = 0$$

in $\mathbb{R}^3$.

Solution. First, it is an easy task to check all three vectors in $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} \}$ satisfy $x + 2y - 3z = 0$. So they all belong to this plane. Let

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}$ is linearly independent. Recall by Theorem 5.4, the dimension of the plane $x+2y-3z=0$ is 2. By Theorem 5.6, $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}$ is a basis of the plane $x+2y-3z=0$. So S is a spanning set of $x+2y-3z=0$. ◀

Summary: How to determine whether a given set $\{v_1, \cdots, v_k\}$ is a spanning set of a subspace S of $\mathbb{R}^n$? (NOT $\mathbb{R}^n$!)

- Find the homogeneous linear system $Ax = 0$, where A is an $m \times n$ matrix, representing the subspace S.
- Verify if v_i's are solutions to $Ax = 0$. If one of v_i's is not a solution, then this is not a spanning set.
- If all v_i's are solutions, then let
$$A = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}.$$
- Pick up a linearly independent subset out of $\{v_1, \cdots, v_k\}$. Recall a linearly independent subset of $\{v_1, \cdots, v_k\}$ consists of the columns containing the leading 1's in ref(A).
- Use Theorem 5.4 to find out the dimension of the subspace S.
- If dim$S$ =the number of vectors in the linearly independent subset of $\{v_1, \cdots, v_k\}$, then $\{v_1, \cdots, v_k\}$ is a spanning set. Otherwise, not.

6. Dimensions and bases

Proposition 6.1. *V is of dimension n. Then*

- *any linearly independent set cannot contain more than n vectors;*
- *any spanning set must contain at least n vectors;*
- *any basis contains exactly n vectors.*

Remark 6.2. *A basis can be considered as a "maximal" linearly independent set, or a "minimal" spanning set.*

Proposition 6.3. *S is a subspace of V. Then* $\dim S \le \dim V$. *If* $\dim S = \dim V$, *then $S = V$.*

Summary: How to determine if a set $\{v_1, \cdots, v_m\}$ is a basis of $\mathbb{R}^n$?

- If $m \neq n$, this is not a basis.

- If $m = n$, let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.
- If $\det(A) \neq 0$, or equivalently, $\operatorname{rank}(A) = n = m$, this is a basis. Otherwise, it is not.

Theorem 6.4. *S is a subspace of V. Then any basis of S can be extended to a basis of V.*

Question: Given a basis of a subspace S, how to extend it to a basis of V?

Example 6.5. *Extend the basis of* $\operatorname{span}(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix})$ *to a basis of* $\mathbb{R}^3$.

Solution. We first check

$$\begin{bmatrix} -2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for $\operatorname{span}(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix})$. To find the third vector extending $\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\}$ into a basis for $\mathbb{R}^3$, we look at

$$A = \begin{bmatrix} -2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The last three column vectors in A is the standard basis for $\mathbb{R}^3$. Thus, the five column vectors of A is a spanning set of $\mathbb{R}^3$. Moreover,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

From this observation, we know that the first three columns of A are linearly independent, and thus$\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$ form a basis for $\mathbb{R}^3$. ◀

Summary: How to extend a basis for a subspace S to a basis $\mathbb{R}^n$?

- Find a basis $\{v_1, \cdots, v_k\}$ for S.
- Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_k & u_1 & u_2 & \cdots & u_n \end{bmatrix}$. Here $\{u_1, \cdots, u_n\}$ is a basis (usually, we take this set to be the standard basis) of $\mathbb{R}^n$.

- A basis of V is the column vectors corresponding to the columns containing the leading 1's in $\text{ref}(A)$ (these columns will include $\{v_1, \cdots, v_k\}$).

Summary: How to find a basis for $\mathbb{R}^n$ or a subspace S of $\mathbb{R}^n$?

- Find a spanning set $\{v_1, \cdots, v_n\}$.
- Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.
- A basis set, or equivalently a linearly independent subset of $\{v_1, \cdots, v_n\}$, is the column vectors corresponding to the columns containing the leading 1's in $\text{ref}(A)$.

7. Relationship between spanning sets, liner independence, and bases

Suppose V is an m-dimensional vector space and $S = \{v_1, \cdots, v_n\}$ is a set of vectors in V. Then

- if $n > m$, then S is linearly dependent;
- if $m > n$, then S is not a spanning set;
- if $n \neq m$, then S is not a basis.

$S = \{v_1, \cdots, v_n\}$ is a basis for V means

- S is a "maximal" linearly independent subset of V, i.e., for any $u \in V$, $\{v_1, \cdots, v_n, u\}$ becomes linearly dependent;
- S is a "minimal" spanning set of V, i.e., after removing any vector v_k from S, the set $\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}$ is not spanning set anymore.

8. Rank, Row and column spaces

$$A_{m\times n} = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} = \begin{bmatrix} c_1 & \cdots c_n \end{bmatrix} \text{ is a } m \times n \text{ matrix.}$$

Note that

$$\text{colspace}(A) = \text{rowspace}(A^T), \quad \text{colspace}(A^T) = \text{rowspace}(A).$$

Remark 8.1.

- *colspace*(A) *is a subspace of* $\mathbb{R}^m$.
- *rowspace*(A) *is a subspace of* $\mathbb{R}^n$.

Remark 8.2. *rank*(A) = *the dimension of colspace*(A) = *the dimension of rowspace*(A) = *rank*(A^T).

Summary: How to find a basis for colspace(A)?

- The basis for colspace(A) consists of the columns containing the leading 1's in ref(A).

Summary: How to find a basis for rowspace(A)?

- The nonzero rows in ref(A) form a basis for rowspace(A). (Note that these rows are not from the original rows of A.) Or
- The columns containing the leading 1's in ref(A) forms a basis for colspace(A^T). Taking transpose of these columns, we obtain a basis for rowspace(A). (These rows are not from the original rows of A.)