EXAMPLES OF SECTIONS 1.6, 1.7

Question 1. Newton's Law of Cooling(changing surrounding temperature). Assuming the temperature of the water in a tank satisfies $T_m(t) = \cos t$. An object with initial temperature T_0 is tossed into the tank. Find the equation for the temperature $T(t)$ of this object.

Question 2. A $100\,\ell$ tank initially contains $10\,kg$ of salt dissolved in $50\,\ell$ of water. Brine containing $1\,kg/\ell$ of salt flows into the tank at the rate $2\,\ell/\min$, and the well-stirred mixture flows out of the tank at the rate $1\,\ell/\min$. Write an initial value problem for the amount of salt in the tank.

Solutions. 1. By Newton's law of cooling, the change rate of the temperature fulfils

$$\frac{dT}{dt} = -k(T - T_m) = -k(T - \cos t) \Longleftrightarrow \frac{dT}{dt} + kT = k\cos t.$$

This is a first order differential equation. Its integrating factor is

$$I(t) = e^{\int k\,dt} = e^{kt}.$$

Multiplying both sides by I gives

$$(e^{kt}T(t))' = ke^{kt}\cos t \Rightarrow T(t) = e^{-kt}(k\int e^{kt}\cos t\,dt + C)$$

To evaluate the integral $\int e^{kt}\cos t\,dt$, we apply integration by parts twice

$$\begin{aligned}\int e^{kt}\cos t\,dt &= \sin t e^{kt} - k\int e^{kt}\sin t\,dt\\ &= \sin t e^{kt} + ke^{kt}\cos t - k^2\int e^{kt}\cos t\,dt,\end{aligned}$$

which implies

$$\int e^{kt}\cos t\,dt = \frac{1}{1+k^2}(\sin t e^{kt} + ke^{kt}\cos t)$$

. We now get

$$T(t) = \frac{k}{1+k^2}(\sin t + \cos t) + Ce^{-kt}.$$

Plugging in the initial condition $T(0) = T_0$

$$T(t) = \frac{k}{1+k^2}(\sin t + \cos t) + (T_0 - \frac{k^2}{1+k^2})e^{-kt}.$$

2. Let $A(t)$ be the amount of salt at time t, measured in kg. Then $\frac{dA}{dt}$ is measured in kg/min, and it is given by

$$\frac{dA}{dt} = in - out.$$

We have

$$in = 1\frac{kg}{\ell} \times 2\frac{\ell}{min} = 2\frac{kg}{min},$$

and

$$out = 1\frac{\ell}{min}\frac{A(t)\,kg}{V(t)\,\ell} = \frac{A(t)}{V(t)}\frac{kg}{\text{min}}.$$

Since $V(t) = 50 + (2-1)t = 50 + t$, we find

$$\frac{dA}{dt} = 2 - \frac{A}{50+t}.$$

Since at time zero there were $10\,kg$ of salt, the initial condition is $A(0) = 10$. Therefore

$$\begin{cases} \frac{dA}{dt} + \frac{A}{50+t} = 2, \\ A(0) = 10. \end{cases}$$

is the sought initial value problem.