EXAMPLES OF SECTIONS 1.8

Question 1. Solve

$$\begin{cases} xy' + (2x-3)y = 5x^5y^4, \\ y(1) = 1. \end{cases}$$

Question 2. Solve

$$(x^4 - 2t^3x)dt + (t^4 - 2tx^3)dx = 0.$$

Solutions.

1. Write the equation as

$$y' + \frac{2x-3}{x}y = 5x^4y^4,$$

for $x \neq 0$, which is a Bernoulli equation with $n = 4$. Set $v = y^{1-n} = y^{-3}$. The equation for v then becomes

$$\frac{dv}{dx} + (1-4)\frac{2x-3}{x}v = (1-4)5x^4,$$

or

$$\frac{dv}{dx} + \Big(\frac{9}{x} - 6\Big)v = -15x^4.$$

Using the formula for first order linear D.E.'s with $p(x) = \frac{9}{x} - 6$ and $q(x) = -15x^4$, we have

$$e^{-\int p(x)\,dx} = e^{6x-9\ln x} = x^{-9}e^{6x},$$
$$e^{\int p(x)\,dx} = e^{-6x+9\ln x} = x^9e^{-6x}.$$

where we assumed $x > 0$ since the problem is defined only for $x > 0$ or $x < 0$ (because $x \neq 0$). From this we get

$$\int q(x)e^{\int p(x)\,dx}\,dx = -15\int x^{13}e^{-6x}\,dx.$$

This integral is done by a tiresome (but not difficult) process of integrating by parts thirteen times. The answer is

$$-\frac{e^{-6x}}{314928}\Big(25025 + 150150x + 450450x^2 + 900900x^3 + 1351350x^4 + 1621620x^5 + 1621620x^6 + 1389960x^7 + 1042470x^8 + 694980x^9 + 416988x^{10} + 227448x^{11} + 113724x^{12} + 52488x^{13}\Big)$$

Denote the above expression by $f(x)$. Then using the formula for solutions of first order linear equations,

$$v(x) = x^{-9} e^{6x} \Big(f(x) + C \Big),$$

from which follows

$$y(x) = \Big[x^{-9} e^{6x} \Big(f(x) + C \Big) \Big]^{-\frac{1}{3}}.$$

To find C use $y(1) = 1$,

$$y(1) = 1 = \Big[e^{6} \Big(f(1) + C \Big) \Big]^{-\frac{1}{3}} \Rightarrow C = e^{-6} - f(1),$$

thus

$$y(x) = \Big[x^{-9} e^{6x} \Big(f(x) + e^{-6} - f(1) \Big) \Big]^{-\frac{1}{3}}.$$

2. We divide both sides of the equation by t^4 and transform it into

$$\frac{dx}{dt} = \frac{(x/t)^4 - 2x/t}{1 - 2(x/t)^3}.$$

This is a homogeneous equation and thus we can make the substitution $V(t) = x/t$.

$$tV' + V = \frac{V^4 - 2V}{1 - 2V^3} \Longrightarrow tV' = 3\frac{V^4 - V}{1 - 2V^3} \Longrightarrow \int \frac{1 - 2V^3}{V^4 - V}\, dV = \int \frac{3}{t}\, dt + C.$$

Using partial fraction and noticing that $V^4 - V = V(V-1)(V^2 + V + 1)$, we have

$$\frac{1 - 2V^3}{V^4 - V} = \frac{1}{V} - \frac{1}{3}\frac{1}{V-1} - \frac{1}{3}\frac{2V+1}{V^2 + V + 1}.$$

This yields

$$\ln(|V| \sqrt[3]{V-1} \sqrt[3]{V^2 + V + 1}) = -9 \ln |t| + C.$$