LOGISTIC POPULATION MODEL

In the logistic population model, the population growth rate obeys

$$\frac{dP}{dt} = r(1 - \frac{P}{c})P, \quad P(0) = P_0.$$

To solve this equation, we proceed as follow

$$\begin{aligned}
\frac{dP}{dt} = r(1 - \frac{P}{c})P \Longleftrightarrow C\frac{dP}{dt} &= r(C-P)P \\
&\Rightarrow \int \frac{C dP}{(C-P)P} = r\int dt + C_0 \\
&\Rightarrow \int (\frac{1}{P} + \frac{1}{C-P})\, dP = rt + C_0 \\
&\Rightarrow \ln|P| - \ln|C-P| = \ln|\frac{P}{P-C}| = rt + C_0 \\
&\Rightarrow |\frac{P}{C-P}| = e^{rt+C_0} = e^{C_0}e^{rt} = C_1 e^{rt} \\
&\Rightarrow \frac{P}{C-P} = \pm C_1 e^{rt} = C_2 e^{rt}.
\end{aligned}$$

C_2 is an unknown constant. In the third line, we have used the partial fraction, i.e., for some constants A, B to be determined

$$\frac{C}{(C-P)P} = \frac{A}{P} + \frac{B}{C-P} = \frac{A(C-P)+BP}{(C-P)P} = \frac{AC - AP + BP}{(C-P)P}.$$

An easy computation shows $A = 1 = B$.

From the last step, we have

$$\begin{aligned}
&\frac{P}{C-P} = C_2 e^{rt} \Longleftrightarrow P = (C-P)C_2 e^{rt} \\
\Longleftrightarrow &P = CC_2 e^{rt} - C_2 e^{rt} P \Longleftrightarrow (C_2 e^{rt} + 1)P = CC_2 e^{rt} \\
\Longleftrightarrow &P = \frac{CC_2 e^{rt}}{1 + C_2 e^{rt}} = \frac{CC_2}{C_2 + e^{-rt}}.
\end{aligned}$$

Plugging in the initial condition $P(0) = P_0$, we obtain

$$P_0 = \frac{CC_2}{C_2 + e^0} = \frac{CC_2}{C_2 + 1} \Rightarrow C_2 = \frac{P_0}{C - P_0},$$

Finally, we reach

$$P(t) = \frac{CP_0}{P_0 + (C - P_0)e^{-rt}}.$$