# EXAMPLES OF SECTIONS 1.4, 1.5

**Question 1.** The intensity $I$ of the light at a depth of $x$ meters below the surface of a lake satisfies the differential equations $I' = -1.4I$.

(a) At what depth is the intensity half of the intensity $I_0$ at the surface of the water?

(b) What is the intensity at a depth of 10 meters?

(c) At what depth will the intensity be 1% of that at the surface?

**Question 2.** According to one cosmological theory, there were equal amounts of the two uranium isotopes $^{235}U$ and $^{238}U$ at the creation of the universe in the big bang. At present there are 137.7 atoms of $^{238}U$ for each atom of $^{235}U$. Using the half-lives $4.51 \times 10^9$ years for $^{238}U$ and $7.10 \times 10^8$ years for $^{235}U$, calculate the age of the universe.

**Definition 0.1.** *The disintegrating rate of radioactive substances obey the Malthusian model $P(t) = P_0 e^{kt}$ for some $k < 0$ varying from substance to substance. The half-life period $t_{1/2}$ of a radioactive substance is the time when the amount of the substance equals the half of its original amount, i.e.,*

$$P(t_{1/2}) = \frac{P_0}{2} = P_0 e^{kt_{1/2}}.$$

*Solving this equation, we get $t_{1/2} = -\frac{1}{k}\ln 2$.*

**Question 3.** A colony of 10 thousand rabbits lives in a confined field and its growth obeys the logistic model.

(a) If after a month the population reaches 12 thousand, and after two month there are 13 thousand rabbits. What is the carrying capacity of the colony?

(b) If the field can support 42 thousand rabbits. After 1 month, the colony reaches 14 thousand rabbits. Use a logistic model to predict when the population will his 21 thousand rabbits.

**Solutions.**

**1. a.** The differential equation

$$I' = -1.4I \tag{1}$$

is separable, and thus we can derive the general solution to (1) as $I(x) = Ce^{-1.4x}$. Plugging in the initial condition $I(0) = I_0$, we get that the intensity at a depth of $x$ meters is $I(x) = I_0 e^{-1.4x}$. Then

$$I(x) = \frac{I_0}{2} = I_0 e^{-1.4x} \Rightarrow x = \frac{\ln 2}{1.4} \approx 0.495 \text{ meters}$$

**b.** Plugging in $t = 10$, $I(10) = I_0 e^{-14} \approx 8.3 \times 10^{-7} I_0$.

**c.** Solving $I_0 e^{-1.4x} = 0.01 I_0$ for $x$ gives $x = \frac{\ln 100}{1.4} \approx 3.29$ meters.

**2.** Let $N_8(t)$ and $N_5(t)$ be the numbers of $^{238}U$ and $^{235}U$ atoms, respectively, $t$ billions of years after the big bang. Since both isotopes follow a radioactive decay model $x' = kx$, whose solution was seen in class to be $x(t) = x_0 e^{kt}$, we have

$$N_8 = N_0 e^{-kt},$$

and

$$N_5 = N_0 e^{-\ell t},$$

where $N_0$ is the initial number of atoms of each isotope, which is the same for both $^{238}U$ and $^{235}U$ by hypothesis. Notice however that the rates of decay, $k$ and $\ell$, differ for these isotopes. Their values are given by

$$N_8(4.51) = \frac{N_0}{2} = N_0 e^{-k \times 4.51} \Rightarrow k = \frac{\ln 2}{4.51},$$
$$N_5(0.71) = \frac{N_0}{2} = N_0 e^{-\ell \times 0.71} \Rightarrow \ell = \frac{\ln 2}{0.71}.$$

We know that for the value of $t$ corresponding to "now" we have $\frac{N_8}{N_5} = 137.7$, hence

$$\frac{N_8}{N_5} = \frac{N_0 e^{-kt}}{N_0 e^{-\ell t}} = e^{(\ell - k)t} = e^{(\frac{\ln 2}{0.71} - \frac{\ln 2}{4.51})t} = 137.7.$$

Solving for $t$ gives

$$t = \frac{\ln 137.7}{\frac{\ln 2}{0.71} - \frac{\ln 2}{4.51}} \approx 5.99.$$

According to this theory, therefore, the universe should be about 6 billion years old.

**3.** **a.** Let $P(t)$ be the population of the colony at time $t$. According to the logistic model and the initial population of the colony,

$$P(t) = \frac{10C}{10 + (C - 10)e^{-rt}}. \tag{2}$$

The given conditions tell us $P(1) = 12$ and $P(2) = 13$, and thus

$$\frac{10C}{10 + (C - 10)e^{-r}} = 12, \quad \frac{10C}{10 + (C - 10)e^{-2r}} = 13.$$

Easy computations show

$$e^{-r} = \frac{10C - 120}{12(c - 10)}, \quad e^{-2r} = \frac{10C - 130}{13(c - 10)}.$$

Since $e^{-2r} = (e^{-r})^2$, we get

$$[\frac{10C - 120}{12(c - 10)}]^2 = \frac{10C - 130}{13(c - 10)} \Longleftrightarrow 144(C^2 - 23C + 130) = 130(C - 12)^2.$$

This is a quadratic equation. Solving it gives $C = 0$ or $C = \frac{192}{14} \approx 13.7143$. The first solution is not physical. Thus the carrying capacity is 13.7143.

**b.** The first condition implies that $C = 42$ in (2) and thus (2) becomes

$$P(t) = \frac{420}{10 + 32e^{-rt}}.$$

By $P(1) = 14$, we have $r = \ln(8/5)$. Solving $21 = \frac{420}{10+32e^{t\ln(5/8)}}$ gives $t = \frac{\ln(5/16)}{\ln(5/8)} \approx 2.4748$.