EXAMPLES OF SECTIONS 4.5

Question 1. Determine whether the vectors $(5, -2, 4)$, $(2, -3, 5)$, and $(4, 5-7)$ are linearly independent or dependent.

Question 2. Verify whether the given vectors $\vec{u} = (7, 3, -1, 9)$, $\vec{v} = (-2, -2, 1, 3)$ are linearly independent. If possible, express $\vec{w} = (4, -4, 3, 3)$ as a linear combination of $\vec{u}$ and $\vec{v}$.

Question 3. Verify if the given vectors $\vec{u} = (1, 0, 0, 3)$, $\vec{v} = (0, 1, -2, 0)$, $\vec{w} = (0, -1, 1, 1)$ are linearly independent. If possible, express $\vec{z} = (2, -3, 2, -3)$ as a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

SOLUTIONS.

1. Denote the vectors by $\vec{u} = (5, -2, 4)$, $\vec{v} = (2, -3, 5)$, and $\vec{w} = (4, 5-7)$. Consider

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}.$$

Recall that the vectors are linearly independent if the only solution of the previous equation is $a = b = c = 0$, and linearly dependent otherwise. The equation can be written as

$$a \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + c \begin{bmatrix} 4 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or in matrix form

$$\begin{bmatrix} 5 & 2 & 4 \\ -2 & -3 & 5 \\ 4 & 5 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The system will have a unique solution provided that the matrix of the system is invertible. But we readily check that

$$\det \begin{bmatrix} 5 & 2 & 4 \\ -2 & -3 & 5 \\ 4 & 5 & -7 \end{bmatrix} = 0,$$

which means that the matrix is not invertible, hence the system does not have a unique solution, and therefore the vectors are linearly dependent.

2. Consider the matrix

$$A = [\,\vec{u}\ \vec{v}\,] = \begin{bmatrix} 7 & -2 \\ 3 & -2 \\ -1 & 1 \\ 9 & -3 \end{bmatrix}.$$

Its submatrix

$$\begin{bmatrix} 7 & -2 \\ 3 & -2 \end{bmatrix}$$

has determinant equal to $(-2) \times 7 - (-2) \times 3 = -14 + 6 = -8 \neq 0$, hence the vectors are linearly independent.

Consider now the system

$$c_1\vec{u} + c_2\vec{v} = \vec{w},$$

or, in matrix form,

$$\begin{bmatrix} 7 & -2 \\ 3 & -2 \\ -1 & 1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 3 \\ 3 \end{bmatrix}.$$

The augmented matrix of the system is

$$\left[\begin{array}{rrcr} 7 & -2 & \vdots & -4 \\ 3 & -2 & \vdots & -4 \\ -1 & 1 & \vdots & 3 \\ 9 & -3 & \vdots & 3 \end{array}\right].$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{cccc} 1 & 0 & \vdots & 2 \\ 0 & 1 & \vdots & 5 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

This means that the system has solution $c_1 = 2$ and $c_2 = 5$, therefore

$$\vec{w} = 2\vec{u} + 5\vec{v}.$$

3. Consider the matrix

$$A = [\,\vec{u}\ \vec{v}\ \vec{w}\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

Its submatrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}.$$

has determinant equal to $1 \times (1 \times 1 - (-1) \times (-2)) = 1 - 2 = -1 \neq 0$, hence the vectors are linearly independent.

Consider now the system

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{z},$$

or, in matrix form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}.$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & -2 & 1 & \vdots & 2 \\ 3 & 0 & 1 & \vdots & -3 \end{bmatrix}.$$

Applying Gauss-Jordan elimination we find

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}.$$

The last row corresponds to

$$0c_1 + 0c_2 + c0c_3 = 1,$$

which of course is contradictory, hence the system has no solution and therefore $\vec{z}$ cannot be expressed as a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Remark. It is important to notice that linear independence *per se* is not a guarantee that the system will always have a solution. More precisely, a set of vectors $f_1, f_2, \dots, f_\ell$ in a vector space V being linearly independent does not automatically guarantee that any $g \in V$ can be written as

$$g = c_1 f_1 + c_2 f_2 + \cdots + c_\ell f_\ell.$$

While the vectors $\vec{u}$ and $\vec{v}$ of problem 1 are linearly independent and it was possible to write $\vec{w}$ as a linear combination of them, the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ of problem 2 are also linearly independent, but the system $\vec{z} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$ had no solution. As another example, think of the vectors $\vec{a} = (1, 0, 0)$ and $\vec{b} = (0, 1, 0)$ in $\mathbb{R}^3$: they are linearly independent, and any vector of the form $(x, y, 0)$ can be written in terms of $\vec{a}$ and $\vec{b}$, but $(0, 0, 1)$ cannot. The situation is different, however, when we have a *basis*: if the vectors f_1, f_2, $\ldots$, f_ℓ form a basis of a vector space V, then not only are they linearly independent but it is also true that any $g \in V$ can be written as

$$g = c_1 f_1 + c_2 f_2 + \cdots + c_\ell f_\ell.$$