EXAMPLES OF SECTIONS 1.11

Question 1. Solve

$$\begin{cases} y'' = 2yy', \\ y(1) = 1,\ y'(1) = 2. \end{cases}$$

Question 2. Solve $y'' + y' \tan x = (y')^2$.

Solutions.

1. Make the substitution $v = y'$. Then

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}y' = \frac{dv}{dy}v,$$

which gives upon plugging into the original equation,

$$v\frac{dv}{dy} = 2yv \Rightarrow \frac{dv}{dy} = 2y.$$

Integrating,

$$v = y^2 + C$$

Since $v = y'$,

$$y' = y^2 + C.$$

From $y(1) = 1$, $y'(1) = 2$, we then have

$$y'(1) = (y(1))^2 + C \Rightarrow 2 = 1 + C \Rightarrow C = 1.$$

So

$$y' = y^2 + 1,$$

or

$$\frac{dy}{y^2+1} = dx.$$

Integrating

$$\arctan(y) = x + C \Rightarrow y = \tan(x + C).$$

Using again $y(1) = 1$:

$$1 = \tan(1 + C) \Rightarrow C = \frac{\pi}{4} - 1,$$

so

$$y = \tan(x + \frac{\pi}{4} - 1).$$

2. Let $u = y'$ and thus $y'' = u'$. The original equation is equivalent to $u' + u\tan x = u^2$. This a Bernoulli equation. We use the change of variables method in Section 1.8 and set $v(x) = u^{-1}(x)$. Then

$$v' - v\tan x = -1.$$

An integrating factor for this equation is $I(x) = e^{-\int \tan x\,dx} = \cos x$. Therefore,

$$\begin{aligned}\frac{d}{dx}(v\cos x) = -\cos x \Rightarrow v\cos x = -\int \cos x\,dx \Rightarrow v = \frac{C_1 - \sin x}{\cos x}\\ \Rightarrow u = \frac{\cos x}{C_1 - \sin x} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{C_1 - \sin x}\\ \Rightarrow y(x) = C_2 - \ln|C_1 - \sin x|.\end{aligned}$$