EXAMPLES OF SECTIONS 1.1, 1.2

Question 1. Classify the following differential equations as linear or nonlinear and determine their orders.

(a) $y' + \cos y = x^3$;

(b) $yy'' + y^2y' + \frac{1}{x} = y$;

(c) $e^{\sin t^2}\frac{d^2x}{dt^2} + \omega^2 t\frac{d^3x}{dt^3} = e^{-t}$.

Question 2. Find the constant r such that $y(t) = e^{rt}$ is a solution to the differential equation $y'' + 2y' - 3y = 0$.

Question 3. Find the constant r such that $y(t) = x^r$ is a solution to the differential equation $x^2y'' + xy' - y = 0$.

Question 4. A projectile is fired straight upward with an initial velocity of $100m/s$ from the top of a building $20m$ high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) the total time in the air.

Solutions.

1.

(a) First order, nonlinear.

(b) Second order, nonlinear.

(c) Third order, linear..

2. Let $y(t) = e^{rt}$. Then $y'' = r^2e^{rt}$ and $y' = re^{rt}$. It yields

$$0 = r^2e^{rt} + 2re^{rt} - 3e^{rt} = e^{rt}(r^2 + 2r - 3).$$

Therefore we have $r^2 + 2r - 3 = 0 \Rightarrow r = 1,$ or $r = -3$.

3. As in Question 2, one checks

$$r(r-1)x^r + rx^r - x^r = 0.$$

Solving it gives $r = \pm 1$.

4. The acceleration of gravity is $g = -9.8m/s$ with the y-axis oriented upward. Since gravity is the only force acting on the projectile,

$$a = \frac{dv}{dt} = g = -9.8 \Rightarrow \int dv = -9.8 \int dt \Rightarrow v = -9.8t + C.$$

But $v(0) = 100$ so

$$v = -9.8t + 100. \tag{1}$$

Integrate again to find the position y:

$$v = \frac{dy}{dt} = -9.8t+100 \Rightarrow \int dy = \int (-9.8t+100)dt \Rightarrow y = -4.9t^2+100t+C.$$

Since $y(0) = 20$, we obtain

$$y = -4.9t^2 + 100t + 20. \tag{2}$$

(a) At the maximum point, $v = 0$. Setting $v = 0$ in (1) gives $t = \frac{100}{9.8}$. Using this into (2) produces $y(\frac{100}{9.8}) = -4.9(\frac{100}{9.8})^2 + 100 \times \frac{100}{9.8} + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, which gives two solutions, $t = 0$ (when the projectile is launched) and $t = \frac{100}{4.9} \approx 20.4$ seconds, which is the desired answer.

(c) It reaches the ground when $y = 0$. Solving $-4.9t^2 + 100t + 20 = 0$ yields $t = 20.61$ seconds and $t = -0.2$ seconds. The second solution is not physical, hence the answer is 20.61 seconds.