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The rough path associated to the multidimensional analytic fBm with any Hurst parameter

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Abstract In this paper, we consider a complex-valued d-dimensional fractional Brownian motion defined on the closure of the complex upper half-plane, called *analytic fractional Brownian motion* and denoted by Γ. This process has been introduced in Unterberger (Ann Probab 37:565–614, 2009), and both its real and imaginary parts, restricted to the real axis, are usual fractional Brownian motions. The current note is devoted to prove that a rough path based on Γ can be constructed for any value of the Hurst parameter in $(0, 1/2)$. We also show how to solve differential equations driven by Γ in a neighborhood of 0 of the complex upper half-plane, by means of elementary arguments.

Keywords Rough paths theory · Stochastic differential equations · Fractional Brownian motion

Mathematics Subject Classification (2000) 60H05 · 60H10 · 60G15

0 Introduction

The (two-sided) fractional Brownian motion $t \to B_t$, $t \in \mathbb{R}$ (fBm for short) with Hurst exponent α, $\alpha \in (0, 1)$, defined as the centered Gaussian process with covariance

$$\mathbf{E}[B_s B_t] = \frac{1}{2}\left(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}\right), \tag{1}$$

is a natural generalization in the class of Gaussian processes of the usual Brownian motion, in the sense that it exhibits two fundamental properties shared with Brownian motion, namely,

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it has stationary increments, viz.

$$\mathbf{E}\left[(B_t - B_s)(B_u - B_v)\right] = \mathbf{E}[(B_{t+a} - B_{s+a})(B_{u+a} - B_{v+a})\right]$$

for every $a, s, t, u, v \in \mathbb{R}$, and it is self-similar, viz.

$$\forall \lambda > 0, \quad (B_{\lambda t}, t \in \mathbb{R}) \overset{(law)}{=} (\lambda^{\alpha} B_t, t \in \mathbb{R}). \tag{2}$$

One may also define a d-dimensional vector Gaussian process (called: *d-dimensional fractional Brownian motion*) by setting $B_t := B_t = (B_t(1), \ldots, B_t(d))$, where $(B_t(i), t \in \mathbb{R})_{i=1,\ldots,d}$ are d independent (scalar) fractional Brownian motions. Its theoretical interest lies in particular in the fact that it is (up to normalization) the only Gaussian process satisfying the two properties (1) and (2). Furthermore, a standard application of Kolmogorov's theorem shows that fBm has a version with $(\alpha - \epsilon)$-Hölder paths for every $\epsilon > 0$. This makes this process amenable to models where a Gaussian process with Hölder continuity exponent different from $1/2$ is needed, and we refer for instance to [3,12,15] for some applications to biophysics.

Consequently, there has been a widespread interest during the past ten years in constructing a stochastic integration theory with respect to fBm and solving stochastic differential equations driven by fBm. The multi-dimensional case is very different from the one-dimensional case. When one tries to integrate for instance a stochastic differential equation driven by a two-dimensional fBm $B = (B(1), B(2))$ by using any kind of Picard iteration scheme, one encounters very soon the problem of defining the Lévy area of B which is the antisymmetric part of

$$\mathcal{A}_{ts} := \int_s^t dB_{t_1}(1) \int_s^{t_1} dB_{t_2}(2). \tag{3}$$

This is the simplest occurrence of iterated integrals of the form

$$B^k_{ts}(i_1, \ldots, i_k) := \int_s^t dB_{t_1}(i_1) \ldots \int_s^{t_{k-1}} dB_{t_k}(i_k),$$

which lie at the heart of the rough path method due to T. Lyons.

Let us describe briefly this method, rephrased in the setting of [8] which is going to be used in the sequel of the paper: assume $X = (X(1), \ldots, X(d))$ is some non-smooth α-Hölder d-dimensional path. Integrals such as $\int f_1(X(t))dX_1(t) + \cdots + f_d(X(t))dX_d(t)$ do not make sense a priori because X is not differentiable (Young's integral works for $\alpha > \frac{1}{2}$ but not beyond). In order to define the integration of a differential form along X, it is enough to define a truncated *multiplicative functional* $(\mathbf{X}^{\mathbf{1}}, \ldots, \mathbf{X}^{\lfloor \mathbf{1}/\alpha \rfloor})$ where $\mathbf{X}^{\mathbf{1}}_{ts} = X_t - X_s$ and each

$$\mathbf{X}^{\mathbf{k}} = (\mathbf{X}^{\mathbf{k}}(i_1, \ldots, i_k))_{1 \le i_1, \ldots, i_k \le d}$$

—a matrix of (increments of) continuous paths—is a *substitute* for the a priori diverging iterated integrals $\int_s^t dX_{t_1}(i_1) \int_s^{t_1} dX_{t_2}(i_2) \ldots \int_s^{t_{k-1}} dX_{t_k}(i_k)$, with the following two properties:

(i) Each component of $\mathbf{X}^{\mathbf{k}}$ is $k\kappa$-Hölder continuous for any $\kappa < \alpha$.
(ii) *Multiplicativity*: letting $(\delta \mathbf{X}^{\mathbf{k}})_{tus} := \mathbf{X}^{\mathbf{k}}_{ts} - \mathbf{X}^{\mathbf{k}}_{tu} - \mathbf{X}^{\mathbf{k}}_{us}$, one requires

$$(\delta \mathbf{X}^{\mathbf{k}})_{tus}(i_1, \ldots, i_k) = \sum_{k_1 + k_2 = k} \mathbf{X}^{\mathbf{k_1}}_{tu}(i_1, \ldots, i_{k_1}) \mathbf{X}^{\mathbf{k_2}}_{us}(i_{k_1+1}, \ldots, i_k). \tag{4}$$

Once these functionals are defined, the theory described in [6,8,13] can be seen as a procedure which allows to define out of these data iterated integrals of any order and to solve differential equations driven by X.

With these preliminary considerations in mind, it is easily conceived that the fundamental problem in order to apply the general theory is to give a suitable definition of the functionals $\mathbf{X}^{\mathbf{k}}$. For any smooth path, $\mathbf{X}^{\mathbf{k}}$ can be defined as a Riemann multiple integral. The multiplicative and Hölder continuity properties are then trivially satisfied, as can be checked by direct computation. So the most natural way to construct such a multiplicative functional is to start from some smooth approximation X^{η}, $\eta \searrow 0$ of X such that each iterated integral $\mathbf{X}^{\mathbf{k},\eta}(i_1,\ldots,i_k)$, $k \le \lfloor 1/\alpha \rfloor$ converges in the $k\kappa$ -Hölder norm for any $\kappa < \alpha$. This general scheme has been applied to fBm in a paper by Coutin and Qian [4], by means of standard n-dyadic piecewise linear approximations $\tilde{B}^{2^{-n}}$ of B. In a later paper, one of the authors [16] tried to tackle the problem by seeing B as the real part of the boundary value of an analytic process Γ living on the upper half-plane $\Pi^+ = \{z \in \mathbb{C} \| \Im z > 0\}$. The time-derivative of this centered Gaussian process has the following hermitian positive-definite covariance kernel:

$$\mathbf{E}\left[\Gamma'_z \overline{\Gamma'_w}\right] = K'(z,\bar{w}) = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}\,(-\imath(z-\bar{w}))^{2\alpha-2}, \quad z,w \in \Pi^+ \tag{5}$$

where $z^{2\alpha-2} := e^{(2\alpha-2)\ln z}$ (with the usual determination of the logarithm) is defined and analytic on the cut plane $\mathbb{C}\backslash\mathbb{R}_-$. Also, by construction, $\mathbf{E}\Gamma'_z\Gamma'_w \equiv 0$ identically. It is essential to understand that K' is a multivalued function on $\mathbb{C}\times\mathbb{C}\backslash\{(z,\bar{z}) \mid z \in \mathbb{C}\}$; on the other hand, for $z,w \in \Pi^+$ we have $\Re(-\imath(z-\bar{w})) > 0$, so the kernel K' is well-defined. Then $B^{\eta}_t := 2\Re\Gamma_{t+\imath\eta}$ is a good approximation of fBm, namely, B^{η} converges a.s. in the κ-Hölder distance to a process B with the same law as fBm for any $\kappa < \alpha$.

Both approximation schemes introduced in [4,16] (see also the recent preprint [5]) lead to the same semi-quantitative result, namely:

- When $\alpha > 1/4$, the Lévy area and volume (in other words, the multiplicative functional truncated to order 3) converge a.s. in the appropriate variation norm. The heart of the proof lies in the study of the Lévy area $\tilde{\mathcal{A}}^{2^{-n}}_{ts}$, resp. $\mathcal{A}^{\eta}_{ts}$ (recall that those quantities have been defined by (3)) of the smooth approximations, $\tilde{B}^{2^{-n}}$, resp. B^{η}; one may prove in particular that $\mathbf{E}[(\tilde{\mathcal{A}}^{2^{-n}}_{ts})^2]$ and $\mathbf{E}[(\mathcal{A}^{\eta}_{ts})^2]$ converge to the same limit when 2^{-n} or η go to 0;
- When $\alpha < 1/4$, $\mathbf{E}[(\tilde{\mathcal{A}}^{2^{-n}}_{ts})^2]$ and $\mathbf{E}[(\mathcal{A}^{\eta}_{ts})^2]$ diverge resp. as $2^{n(1-4\alpha)}$ and $\eta^{-(1-4\alpha)}$. Hence the methods alluded to above fail.

The latter result is of course unsatisfactory. Actually, a rough path has been constructed in a recent paper by one of the authors [18,19] above fBm with arbitrary Hurst index; however, the rough path is not obtained by an approximation procedure, but through an algebraic regularization procedure. To the best of our knowledge, there is no explicit example in the literature of a construction by approximation of a rough path over a d-dimensional (with $d > 1$) process with Hölder regularity $\alpha < 1/4$.

In the current article, we propose to consider the complex-valued process $(\Gamma_z)_{z\in\bar{\Pi}^+}$ over the closed upper half-plane $\bar{\Pi}^+ := \{z \in \mathbb{C} \mid \Im z \ge 0\}$ for its own sake, and show how to construct a rough path over Γ by using its analytic approximation $\Gamma^{\eta}_z := \Gamma_{z+\imath\eta}$, $\eta > 0$. An adequate limiting procedure for $\eta \to 0^+$ will allow us to prove the following main result: the iterated integrals $\mathbf{\Gamma}^{\mathbf{k},\eta}(i_1,\ldots,i_k)$, $k \le \lfloor 1/\alpha \rfloor$ converge in the $(k\kappa)$-Hölder norm for any $\kappa < \alpha$ and *any* Hurst index $\alpha > 0$. The limiting objects $\mathbf{\Gamma}^{\mathbf{k}}$ satisfy our conditions (i) and (ii) above, which yields the construction of a rough path above the process Γ. It should be noticed that, in the analytic context we are dealing with, our rough path will be indexed by $\bar{\Pi}^+$. We will also show how to solve (locally in $\bar{\Pi}^+$) some differential equations driven by Γ.

Let us make a few comments on these results:
(i) An appropriate name for the Γ-process could be *analytic fractional Brownian motion* (analytic fBm or afbm for short). This is the name we shall use throughout the article. Yet the reader should be warned against two possible misunderstandings:

- Γ is analytic only in the (open) upper half-plane. When we consider its restriction to $\mathbb{R}$ (its boundary value on $\mathbb{R}$, one might say) it is merely a continuous process with the same Hölder continuity as the usual fBm. The fact that Γ is very irregular on $\mathbb{R}$ makes it interesting to be able to solve stochastic differential equations driven by Γ, whereas they are almost trivially solved on Π^+;
- considering the restriction of Γ to $\mathbb{R}$, one may be tempted to write $\Gamma_t = \Re\Gamma_t + \imath\Im\Gamma_t$ and to consider separately the real and the imaginary part. Elementary computations show that both $\Re\Gamma$ and $\Im\Gamma$ have the same law as fBm. But Γ is not merely a *complex* fBm since $\Re\Gamma$ and $\Im\Gamma$ are *not* independent (see Sect. 1). It is the correlation between $\Re\Gamma$ and $\Im\Gamma$ that cancels the singularities for small Hurst indices.

(ii) It can be shown, thanks to the fact that a geometric rough path above Γ exists, that stochastic differential equations on $\bar{\Pi}^+$ of the type

$$z_t = a + \int_{\gamma_t} b(z_u)du + \int_{\gamma_t} \sigma(z_u)d\Gamma_u, \quad a \in \mathbb{C}^n, \quad t \in \bar{\Pi}^+ \tag{6}$$

can be handled, where b and σ are vector-valued, resp. matrix-valued analytic functions on a complex neighborhood of a, and where the integrals $\int_{\gamma_t}$ are understood as integrals along any continuous path γ_t from 0 to t lying inside $\bar{\Pi}^+$, in a sense which is compatible with the rough paths theory. We shall see however at Sect. 4 that, in spite of the fact that Γ is irregular on $\mathbb{R}$, Eq. 6 can be solved in a rather easy way thanks to a limiting procedure involving Lebesgue type integrals only. From our point of view, the interest of the current paper rather lies in the construction of iterated integrals with respect to Γ, which is based on elegant complex analytic methods and yields sharp Hölder estimates.
(iii) Let us try to explain briefly why the regularized Levy area is divergent for the real fBm and $\alpha \le 1/4$, while it converges for the analytic one for any $\alpha > 0$. Let us call then $\mathcal{A}^{\eta}_{ts}$ the regularized Levy area for $\Re\Gamma$, and let us compute $\mathbf{E}(\mathcal{A}^{\eta}_{ts})^2$ for $t, s \in \mathbb{R}$: by definition (recall that $\mathbf{E}[\Gamma'(z)\Gamma'(w)] \equiv 0$ identically)

$$\begin{aligned}\mathbf{E}\left[(\mathcal{A}^{\eta}_{ts})^2\right] &= 2\mathbf{E}\left(\int_s^t d\Gamma_{x_1+\imath\eta}(1)\int_s^{x_1} d\Gamma_{x_2+\imath\eta}(2)\right)\left(\int_s^t d\bar{\Gamma}_{y_1+\imath\eta}(1)\int_s^{y_1} d\bar{\Gamma}_{y_2+\imath\eta}(2)\right)\\ &\quad +2\Re\mathbf{E}\left(\int_s^t d\Gamma_{x_1+\imath\eta}(1)\int_s^{x_1} d\bar{\Gamma}_{x_2+\imath\eta}(2)\right)\left(\int_s^t d\bar{\Gamma}_{y_1+\imath\eta}(1)\int_s^{y_1} d\Gamma_{y_2+\imath\eta}(2)\right)\\ &=: \mathcal{V}_1(\eta) + \mathcal{V}_2(\eta).\end{aligned} \tag{7}$$

The first term in the right-hand side writes

$$\begin{aligned}&\mathcal{V}_1(\eta)\\ &= C\int_s^t dx_1 \int_s^{x_1} dx_2 \int_s^t dy_1 \int_s^{y_1} dy_2 (-\imath(x_1-y_1)+2\eta)^{2\alpha-2}(-\imath(x_2-y_2)+2\eta)^{2\alpha-2}\end{aligned}$$

$$= C' \int_s^t dx_1 \int_s^t dy_1 (-\imath(x_1 - y_1) + 2\eta)^{2\alpha-2}$$
$$\times \left[(-\imath(x_1 - y_1) + 2\eta)^{2\alpha} - (-\imath x_1 + 2\eta)^{2\alpha} - (\imath y_1 + \eta)^{2\alpha} \right],$$

while the second term writes

$$\mathcal{V}_2(\eta) = C' \int_s^t dx_1 \int_s^t dy_1 (-\imath(x_1 - y_1) + 2\eta)^{2\alpha-2}$$
$$\times \left[(\imath(x_1 - y_1) + 2\eta)^{2\alpha} - (\imath x_1 + 2\eta)^{2\alpha} - (-\imath y_1 + \eta)^{2\alpha} \right].$$

Both integrals look the same *except* that $\mathcal{V}_2$ (contrary to $\mathcal{V}_1$) involves both $-\imath x_1$ and $\imath x_1$, and similarly for y_1. This seemingly insignificant difference is essential, since $\mathcal{V}_1$ can be shown to have a bounded limit when $\eta \to 0$ by using a contour deformation in $\Pi^+ \times \Pi^+$ which avoids the real axis where singularities live, while this is impossible for $\mathcal{V}_2$. Namely, $(-\imath(x_1 - \bar{y}_1) + 2\eta)^{2\alpha-2}$ is well-defined if (x_1, y_1) is in the closure of $\Pi^+ \times \Pi^+$, while $(\imath(x_1 - \bar{y}_1) + 2\eta)^{2\alpha}$ for instance is well-defined on the closure of $\Pi^- \times \Pi^-$, where Π^- is the *lower* half-plane. In fact, explicit computations prove that $\mathcal{V}_2(\eta)$ diverges in the limit $\eta \to 0$ when $\alpha < 1/4$. Now, the integral $\mathcal{V}_1(\eta)$ is the one which appears in the computations concerning the analytic fBm Γ, while the additional integral $\mathcal{V}_2(\eta)$ is needed in order to handle the case of the real-valued fBm $\Re\Gamma$. This fact had already been noted in [16], where $\mathcal{V}_1(\eta)$ (as part of the calculations needed to compute $\mathcal{A}^{\eta}_{ts}$) is evaluated in closed form involving Gauss' hypergeometric function (see [16, proof of Theorem 4.4])—in fact (see [16], formula (4.36))

$$\mathcal{V}_1(\eta) \overset{\eta\to 0}{\to} \frac{\alpha(2\alpha-1)}{4\cos^2 \pi\alpha} . \left[\frac{2\Gamma(2\alpha-1)\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} + \frac{\cos 2\pi\alpha}{(2\alpha-1)(4\alpha-1)} \right] |t-s|^{4\alpha} \tag{8}$$

(a regular expression when $\alpha \to 1/4$ or $1/2$ as Taylor's formula proves). In the same article, more general iterated integrals of the process Γ^η are introduced *en passant* under the name of *analytic iterated integrals* and shown to converge in the limit $\eta \to 0$; we reproduce these crucial results here. On the other hand, singularities of non-analytic iterated integrals for $\alpha < 1/4$ are analyzed in great details in [17].

Here is how our article is structured. In Sect. 1, we recall the basic features of the algebraic integration theory. The remaining sections deal with the rough path construction for the analytic fBm Γ: Sect. 2.1 is concerned with the definition of this process, Sect. 2.2 with the proof of some general regularity results for increments. Some useful (yet elementary) complex analysis preliminaries are given in Sect. 2.3. We then proceed to prove the convergence of our approximations based on Γ^η: Sect. 3.1 deals with Γ^η itself, Sect. 3.2 handles the case of the Levy area, while the general multiple integral case is treated in Sect. 3.3. Finally, at Sect. 4, we show how to solve a differential equation driven by Γ thanks to simple considerations.

Notations: Starting from Sect. 2 and throughout the paper, the following notations concerning processes will be used. A generic μ-Hölder function will be denoted by X, while general variables on $\bar{\Pi}^+$ are denoted by s, t, u, z, etc., whereas variables on the real boundary $\mathbb{R}$ are usually denoted by x, x', etc. The analytic fBm defined on the complex upper half-plane is written $\Gamma = \{\Gamma_t; t \in \bar{\Pi}^+\}$, and its smooth approximation is denoted by Γ^ε or Γ^η. If $s, t \in \bar{\Pi}^+$, then $[s, t] = \{\lambda s + (1-\lambda)t | \lambda \in [0, 1]\} \subset \bar{\Pi}^+$ is the segment between s and t. Generally speaking, Ω will denote a *bounded* neighborhood of 0 in the closure of the upper half-plane $\bar{\Pi}^+$. Since this notation is generally used for probability spaces, we shall call

$(\mathcal{U}, \mathcal{F}, \mathbf{P})$ the probability space under consideration here. Here is also a convention which will be used throughout the paper: for two real positive numbers, the relation $a \lesssim b$ stands for $a \leq C\, b$, where C is a given universal constant (possibly depending continuously on $\alpha \in (0,1)$).

1 Algebraic integration

Algebraic integration theory is conceived as an alternative to the popular rough paths analysis, and aims at solving differential equations driven by irregular processes with a minimal theoretical apparatus. Introduced in [8] for a Hölder regularity of the driving noise $\mu > 1/3$, it has then been extended to arbitrary $\mu > 0$ in a quite general setting (far beyond the geometric case) in [9]. Our aim is more modest here and we will merely recall some basic vocabulary from the theory, in the unusual context of processes indexed by the complex plane, allowing to define the objects we shall consider in the sequel. Notice that all the functions considered in the remainder of the paper will be *analytic* on the open upper half-plane Π^+.

1.1 Increments

The extended integral we deal with is based on the notion of increment, together with an elementary operator δ acting on them. These first notions are specifically introduced in [8,10], and we shall merely recall here their definition in the complex plane context. Consider an arbitrary neighborhood Ω of 0 in the closure of the upper half-plane $\bar{\Pi}^+ = \{z \in \mathbb{C} \Im z \geq 0\}$. Then, for a complex vector space V, and an integer $k \geq 1$, we denote by $\mathcal{C}_k(\Omega; V)$ the set of functions $g : \Omega^k \to V$, *analytic* on $(\Pi^+)^k$, such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$*-increment*, and we shall set $\mathcal{C}_*(V) = \cup_{k \geq 1} \mathcal{C}_k(\Omega; V)$. The operator δ alluded to above can be seen as an operator acting on k-increments, and is defined as follows on $\mathcal{C}_k(\Omega; V)$:

$$\delta : \mathcal{C}_k(\Omega; V) \to \mathcal{C}_{k+1}(\Omega; V), \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{9}$$

where $\hat{t}_i$ means that this particular argument is omitted. Then a fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(\Omega; V)$ to $\mathcal{C}_{k+2}(\Omega; V)$, so $(\mathcal{C}_*(\Omega; V), \delta)$ is a cochain complex. We shall denote $\mathcal{ZC}_k(\Omega; V) = \mathcal{C}_k(\Omega; V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(\Omega; V) = \mathcal{C}_k(\Omega; V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ, which will be the ones we shall really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $t, u, s \in \Omega$, we have

$$(\delta g)_{ts} = g_t - g_s, \quad \text{and} \quad (\delta h)_{tus} = h_{ts} - h_{tu} - h_{us}. \tag{10}$$

Furthermore, it is readily checked that the complex $(\mathcal{C}_*, \delta)$ is *acyclic*, i.e. $\mathcal{ZC}_k(\Omega; V) = \mathcal{BC}_k(\Omega; V)$ for any $k \geq 1$.

Let us mention at this point some conventions on products of increments which will be used in the sequel: assuming for the moment that $V = \mathbb{C}$, set $\mathcal{C}_k(\Omega; \mathbb{C}) = \mathcal{C}_k(\Omega)$. Then the complex $(\mathcal{C}_*(\Omega), \delta)$ is an (associative, non-commutative) graded algebra once endowed with the following product: for $g \in \mathcal{C}_n(\Omega)$ and $h \in \mathcal{C}_m(\Omega)$ let $gh \in \mathcal{C}_{n+m-1}(\Omega)$ be the element defined by

$$(gh)_{t_1,\ldots,t_{m+n-1}} = g_{t_1,\ldots,t_n} h_{t_n,\ldots,t_{m+n-1}}, \quad t_1, \ldots, t_{m+n-1} \in \Omega. \tag{11}$$

The pointwise multiplication of $g, \hat{g} \in \mathcal{C}_n(\Omega)$, denoted by $g \circ \hat{g}$, is also defined by:

$$(g \circ \hat{g})_{t_1,\dots,t_n} = g_{t_1,\dots,t_n} \hat{g}_{t_1,\dots,t_n}, \quad t_1, \dots, t_n \in \Omega.$$

Our future discussions will mainly rely on k-increments with $k \le 2$, for which we shall use some analytic assumptions. Namely, sticking to the case $V = \mathbb{C}^d$ for $d \ge 1$, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(\Omega; V)$ let

$$\|f\|_\mu \equiv \sup_{s,t\in\Omega} \frac{|f_{ts}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \left\{ f \in \mathcal{C}_2(\Omega; V); \|f\|_\mu < \infty \right\}.$$

In the same way, for $h \in \mathcal{C}_3(\Omega; V)$, set

$$\begin{aligned} \|h\|_{\mu,\rho} &= \sup_{s,u,t\in\Omega} \frac{|h_{tus}|}{|u-s|^\mu |t-u|^\rho} \\ \|h\|_\nu &\equiv \inf \left\{ \sum_i \|h_i\|_{\rho_i, \nu-\rho_i};\, h = \sum_i h_i,\ 0 < \rho_i < \nu \right\}, \end{aligned} \tag{12}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(\Omega; V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, z)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(\Omega; V)$, and we set

$$\mathcal{C}_3^\nu(V) := \{h \in \mathcal{C}_3(\Omega; V); \|h\|_\nu < \infty\}.$$

Notice that, in order to avoid ambiguities, we shall use the notation $\mathcal{N}[\cdot; \mathcal{C}_k^\mu(\Omega; V)]$, instead of $\|\cdot\|_\mu$, to denote the Hölder norms in the spaces $\mathcal{C}_k(\Omega; V)$. It also turns out to be useful to consider the spaces of continuous $(k-1)$ increments $\mathcal{C}_k^0(\Omega; V)$, equipped with the norm $\mathcal{N}[h; \mathcal{C}_k^0(\Omega; V)] = \sup_{t_1,\dots,t_k\in\Omega} |h_{t_1,\dots,t_k}|$. Let us mention once and for all that all the Hölder spaces we are considering in this article are complete.

1.2 Iterated integrals on the open upper-half plane

The iterated integrals of analytic functions on $\Omega \cap \Pi^+$ are particular cases of elements of $\mathcal{C}_*$ which will be of interest for us. Let us recall some basic rules for these objects. Set $\mathcal{C}_k^\omega(\Omega \cap \Pi^+) = \mathcal{C}_k^\omega(\Omega \cap \Pi^+; \mathbb{C})$ for the set of analytic $(k-1)$-increments from $\Omega \cap \Pi^+$ to $\mathbb{C}$, and consider $f, g \in \mathcal{C}_1^\omega(\Omega \cap \Pi^+)$. Then the integral $\int dg\, f$, which will also be denoted by $\mathcal{J}(dg\, f)$, can be considered as an element of $\mathcal{C}_2^\omega(\Omega \cap \Pi^+) \equiv \mathcal{C}_2^\omega(\Omega \cap \Pi^+; \mathbb{C})$. That is, for $s, t \in \Omega \cap \Pi^+$, we set

$$\mathcal{J}_{ts}(dg\, f) = \left(\int dg f \right)_{ts} = \int_{[s,t]} dg_u\, f_u = \int_{\gamma_{st}} dg_u\, f_u,$$

where γ_{st} is any continuous path in Π^+ joining s and t.

The multiple integrals can also be defined in the following way: given a smooth element $h \in \mathcal{C}_2^\omega$ and $s, t \in \Omega \cap \Pi^+$, we set

$$\mathcal{J}_{ts}(dg\, h) \equiv \left(\int dgh \right)_{ts} = \int_{[s,t]} dg_u h_{us}.$$

In particular, the double integral $\mathcal{J}_{ts}(df^3 df^2\, f^1)$ is defined, for $f^1, f^2, f^3 \in \mathcal{C}_1^{\omega}(\Omega \cap \Pi^+)$, as

$$\mathcal{J}_{ts}(df^3 df^2\, f^1) = \left(\int df^3 df^2\, f^1 \right)_{ts} = \int_{[s,t]} df_u^3\, \mathcal{J}_{us}\left(df^2 f^1\right).$$

Now, suppose that the nth order iterated integral of $df^n \cdots df^2 f^1$, still denoted by $\mathcal{J}(df^n \cdots df^2 f^1)$, has been defined for $f^1, f^2 \ldots, f^n \in \mathcal{C}_1^{\omega}(\Omega \cap \Pi^+)$. Then, if $f^{n+1} \in \mathcal{C}_1^{\omega}(\Omega \cap \Pi^+)$, we set

$$\mathcal{J}_{ts}(df^{n+1} df^n \cdots df^2 f^1) = \int_{[s,t]} df_u^{n+1} \mathcal{J}_{us}\left(df^n \cdots df^2 f^1\right), \tag{13}$$

which defines the iterated integrals of smooth functions recursively. Observe that a nth order integral $\mathcal{J}(df^n \cdots df^2 df^1)$, where we have simply replaced f^1 by df^1, could be defined along the same lines.

The following relations between multiple integrals and the operator δ are easily checked for analytic functions. They are also a prototype of the algebraic relations we shall impose in the rough setting:

Proposition 1.1 *Let f, g be two elements of $\mathcal{C}_1^{\omega}(\Omega \cap \Pi^+)$. Then, recalling the convention* (11)*, it holds that*

$$\delta f = \mathcal{J}(df), \qquad \delta\left(\mathcal{J}(dgf)\right) = 0, \quad \delta\left(\mathcal{J}(dgdf)\right) = (\delta g)(\delta f) = \mathcal{J}(dg)\mathcal{J}(df),$$

and, in general,

$$\delta\left(\mathcal{J}(df^n \cdots df^1)\right) = \sum_{i=1}^{n-1} \mathcal{J}\left(df^n \cdots df^{i+1}\right) \mathcal{J}\left(df^i \cdots df^1\right)$$

which is simply a way of rewriting the multiplicative property (4).

2 Analytic fractional Brownian motion and preliminaries

We review in this section the construction of the analytic fBm Γ, and we also include here some useful preliminary results concerning the regularity of increments in Ω, and some complex analysis estimates for the kernel $(-\imath(z - \bar{w}))^{2\alpha-2}$ defined on Π^+.

2.1 Definition of the analytic fBm

‘ As mentioned in the Introduction, the article [16] is an elaboration of a stochastic calculus with respect to the fractional Brownian motion by analytic continuation. More specifically, a complex-valued processed indexed by $z \in \Pi^+$, called Γ, is introduced there. This process is analytic on Π^+ and converges uniformly over every compact in probability, and also in $L^2(\mathcal{U})$ for the rough path distance, see [16], to a continuous process with real-time parameter (still denoted by Γ) when the imaginary part of z goes to 0. The current section is devoted to recall this formalism, which is the one we shall adopt in order to construct a fractional rough path for Γ for any Hurst parameter $\alpha \in (0, 1)$. Notice that, since the case $\alpha > 1/2$ is trivial from the rough path analysis point of view, we shall assume in the sequel that $\alpha \in (0, 1/2)$. The Brownian case $\alpha = 1/2$ may be seen as a limit.

Let us first recall some classical notations of complex analysis: for $x \in \mathbb{R}$ and $k \in \mathbb{N}$, the Pochhammer symbol $(x)_k$ is defined by: $(x)_k = \prod_{j=0}^{k-1}(x+j)$. Recall that we denote by $\Pi^+ = \{z = x + \imath y \mid x \in \mathbb{R}, y > 0\}$, resp. $\bar{\Pi}^+ = \{z = x + \imath y \mid x \in \mathbb{R}, y \geq 0\}$ the open, resp. closed upper half-plane in $\mathbb{C}$. Similarly, Π^-, resp. $\bar{\Pi}^-$ stand for the open, resp. closed lower half-planes.

With these notations in mind, the easiest way to define $\Gamma = \{\Gamma_z; z \in \Pi^+\}$ makes use of a series expansion involving the analytic functions $\{f_k; k \geq 0\}$, defined on Π^+ by:

$$f_k(z) = 2^{\alpha-1}\left[\frac{\alpha(1-2\alpha)(2-2\alpha)_k}{2\cos(\pi\alpha)k!}\right]^{1/2}\left[\frac{z+\imath}{2\imath}\right]^{2\alpha-2}\left[\frac{z-\imath}{z+\imath}\right]^k.$$

It is shown in [16] that the series $\sum_{k\geq 0} f_k(z)\overline{f_k(w)}$ converges in absolute value for $z, w \in \Pi^+$, and that the following identity holds true:

$$\sum_{k\geq 0} f_k(z)\overline{f_k(w)} = \frac{\alpha(1-2\alpha)}{2\cos(\pi\alpha)}\left(-\imath(z-\bar{w})\right)^{2\alpha-2}. \tag{14}$$

This fact allows to define the process Γ in the following way:

Proposition 2.1 *Let $\{\xi_k^1, \xi_k^2; k \geq 0\}$ be two families of independent standard Gaussian random variables, defined on a complete probability space $(\mathcal{U}, \mathcal{F}, \mathbf{P})$, and for $k \geq 0$, set $\xi_k = \xi_k^1 + \imath\xi_k^2$. Consider the process Γ' defined for $z \in \Pi^+$ by $\Gamma'_z = \sum_{k\geq 0} f_k(z)\xi_k$. Then:*

(1) *Γ' is a well-defined analytic process on Π^+, i.e. $z \to \Gamma'_z$, $z \in \Pi^+$ is a.s. analytic.*
(2) *Let $\gamma : (0,1) \to \Pi^+$ be any continuous path with endpoints $\gamma_0 = 0$ and $\gamma(1) = z$, and set $\Gamma_z = \int_\gamma \Gamma'_u\, du$. Then Γ is an analytic process on Π^+. Furthermore, as z runs along any path in Π^+ going to $x \in \mathbb{R}$, the random variables Γ_z converge almost surely to a random variable called again Γ_x.*
(3) *The family $\{\Gamma_x; x \in \mathbb{R}\}$ defines a Gaussian centered complex-valued process, whose covariance function is given by:*

$$\mathbf{E}[\Gamma_x\Gamma_{x'}] = 0,$$
$$\mathbf{E}[\Gamma_x\bar{\Gamma}_{x'}] = \frac{e^{-\imath\pi\alpha\,\mathrm{sgn}(x)}|x|^{2\alpha} + e^{\imath\pi\alpha\,\mathrm{sgn}(x')}|x'|^{2\alpha} - e^{\imath\pi\alpha\,\mathrm{sgn}(x'-x)}|x'-x|^{2\alpha}}{4\cos(\pi\alpha)}.$$

The paths of this process are almost surely κ-Hölder for any $\kappa < \alpha$.
(4) *Both real and imaginary parts of $\{\Gamma_x; x \in \mathbb{R}\}$ are (non independent) fractional Brownian motions indexed by $\mathbb{R}$, with covariance given by*

$$\mathbf{E}[\Re\Gamma_x\Im\Gamma_{x'}] = -\frac{\tan\pi\alpha}{8}\left[-\mathrm{sgn}(x)|x|^{2\alpha} + \mathrm{sgn}(x')|x'|^{2\alpha} - \mathrm{sgn}(x'-x)|x'-x|^{2\alpha}\right]. \tag{15}$$

Remark 2.2 It should be stressed at this point that the paper [16] mainly focuses on the real part of Γ, that is a standard fractional Brownian motion. We shall see however that Γ is an interesting process in its own right, insofar as it allows the construction of a rough path for any value of the Hurst parameter $\alpha \in (0,1)$.

Let us also recall some basic facts about Γ which will be used extensively in the sequel: first, according to (14), the (Hermitian) covariance between Γ'_z and Γ'_w for $z, w \in \Pi^+$ is given by:

$$\mathbf{E}\left[\Gamma'_z\bar{\Gamma}'_w\right] = \frac{\alpha(1-2\alpha)}{2\cos(\pi\alpha)}\left(-\imath(z-\bar{w})\right)^{2\alpha-2}. \tag{16}$$

Some more general line integrals with respect to Γ will also be needed, and let us define them more specifically: in general, an integral over any continuous path $\gamma = \gamma_{st}$ in Ω joining s and t is constructed thanks to a simple limiting procedure. Indeed, for any $\varepsilon > 0$ and any function z which is analytic on the upper half plane Π^+, the integral $\mathcal{J}_{t+\imath\varepsilon,s+\imath\varepsilon}(d\Gamma^* z)$ is analytic, which yields:

$$\mathcal{J}_{t+\imath\varepsilon,s+\imath\varepsilon}(d\Gamma^* z) = \int_{\gamma+\imath\varepsilon} d\Gamma_u^* z_u,$$

where the last integral is understood in the Riemann sense. Furthermore, this integral only depends on the ending points $t+\imath\varepsilon$, $s+\imath\varepsilon$. If one is then allowed to pass to the limit $\varepsilon \to 0$, we still write $\mathcal{J}_\gamma(d\Gamma^* z) = \mathcal{J}_{ts}(d\Gamma^* z)$ in a natural way.

This being said, the following formula will be used throughout the article: for a continuous path $\gamma : (0, 1) \to \Pi^+$, we have:

$$\mathbf{E}\left[\int_\gamma \Gamma_z' dz \int_\gamma \bar{\Gamma}_w' dw\right] = \frac{\alpha(1-2\alpha)}{2\cos(\pi\alpha)} \int_\gamma dz \int_{\bar{\gamma}} d\bar{w}\,(-\imath(z-\bar{w}))^{2\alpha-2}. \tag{17}$$

2.2 Garsia–Rodemich–Rumsey type lemmas

This section is devoted to recall or give some deterministic regularity results for increments, which will be essential in order to quantify the convergence of the approximations of our process Γ. First let us recall a particular case of a classical lemma due to Garsia [7, Lemma 2]:

Lemma 2.3 *Let f be a continuous function defined on a compact set $D \subset \mathbb{R}^d$ for $d \geq 1$, and set, for $p \geq 1$*

$$U_{\kappa,p}(f) = \left(\int_D \int_D \frac{|(\delta f)_{wv}|^{2p}}{|w-v|^{2\kappa p+2d}}\right)^{1/2p}.$$

Then $\mathcal{N}[f; \mathcal{C}_1^\kappa(D)] \leq c U_{\kappa,p}(f)$, for a universal positive constant c.

When $D = \Omega \subset \bar{\Pi}^+$, we need an extension of this lemma to increments which are not necessarily written as δf for functions $f \in \mathcal{C}_1$:

Proposition 2.4 *Let $\Omega := B(0,r) \cap \bar{\Pi}^+$ be a neighborhood of 0 in $\bar{\Pi}^+$, and $\mathcal{R} \in C_2(\Omega; \mathbb{C}^n)$ for $n \geq 1$ such that $\delta\mathcal{R} \in C_3^\kappa(\Omega; \mathbb{C}^n)$. Set for $p \geq 1$*

$$U_{\kappa,p}(\Omega; \mathcal{R}) := \left(\int_\Omega \int_\Omega \frac{|\mathcal{R}_{wv}|^{2p}}{|w-v|^{2\kappa p+4}}\, dv dw\right)^{1/2p}, \tag{18}$$

where $dv = d\Re v\, d\Im v$, resp. $dw = d\Re w\, d\Im w$ is the usual area element of $\mathbb{R}^2 \simeq \mathbb{C}$, and assume $U_{\kappa,p}(\Omega; \mathcal{R}) < \infty$. Then $\mathcal{R} \in C_2^\kappa(\Omega; \mathbb{C}^n)$; more precisely,

$$\mathcal{N}[\mathcal{R}; \mathcal{C}_2^\kappa(\Omega; \mathbb{C}^n)] \leq c\left(U_{\kappa,p}(\Omega; \mathcal{R}) + \mathcal{N}[\delta\mathcal{R}; \mathcal{C}_3^\kappa(\Omega; \mathbb{C}^n)]\right), \tag{19}$$

for a universal constant $c > 0$.

Proof Let $s,t\in\Omega' := B(0,r/4)\cap\bar{\Pi}^+\subset\Omega$, so that $|t-s|<r/2$. We wish to show that

$$\mathcal{R}_{ts}\le c\left(U_{\kappa,p}(\Omega;\mathcal{R})+\mathcal{N}[\delta\mathcal{R};\mathcal{C}_3^{\kappa}(\Omega;\mathbb{C}^n)]\right)|t-s|^{\kappa}. \tag{20}$$

To this end, let us construct a sequence of points $(s_k)_{k\ge 0}$, $s_k\in\Omega$ converging to t in the following way: set $s_0=t$, suppose by induction that $s_0,\dots,s_k\in\bar{\Pi}^+$ with $|s_0-s|,\dots,|s_k-s|<r/2$ (so that, in particular, $s_0,\dots,s_k\in\Omega$) have been constructed, and let $V_k := B(s,\frac{|s_k-s|}{2})\cap\bar{\Pi}^+$. Note that, since we are working on the upper half plane Π^+, the area $\mu(V_k)$ of V_k is at least $\frac{1}{2}\mu(B(s,\frac{|s_k-s|}{2}))=\frac{\pi}{8}|s_k-s|^2$. Define then

$$A_k := \left\{v\in V_k \mid I(v)>\frac{16}{\pi}\frac{U_{\kappa,p}^{2p}(\Omega;\mathcal{R})}{|s_k-s|^2}\right\} \tag{21}$$

and

$$B_k := \left\{v\in V_k \mid \frac{|\mathcal{R}_{s_k v}|^{2p}}{|s_k-v|^{2\kappa p+4}}>\frac{16}{\pi|s_k-s|^2}I(s_k)\right\} \tag{22}$$

where we have set

$$I(v) := \int_{B(s,|v-s|)\cap\bar{\Pi}^+}\frac{|\mathcal{R}_{uv}|^{2p}}{|v-u|^{2\kappa p+4}}\,du.$$

Let us prove now that $V_k\backslash(A_k\cup B_k)$ is not empty: observe that

$$U_{\kappa,p}^{2p}(\Omega;\mathcal{R})\ge\int_{A_k}dv I(v)>\frac{16}{\pi}\frac{U_{\kappa,p}^{2p}(\Omega;\mathcal{R})}{|s_k-s|^2}\mu(A_k)$$

and

$$I(s_k)\ge\int_{B_k}\frac{|\mathcal{R}_{us_k}|^{2p}}{|s_k-u|^{2\kappa p+4}}\,du>\frac{16}{\pi}\frac{\mu(B_k)}{|s_k-s|^2}I(s_k).$$

All together one has obtained $\mu(A_k),\mu(B_k)<\frac{\pi}{16}|s_k-s|^2$ so $\mu(A_k)+\mu(B_k)<\mu(V_k)$. One now chooses s_{k+1} arbitrarily in $V_k\backslash(A_k\cup B_k)$. Note that, by construction, $|t-s|<r/2$, and $\left|\frac{s_{k+1}-s}{s_k-s}\right|<1/2$ so $s_k\to s$ while staying inside Ω.

Now decompose (by using a number of times the operator δ) $\mathcal{R}_{s_0,s}$ into

$$\mathcal{R}_{s_0 s}=\mathcal{R}_{s_{n+1}s}+\sum_{k=0}^{n}\left(\mathcal{R}_{s_k s_{k+1}}+(\delta\mathcal{R})_{s_k s_{k+1} s}\right). \tag{23}$$

Applying $(22)_k$ and $(21)_{k-1}$, one gets

$$\frac{|\mathcal{R}_{s_k s_{k+1}}|^{2p}}{|s_k-s_{k+1}|^{2\kappa p+4}}<\frac{16}{\pi|s_k-s|^2}I(s_k)<\frac{256}{\pi^2}U_{\kappa,p}^{2p}(\Omega;\mathcal{R})|s_k-s|^{-4}.$$

Recalling our convention $a\lesssim b$ for the relation $a\le Cb$, where C is a given universal constant, we obtain $|\mathcal{R}_{s_k s_{k+1}}|\lesssim U_{\kappa,p}(\Omega;\mathcal{R})|s_k-s|^{\kappa}$. Furthermore, we have by construction $|s_k-s|\lesssim 2^{-n}|t-s|$, and thus

$$\left|\sum_{k=0}^{n}\mathcal{R}_{s_k,s_{k+1}}\right|\lesssim U_{\kappa,p}(\Omega;\mathcal{R})|t-s|^{\kappa}. \tag{24}$$

Turning now to $\delta\mathcal{R}$, it is easily seen that $|\delta\mathcal{R}_{s_k s_{k+1},s}| \lesssim \mathcal{N}[\delta\mathcal{R};\, \mathcal{C}_3^\kappa(\Omega;\mathbb{C}^n)]|s_k - s|^\kappa$. Invoking again the relation $|s_k - s| \lesssim 2^{-n}|t-s|$, we end up with

$$\sum_{k=0}^{n} |\delta\mathcal{R}_{s_k s_{k+1},s}| \lesssim \mathcal{N}[\delta\mathcal{R};\, \mathcal{C}_3^\kappa(\Omega;\mathbb{C}^n)]|t-s|^\kappa. \tag{25}$$

Finally, plugging relations (24)–(25) into (23) and letting $n\to\infty$, we easily get the announced bound (20), which ends the proof. □

2.3 Complex analysis preliminaries

The identity (17) involves integrals along some continuous paths in $\mathbb{C}$, which have to be estimated. We summarize in this section the upper bounds which will be needed later on.

First of all, the integral appearing in (17) can be estimated thanks to the following lemma borrowed from [16, Lemma 1.5]:

Lemma 2.5 *Let $\gamma : (0,1)\to\Pi^+$ be a piecewise smooth, continuous path, with ends $\gamma(0), \gamma(1) \in \bar{\Pi}^+$. Then*

$$\left|\mathbf{E}\int_\gamma \Gamma'_z dz \int_\gamma \bar{\Gamma}'_w dw\right| = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}\left|\int_\gamma dz \int_{\bar{\gamma}} d\bar{w}\, (-\imath(z-\bar{w}))^{2\alpha-2}\right| \le c|\gamma(1)-\gamma(0)|^{2\alpha},$$

for a universal positive constant c.

The following bound on iterated integrals of the process Γ, shown in [16, Theorem 3.4] (where they are called *analytic iterated integrals*) can then be seen as an extension of the previous lemma.

Lemma 2.6 (analytic iterated integrals) *Consider s, t in a fixed bounded neighborhood of* 0 *in $\bar{\Pi}^+$, and let $f_1,\dots,f_n$ and $g_1,\dots,g_n$ be analytic functions defined on a neighborhood V of the closed strip*

$$\bar{\Pi}^+_{s,t} := \{z\in\mathbb{C} | z = \lambda s + (1-\lambda)t + \imath\mu|t-s|, \quad \lambda,\mu\in[0,1]\}.$$

Let also $\Gamma = (\Gamma(1),\dots,\Gamma(d))$ be an analytic fractional Brownian motion with d independent components, where each component $\Gamma(j)$ is defined as in Sect. 2.1. *For ε,η small enough, define $\mathcal{V}_{s,t}(\varepsilon,\eta)$ by*

$$\begin{aligned}\mathcal{V}_{ts}(\varepsilon,\eta) &= \mathbf{E}[Z_1 Z_2]\\ &= \left(\int_{[s,t]} du_1 \int_{[s,u_1]} du_2 \dots \int_{[s,u_{n-1}]} du_n\right)\left(\int_s^t dv_1 \int_{[s,v_1]} dv_2 \dots \int_{[s,v_{n-1}]} dv_n\right)\\ &\quad\times\prod_{j=1}^{n} f_j(u_j+\imath\varepsilon)\overline{g_j(v_j+\imath\eta)}(-\imath(u_j-\bar{v}_j)+\varepsilon+\eta)^{2\alpha-2} du_j\, dv_j,\end{aligned}$$

where Z_1 is defined by

$$\int_{[s,t]} f_1(u_1+\imath\varepsilon) d\Gamma_{u_1+\imath\varepsilon}(1) \int_{[s,u_1]} f_2(u_2+\imath\varepsilon) d\Gamma_{u_2+\imath\varepsilon}(2)\dots \int_{[s,u_{n-1}]} f_n(u_n+\imath\varepsilon) d\Gamma_{u_n+\imath\varepsilon}(n),$$

and Z_2 can be written as:

$$\int_{[s,t]} \overline{g_1(v_1+\imath\varepsilon)}d\bar{\Gamma}_{v_1+\imath\varepsilon}(1) \int_{[s,v_1]} \overline{g_2(v_2+\imath\varepsilon)}d\bar{\Gamma}_{v_2+\imath\varepsilon}(2)\dots \int_{[s,v_{n-1}]} \overline{g_n(v_n+\imath\varepsilon)}d\bar{\Gamma}_{v_n+\imath\varepsilon}(n).$$

Then the following bound holds true:

$$|\mathcal{V}_{ts}(\varepsilon,\eta)| \lesssim \prod_{j=1}^{n} \sup_{z\in\bar{\Pi}^+_{s,t}} |f_j(z)| \prod_{j=1}^{n} \sup_{z\in\bar{\Pi}^+_{s,t}} |g_j(z)||t-s|^{2\alpha n}.$$

The last ingredient we need for our computations is a specific bound for analytic functions integrated with respect to the kernel $(-\imath(x-y))^{2\alpha-2}$. Observe that this bound will not be used directly in the sequel, but will serve as a prototype for our future computations.

Lemma 2.7 *Let s,t in a fixed bounded neighborhood of 0 in $\bar{\Pi}^+$, and $\phi(z,\bar{w})$ be an analytic function on a neighborhood of $\bar{\Pi}^+_{(s,t)}\times\bar{\Pi}^-_{(s,t)}$, where*

$$\bar{\Pi}^+_{s,t} := \{z\in\mathbb{C}|z=\lambda s+(1-\lambda)t+\imath\mu|t-s|,\quad \lambda,\mu\in[0,1]\},\quad \bar{\Pi}^-_{s,t} := \{\bar{z}|z\in\bar{\Pi}^+_{s,t}\}. \tag{26}$$

For $\varepsilon,\eta>0$, define $\Theta(\phi)$ as:

$$\begin{aligned}&[\Theta(\phi)](\varepsilon,\eta;s,t)\\ &:= \int_{[s,t]} dz \int_{[\bar{s},\bar{t}]} d\bar{w}\left[(-\imath(z-\bar{w})+2\varepsilon)^{2\alpha-2}-(-\imath(z-\bar{w})+\varepsilon+\eta)^{2\alpha-2}\right]\phi(z,\bar{w}).\end{aligned}$$

Then, for every $\rho\in(0,2\alpha)$, there exists C_ρ such that

$$|[\Theta(\phi)](\varepsilon,\eta;s,t)| \le C_\rho|\varepsilon-\eta|^\rho|t-s|^{2\alpha-\rho}M^\phi_{ts}, \tag{27}$$

where

$$M^\phi_{ts} \triangleq \sup\left\{|\phi(u,\bar{v})|;\,(u,\bar{v})\in\bar{\Pi}^+_{(s,t)}\times\bar{\Pi}^-_{(s,t)}\right\}.$$

Proof Without restriction of generality we may assume that $\varepsilon>\eta>0$. We use the following contour of integration in $\bar{\Pi}^+_{(s,t)}\times\bar{\Pi}^-_{(s,t)}$:

$$\begin{aligned}\Delta &:= \gamma\times\bar{\gamma},\quad \gamma=\gamma_1\cup\gamma_2\cup\gamma_3\\ &:= [s,s+\imath|t-s|]\cup[s+\imath|t-s|,t+\imath|t-s|]\cup[t+\imath|t-s|,t].\end{aligned} \tag{28}$$

Set $\Delta_{i,j}=\gamma_i\times\bar{\gamma}_j$ so that $\Delta=\cup_{1\le i,j\le 3}\Delta_{i,j}$. Let $I_{i,j}$ be the integral over $\Delta_{i,j}$ of the function $(z,\bar{w})\mapsto\left[(-\imath(z-\bar{w})+2\varepsilon)^{2\alpha-2}-(-\imath(z-\bar{w})+\varepsilon+\eta)^{2\alpha-2}\right]\phi(z,\bar{w})$. We shall give a bound of type (27) for each $I_{i,j}$. The proof relies on the following observation: if $\varepsilon,\eta>0$ and $z\in\mathbb{C}$, $\Re z>0$, then (for any $\rho\in(0,1)$)

$$|(z+\varepsilon)^{2\alpha-2}-(z+\eta)^{2\alpha-2}| \le C|\varepsilon-\eta|^\rho|z|^{2\alpha-2-\rho}. \tag{29}$$

By symmetry we only need to consider the following four cases (only the fourth one is non-trivial since z and $\bar{w}$ may be ε-close):

Case 1 $i = j = 2$.

$$
\begin{aligned}
|I_{2,2}| = \Big| \int_{[s,t]} dz \int_{[\bar{s},\bar{t}]} d\bar{w} \big[& (-\imath(z-\bar{w}) + 2|t-s| + 2\varepsilon)^{2\alpha-2} \\
& -(-\imath(z-\bar{w}) + 2|t-s| + \varepsilon + \eta)^{2\alpha-2} \big] \phi(z + \imath|t-s|, \bar{w} - \imath|t-s|) \Big| \qquad (30) \\
\le C_\rho \int_{[s,t]} |dz| \int_{[\bar{s},\bar{t}]} & |d\bar{w}||t-s|^{2\alpha-2-\rho}(\varepsilon-\eta)^\rho\, M^\phi_{ts} = C'_\rho |t-s|^{2\alpha-\rho}(\varepsilon-\eta)^\rho\, M^\phi_{ts}.
\end{aligned}
$$

Case 2 $i = 1$, $j = 3$.

$$
\begin{aligned}
|I_{1,3}| &= \Big| \int_0^{|t-s|} dx \int_0^{|t-s|} dy \big[(-\imath(s-\bar{t}) + x + y + 2\varepsilon)^{2\alpha-2} \\
&\qquad -(-\imath(s-\bar{t}) + x + y + \varepsilon + \eta)^{2\alpha-2} \big] \phi(s + \imath x, t - \imath y) \Big| \\
&\le C_\rho \int_0^{|t-s|} dx \int_0^{|t-s|} dy |t-\bar{s}|^{2\alpha-2-\rho}(\varepsilon-\eta)^\rho\, M^\phi_{ts} \\
&\le C'_\rho |t-s|^{2\alpha-\rho}(\varepsilon-\eta)^\rho\, M^\phi_{ts}, \qquad (31)
\end{aligned}
$$

since $|t-\bar{s}| \ge |t-s|$.
Case 3 $i = 1$, $j = 2$.

$$
\begin{aligned}
|I_{1,2}| = \Big| \int_0^{|t-s|} dx \int_{[\bar{s},\bar{t}]} d\bar{w} \big[& (-\imath(s-\bar{w}) + |t-s| + x + 2\varepsilon)^{2\alpha-2} \\
& -(-\imath(s-\bar{w}) + |t-s| + x + \varepsilon + \eta)^{2\alpha-2} \big] \phi(s + \imath x, \bar{w} - \imath|t-s|) \Big| \\
\le C_\rho \int_0^{|t-s|} dx \int_{[\bar{s},\bar{t}]} & |d\bar{w}||t-s|^{2\alpha-2-\rho}(\varepsilon-\eta)^\rho\, M^\phi_{ts} = C'_\rho |t-s|^{2\alpha-\rho}(\varepsilon-\eta)^\rho\, M^\phi_{ts}.
\end{aligned}
\qquad (32)
$$

Case 4 $i = 1$, $j = 1$.

$$
\begin{aligned}
|I_{1,1}| &= \Big| \int_0^{|t-s|} dx \int_0^{|t-s|} dy \big[K_{2\varepsilon}(x,y) - K_{\varepsilon+\eta}(x,y) \big] \phi(s + \imath x, s - \imath y) \Big| \\
&\le C_\rho \int_0^{|t-s|} dx \int_0^{|t-s|} dy (x+y)^{2\alpha-2-\rho}(\varepsilon-\eta)^\rho M^\phi_{ts} \le C'_\rho |t-s|^{2\alpha-\rho}(\varepsilon-\eta)^\rho M^\phi_{ts},
\end{aligned}
\qquad (33)
$$

where we have set $K_a(x,y) = (2\Im s + x + y + a)^{2\alpha-2}$ for any positive a. It should be noticed at this point that the last integral converges only if $\rho < 2\alpha$, which is one of our standing assumptions. Now, putting together the estimates (30), (31), (32) and (33), we get the desired result. □

Remark 2.8 The kernel $(x, y) \to (x+y)^{2\alpha-2-\rho}$ appearing in the last case is singular only at the point $(x, y) = (0, 0)$, whereas the usual kernel $(x, y) \to (\pm\imath(x-y))^{2\alpha-2}$ is singular on the diagonal. This simple fact explains why our estimates work (and why the deformation of contour is so important). Note that the absolute value should *not* be placed inside the integral *before* the deformation of contour (otherwise the integrals become most of the time infinite in the limit $\eta \to 0$).

3 The rough path associated to Γ

We proceed in this section to the definition of a rough path above the process Γ defined at Sect. 2.1. To this purpose, let us first recall what is meant by rough path in our complex plane context:

Definition 3.1 Let Ω be a neighborhood of 0 in Π^+, $X : \Omega \to \mathbb{C}^d$ be a μ-Hölder path with $0 < \mu < 1$, whose restriction to Π^+ is analytic, and set $N = \lfloor 1/\mu \rfloor$. We say that X generates a rough path if there exists a family $\{\mathbf{X^n};\ 1 \le n \le N\}$ defined on Ω, satisfying:

(i) $\mathbf{X^n}$ is a 1-increment with Hölder regularity $n\mu$, taking values in $\mathbb{C}^{d^n}$, that is $\mathbf{X^n}$ is an element of $\mathcal{C}_2^{n\mu}(\Omega; \mathbb{C}^{d^n})$. Recall in particular that $\mathbf{X^n}$ is by definition analytic on Π^+.

(ii) We have $\mathbf{X^1} := X$, and the algebraic relations satisfied by the $\mathbf{X^n}$ are the same as those of Proposition 1.1: for any $n \le N$, $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$, we have the multiplicative property

$$\delta \mathbf{X^n}(i_1, \ldots, i_n) = \sum_{j=1}^{n-1} \mathbf{X^j}(i_1, \ldots, i_j) \mathbf{X^{n-j}}(i_{j+1}, \ldots, i_n), \tag{34}$$

that is $\delta \mathbf{X^n}_{tus}(i_1, \ldots, i_n) = \sum_{j=1}^{n-1} \mathbf{X^j}_{tu}(i_1, \ldots, i_j) \mathbf{X^{n-j}}_{us}(i_{j+1}, \ldots, i_n)$, for any $t, u, s \in \Omega$.

(iii) Furthermore, the rough path generated by X is said to be *of geometric type* under the following additional condition: for any n, m such that $n + m \le N$, we have:

$$\mathbf{X^n}(i_1, \ldots, i_n) \circ \mathbf{X^m}(j_1, \ldots, j_m) = \sum_{\bar{k} \in \mathrm{Sh}(\bar{\imath}, \bar{\jmath})} \mathbf{X^{n+m}}(k_1, \ldots, k_{n+m}), \tag{35}$$

where, for two tuples $\bar{\imath}, \bar{\jmath}$, $\Sigma_{(\bar{\imath}, \bar{\jmath})}$ stands for the set of permutations of the indices contained in $(\bar{\imath}, \bar{\jmath})$, and $\mathrm{Sh}(\bar{\imath}, \bar{\jmath})$ is a subset of $\Sigma_{(\bar{\imath}, \bar{\jmath})}$ defined by:

$$\mathrm{Sh}(\bar{\imath}, \bar{\jmath}) = \left\{\sigma \in \Sigma_{(\bar{\imath}, \bar{\jmath})};\ \sigma \text{ does not change the orderings of } \bar{\imath} \text{ and } \bar{\jmath}\right\}.$$

For instance, Eq. 35 reads for $n = 2$ and $m = 1$

$$\mathbf{X^2}(i_1, i_2) \circ \mathbf{X^1}(j_1) = \mathbf{X^3}(i_1, i_2, j_1) + \mathbf{X^3}(i_1, j_1, i_2) + \mathbf{X^3}(j_1, i_1, i_2). \tag{36}$$

As mentioned in the introduction, the construction of the rough path above our analytic fBm will be achieved by regularizing Γ into a process Γ^ε defined on Π^+ by $\Gamma_t^\varepsilon = \Gamma_{t+\imath\varepsilon}$. The latter process is analytic on an open neighborhood of $\bar{\Pi}^+$, which allows to define any iterated integral of Γ^ε in the Riemann sense. Then the convergence of these integrals in some suitable Hölder spaces is obtained by combining the Garsia type result of Proposition 2.4 and some moment estimates similar to Lemma 2.7.

3.1 Convergence of Γ^ε

A very first step in the analysis of Γ consists in getting some convergence results for Γ^ε itself towards Γ, in Hölder spaces. In order to obtain this (intuitively trivial) convergence, we shall use the following estimates, which are immediate consequences of Lemma 2.5: there exists a constant c such that, for all $\varepsilon, \eta > 0$, and $s, t \in \bar{\Pi}^+$, we have

$$\mathbf{E}\left[\left|\Gamma_t^\varepsilon - \Gamma_s^\varepsilon\right|^2\right] \le c\,|t-s|^{2\alpha}, \quad \text{and} \quad \mathbf{E}\left[\left|\Gamma_t^\varepsilon - \Gamma_t^\eta\right|^2\right] \le c\,|\varepsilon-\eta|^{2\alpha}. \tag{37}$$

We are now ready to study the convergence of Γ^ε on our fixed neighborhood Ω (recall also that we work on a complete probability space $(\mathcal{U}, \mathcal{F}, \mathbf{P})$):

Lemma 3.2 *As $\varepsilon \to 0$, the process Γ^ε converges in $L^1(\mathcal{U}; \mathcal{C}_1^\mu(\Omega))$, for any $\mu < \alpha$ and $T > 0$. Its limit is the analytic fractional Brownian motion Γ.*

Proof We shall divide this proof into two steps:
Step 1: Reduction to moment estimates. We shall prove that $\{\Gamma^\varepsilon;\ \varepsilon > 0\}$ is a Cauchy sequence in $L^1(\mathcal{U}; \mathcal{C}_1^\mu(\Omega))$, and in order to estimate $\mathcal{N}[\Gamma^\varepsilon - \Gamma^\eta; \mathcal{C}_1^\mu]$, we shall resort to Lemma 2.3 with $f = \Gamma^\varepsilon - \Gamma^\eta$. This yields, for $p > 1$ and $\mu < \alpha$,

$$\mathcal{N}[\Gamma^\varepsilon - \Gamma^\eta; \mathcal{C}_1^\mu] \le c U_{\mu,p}\left(\delta(\Gamma^\varepsilon - \Gamma^\eta)\right) = c\left(\int_\Omega\int_\Omega \frac{|\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^{2p}}{|t-s|^{2\mu p+4}}\,dsdt\right)^{1/2p}.$$

Hence, invoking Jensen's inequality, we obtain:

$$\begin{aligned}\mathbf{E}\left[\mathcal{N}[\Gamma^\varepsilon - \Gamma^\eta; \mathcal{C}_1^\mu]\right] &\lesssim \left(\int_\Omega\int_\Omega \frac{\mathbf{E}\left[|\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^{2p}\right]}{|t-s|^{2\mu p+4}}\,dsdt\right)^{1/2p}\\ &\lesssim \left(\int_\Omega\int_\Omega \frac{\mathbf{E}^p\left[|\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^{2}\right]}{|t-s|^{2\mu p+4}}\,dsdt\right)^{1/2p},\end{aligned}$$

where we have used the fact that $\Gamma^\varepsilon, \Gamma^\eta$ are Gaussian processes in the last inequality. By considering p large enough in the relation above, it is thus easily seen that, if we can prove that

$$\mathbf{E}[|\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^2] \le c|t-s|^{2\hat{\mu}}|\varepsilon-\eta|^\beta \tag{38}$$

for a certain $\beta > 0$ and $\mu < \hat{\mu} < \alpha$, then the following relation holds true:

$$\mathbf{E}\left[\mathcal{N}[\Gamma^\varepsilon - \Gamma^\eta; \mathcal{C}_1^\mu]\right] \lesssim |\varepsilon-\eta|^{\beta/2}.$$

Thus, we get that the family $\{\Gamma^\varepsilon; \varepsilon > 0\}$ is a Cauchy sequence in $L^1(\mathcal{U}; \mathcal{C}_1^\mu([0,T]))$, whose limit is the analytic fBm Γ, *provided* we can prove (38). The remainder of the proof is thus devoted to show the latter relation.
Step 2: Moment estimates. Set $U_{ts} = \mathbf{E}[|\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^2]$. We shall now prove that $U_{ts} \le c_\rho|t-s|^{2\alpha\rho}|\varepsilon-\eta|^{2\alpha(1-\rho)}$ for every $\rho \in (0,1)$. To this purpose, notice that:

$$|\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^2 \lesssim |\delta\Gamma^\varepsilon_{ts}|^2 + |\delta\Gamma^\eta_{ts}|^2, \quad \text{and} \quad |\delta(\Gamma^\varepsilon - \Gamma^\eta)_{ts}|^2 \lesssim |\Gamma^\varepsilon_t - \Gamma^\eta_t|^2 + |\Gamma^\varepsilon_s - \Gamma^\eta_s|^2.$$

This allows to write, for an arbitrary exponent $\rho \in (0,1)$,

$$U_{ts} \lesssim \left(\mathbf{E}\left[|\delta\Gamma^\varepsilon_{ts}|^2 + |\delta\Gamma^\eta_{ts}|^2\right]\right)^\rho \left(\mathbf{E}\left[|\Gamma^\varepsilon_t - \Gamma^\eta_t|^2 + |\Gamma^\varepsilon_s - \Gamma^\eta_s|^2\right]\right)^{1-\rho}.$$

A direct application of the estimates given at the beginning of the present paragraph yields:

$$U_{ts} \lesssim |t-s|^{2\alpha\rho} |\varepsilon - \eta|^{2\alpha(1-\rho)}, \tag{39}$$

which ends the proof, since $\mu \equiv \alpha\rho$ can be taken as close as we wish to α. □

Remark 3.3 A slight extension of the computations above allows to prove that in fact, Γ^ε converges in $L^p(\mathcal{U}; \mathcal{C}_1^\mu(\Omega))$ for any $p > 1$.

3.2 Convergence of Lévy's area

Consider a two-dimensional analytic fBm $\Gamma = (\Gamma(1), \Gamma(2))$ with independent components, and the associated approximation $\Gamma^\varepsilon = (\Gamma^\varepsilon(1), \Gamma^\varepsilon(2))$. We then set

$$\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(j_1, j_2) = \int_{[s,t]} d\Gamma^\varepsilon_{u_1}(j_1) \int_{[s,u_1]} d\Gamma^\varepsilon_{u_2}(j_2), \quad \text{for} \quad j_1, j_2 \in \{1,2\},\ s,t \in \Omega, \tag{40}$$

where the above iterated integral is understood in the Riemann sense. This section is devoted to prove that $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}$ is a convergent sequence in $L^1(\mathcal{U}; \mathcal{C}_2^{2\mu}(\Omega))$, and that its limit $\Gamma^{\mathbf{2}}$ satisfies $\delta\Gamma^{\mathbf{2}} = \delta B \otimes \delta B$ as in Definition 3.1. We shall study the convergence for $j_1 = j_2$ and $j_1 \neq j_2$ separately.

Proposition 3.4 *The increments* $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1)$ *and* $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(2,2)$ *converge in* $L^1(\mathcal{U}; \mathcal{C}_2^{2\mu}(\Omega))$.

Proof For notational sake, we shall write $\mathcal{C}_2^{2\mu}, \mathcal{C}_3^{2\mu}$ instead of $\mathcal{C}_2^{2\mu}(\Omega), \mathcal{C}_3^{2\mu}(\Omega)$ in the sequel. In order to prove that $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1)$ is a Cauchy sequence in $L^1(\mathcal{U}; \mathcal{C}_2^{2\mu}(\Omega))$, we invoke Proposition 2.4, which can be read here as:

$$\begin{aligned}
&\mathcal{N}[\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1) - \mathbf{\Gamma}^{\mathbf{2},\eta}(1,1); \mathcal{C}_2^{2\mu}] \\
&\quad \lesssim U_{2\mu,p}(\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1) - \mathbf{\Gamma}^{\mathbf{2},\eta}(1,1)) + \mathcal{N}[\delta\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1) - \delta\mathbf{\Gamma}^{\mathbf{2},\eta}(1,1); \mathcal{C}_3^{2\mu}] \equiv A_1 + A_2.
\end{aligned}$$

In order to estimate the term A_1 above, notice first that, since Γ^ε is a regular path, we have: $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}_{ts}(1,1) = \frac{1}{2}\left[\delta\Gamma^\varepsilon_{ts}(1)\right]^2$. Hence,

$$A_1 = \frac{1}{2}\left(\int_\Omega \int_\Omega \frac{|[\delta\Gamma^\varepsilon_{ts}(1)]^2 - [\delta\Gamma^\eta_{ts}(1)]^2|^{2p}}{|t-s|^{4\mu p+4}}\, ds dt\right)^{1/2p}.$$

This integral can now be bounded as in Lemma 3.2, by means of an inequality similar to (39).

Let us turn now to the evaluation of the term A_2. Owing to the fact that $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}$ is defined by (40), where Γ^ε is a regular process, the following particular case of (4) is readily checked:

$$\left[\delta\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1) - \delta\mathbf{\Gamma}^{\mathbf{2},\eta}(1,1)\right]_{tus} = \delta\Gamma^\varepsilon_{tu}(1)\delta\Gamma^\varepsilon_{us}(1) - \delta\Gamma^\eta_{tu}(1)\delta\Gamma^\eta_{us}(1).$$

Hence, for an arbitrary coefficient $\rho \in (0,1)$ and $0 < \mu < \hat{\mu} < \alpha$, we end up with:

$$\begin{aligned}
\left|\left[\delta\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,1) - \delta\mathbf{\Gamma}^{\mathbf{2},\eta}(1,1)\right]_{tus}\right| \le & \left[(\mathcal{N}[\Gamma^\varepsilon; \mathcal{C}_1^{\hat{\mu}}] + \mathcal{N}[\Gamma^\eta; \mathcal{C}_1^{\hat{\mu}}])|t-u|^{\hat{\mu}}|u-s|^{\hat{\mu}}\right]^\rho \\
& \times\Big[\delta\Gamma^\varepsilon_{tu}(1)\left(\delta\Gamma^\varepsilon_{us}(1) - \delta\Gamma^\eta_{us}(1)\right) \\
& + \left(\delta\Gamma^\varepsilon_{tu}(1) - \delta\Gamma^\eta_{tu}(1)\right)\delta\Gamma^\eta_{us}(1)\Big]^{1-\rho},
\end{aligned}$$

and thus, a standard application of the Cauchy–Schwarz inequality yields:

$$\begin{aligned}\mathbf{E}[A_2] &= \mathbf{E}\left[\mathcal{N}[\delta\boldsymbol{\Gamma}^{\mathbf{2},\varepsilon}(1,1)-\delta\boldsymbol{\Gamma}^{\mathbf{2},\eta}(1,1);\mathcal{C}_3^{2\mu}]\right]\\ &\lesssim \mathbf{E}^{1/2}\left[\mathcal{N}^{2\rho}[\Gamma^{\varepsilon}(1);\mathcal{C}_1^{\hat{\mu}}]+\mathcal{N}^{2\rho}[\Gamma^{\eta}(1);\mathcal{C}_1^{\hat{\mu}}]\right]\mathbf{E}^{1/2}\left[\mathcal{N}^{2(1-\rho)}[\Gamma^{\varepsilon}(1)-\Gamma^{\eta}(1);\mathcal{C}_1^{\hat{\mu}}]\right]\\ &\lesssim |\varepsilon-\eta|^{\beta},\end{aligned}$$

for a certain $\beta > 0$, according to Remark 3.3 and Eq. 39. Our claim is now easily deduced from our estimates on A_1 and A_2. □

Let us begin the preliminary steps for the convergence of the crossed terms $\boldsymbol{\Gamma}^{\mathbf{2},\varepsilon}(1,2)$ and $\boldsymbol{\Gamma}^{\mathbf{2},\varepsilon}(2,1)$, for which the following notation will be needed

Notation 3.5 For $\varepsilon_1,\varepsilon_2>0$ and $(x,\bar{y})\in\bar{\Pi}^+\times\bar{\Pi}^-$, we set

$$F_{\varepsilon_1,\varepsilon_2}(x,\bar{y}) := (-\imath(x-\bar{y})+\varepsilon_1+\varepsilon_2)^{2\alpha-2}.$$

With this notation in hand, one can estimate the increments of $\boldsymbol{\Gamma}^{\mathbf{2},\varepsilon}$ as follows:

Lemma 3.6 *For any $s,t\in\Omega$ and $\rho\in(0,1)$, there exists a positive constant $c_\rho>0$ such that*

$$\mathbf{E}[|(\boldsymbol{\Gamma}^{\mathbf{2},\varepsilon}(1,2)-\boldsymbol{\Gamma}^{\mathbf{2},\eta}(1,2))_{ts}|^2] \le c_\rho|t-s|^{4\alpha(1-\rho)}|\varepsilon-\eta|^{4\alpha\rho}.$$

Proof According to identity (17), we have:

$$\begin{aligned}&\mathbf{E}[|(\boldsymbol{\Gamma}^{\mathbf{2},\varepsilon}(1,2)-\boldsymbol{\Gamma}^{\mathbf{2},\eta}(1,2))_{ts}|^2]\\ &=\mathbf{E}\left[\left(\int_{[s,t]}d\Gamma_{u_1+\imath\varepsilon}(1)\int_{[s,u_1]}d\Gamma_{u_2+\imath\varepsilon}(2)-\int_{[s,t]}d\Gamma_{u_1+\imath\eta}(1)\int_{[s,u_1]}d\Gamma_{u_2+\imath\eta}(2)\right)\right.\\ &\quad\left.\times\left(\int_{[s,t]}d\bar{\Gamma}_{v_1+\imath\varepsilon}(1)\int_{[s,v_1]}d\bar{\Gamma}_{v_2+\imath\varepsilon}(2)-\int_{[s,t]}d\bar{\Gamma}_{v_1+\imath\eta}(1)\int_{[s,v_1]}d\bar{\Gamma}_{v_2+\imath\eta}(2)\right)\right]\\ &=\int_{[s,t]}du_1\int_{[s,u_1]}du_2\int_{[\bar{s},\bar{t}]}d\bar{v}_1\int_{[\bar{s},\bar{v}_1]}d\bar{v}_2 F^{(2)}_{\varepsilon,\eta}(u_1,\bar{v}_1;u_2,\bar{v}_2),\end{aligned}$$

where, recalling Notation 3.5, the function $F^{(2)}_{\varepsilon,\eta}$ is defined by:

$$\begin{aligned}F^{(2)}_{\varepsilon,\eta}(u_1,\bar{v}_1;u_2,\bar{v}_2) &= F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)F_{\varepsilon,\varepsilon}(u_2,\bar{v}_2)\\ &\quad+F_{\eta,\eta}(u_1,\bar{v}_1)F_{\eta,\eta}(u_2,\bar{v}_2)-2F_{\varepsilon,\eta}(u_1,\bar{v}_1)F_{\varepsilon,\eta}(u_2,\bar{v}_2)\\ &= F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)\left[F_{\varepsilon,\varepsilon}(u_2,\bar{v}_2)-F_{\varepsilon,\eta}(u_2,\bar{v}_2)\right]\\ &\quad+F_{\varepsilon,\eta}(u_2,\bar{v}_2)\left[F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)-F_{\varepsilon,\eta}(u_1,\bar{v}_1)\right]\\ &\quad+F_{\eta,\eta}(u_1,\bar{v}_1)\left[F_{\eta,\eta}(u_2,\bar{v}_2)-F_{\eta,\varepsilon}(u_2,\bar{v}_2)\right]\\ &\quad+F_{\eta,\varepsilon}(u_2,\bar{v}_2)\left[F_{\eta,\eta}(u_1,\bar{v}_1)-F_{\eta,\varepsilon}(u_1,\bar{v}_1)\right].\end{aligned}$$

We now have to control a sum made of many terms exhibiting the same level of difficulty. We shall thus focus on one of them, namely:

$$I_1^{(2)} \triangleq \int_{[s,t]}du_1\int_{[s,u_1]}du_2\int_{[\bar{s},\bar{t}]}d\bar{v}_1\int_{[\bar{s},\bar{v}_1]}d\bar{v}_2 F_{\varepsilon,\eta}(u_2,\bar{v}_2)\left[F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)-F_{\varepsilon,\eta}(u_1,\bar{v}_1)\right].$$

For the control of $I_1^{(2)}$, as in Lemma 2.7, we introduce the contour of integration

$$\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3 = [s, s + \imath|t-s|] \cup [s + \imath|t-s|, t + \imath|t-s|] \cup [t + \imath|t-s|, t].$$

If $z \in \gamma$, let $\gamma(z)$ be the section of the path γ comprised between s and z. Then (by Cauchy's theorem)

$$I_1^{(2)} = \int_{\gamma} dz_1 \int_{\gamma(z_1)} dz_2 \int_{\bar{\gamma}} d\bar{w}_1 \int_{\overline{\gamma(w_1)}} d\bar{w}_2 F_{\varepsilon,\eta}(z_2, \bar{w}_2) \left[F_{\varepsilon,\varepsilon}(z_1, \bar{w}_1) - F_{\varepsilon,\eta}(z_1, \bar{w}_1) \right].$$

As in Lemma 2.7, 9 terms should be controlled in order to achieve the desired bound. We shall treat the most divergent of them, that is:

$$J_1^{(2)} = \int_{\gamma_1} dz_1 \int_{\gamma(z_1)} dz_2 \int_{\bar{\gamma}_1} d\bar{w}_1 \int_{\overline{\gamma(w_1)}} d\bar{w}_2 F_{\varepsilon,\eta}(z_2, \bar{w}_2) \left[F_{\varepsilon,\varepsilon}(z_1, \bar{w}_1) - F_{\varepsilon,\eta}(z_1, \bar{w}_1) \right].$$

On γ_1, the change of variable $z_1 = s + \imath u_1$ for $0 \le u_1 \le |t-s|$, $z_2 = s + \imath u_2$ for $0 \le u_2 \le u_1$, and the same kind of transformations for $\bar{w}_1, \bar{w}_2$, yield:

$$\begin{aligned} J_1^{(2)} = & \int_0^{|t-s|} du_1 \int_0^{u_1} du_2 \int_0^{|t-s|} dv_1 \int_0^{v_1} dv_2 \\ & \left[(2\Im s + u_1 + v_1 + 2\varepsilon)^{2\alpha-2} - (2\Im s + u_1 + v_1 + \varepsilon + \eta)^{2\alpha-2} \right] \\ & \times (2\Im s + u_2 + v_2 + 2\varepsilon)^{2\alpha-2}, \end{aligned}$$

and hence:

$$\begin{aligned} |J_1^{(2)}| \le & \int_0^{|t-s|} du_1 \int_0^{|t-s|} dv_1 \left| (u_1 + v_1 + 2\varepsilon)^{2\alpha-2} - (u_1 + v_1 + \varepsilon + \eta)^{2\alpha-2} \right| \\ & \times \int_0^{|t-s|} du_2 \int_0^{|t-s|} dv_2 \left| (u_2 + v_2 + 2\varepsilon)^{2\alpha-2} \right|. \end{aligned}$$

As in Lemma 2.7, we can now easily conclude, for an arbitrary constant $0 < \rho < 1$, that:

$$|J_1^{(2)}| \lesssim |\varepsilon - \eta|^{4\alpha\rho} (t-s)^{4\alpha(1-\rho)}.$$

We may now treat the other terms appearing in the analysis of $I_1^{(2)}$ (and more generally of $F^{(2)}$) in the same way, which ends the proof. $\square$

We are now ready to analyze the convergence of the crossed terms $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2)$ and $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(2,1)$:

Proposition 3.7 *The increments* $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2)$ *and* $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(2,1)$ *converge in* $L^1(\mathcal{U}; \mathcal{C}_2^{2\gamma}(\Omega))$.

Proof The beginning of the proof goes exactly along the same lines as for Proposition 3.4. Let us write $\mathcal{C}_2^{2\mu}, \mathcal{C}_3^{2\mu}$ for $\mathcal{C}_2^{2\mu}(\Omega), \mathcal{C}_3^{2\mu}(\Omega)$. In order to prove that $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2)$ is a Cauchy sequence in $L^1(\mathcal{U}; \mathcal{C}_2^{2\mu}(\Omega))$, we invoke Proposition 2.4:

$$\begin{aligned} & \mathcal{N}[\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2) - \mathbf{\Gamma}^{\mathbf{2},\eta}(1,2); \mathcal{C}_2^{2\mu}] \\ & \quad \lesssim U_{2\mu,p}(\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2) - \mathbf{\Gamma}^{\mathbf{2},\eta}(1,2)) + \mathcal{N}[\delta\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2) - \delta\mathbf{\Gamma}^{\mathbf{2},\eta}(1,2); \mathcal{C}_3^{2\mu}] \equiv A_1 + A_2. \end{aligned}$$

The term A_2 can now be bounded as in Proposition 3.4, owing to the fact that $\delta\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2) = \delta\Gamma^{\varepsilon}(1)\delta\Gamma^{\varepsilon}(2)$. We thus get

$$\mathbf{E}[A_2] \lesssim |\varepsilon-\eta|^{\beta},$$

for a certain $\beta > 0$.

The term A_1 can be handled in the following way: by definition, we have

$$A_1 = \left(\int_{\Omega}\int_{\Omega} \frac{|\mathbf{\Gamma}^{\mathbf{2},\varepsilon}_{ts}(1,2) - \mathbf{\Gamma}^{\mathbf{2},\eta}_{ts}(1,2)|^{2p}}{|t-s|^{4\mu p+4}}\, dsdt \right)^{1/2p}.$$

We can now apply Jensen's inequality as in Lemma 3.2. Furthermore, $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}_{ts}(1,2)$ is a random variable in the second chaos of the fractional Brownian motion Γ, and since all the L^p norms on any given fixed chaos are equivalent, we obtain:

$$\mathbf{E}[A_1] \lesssim \int_{\Omega}\int_{\Omega} \frac{\mathbf{E}^p\left[|\mathbf{\Gamma}^{\mathbf{2},\varepsilon}_{ts}(1,2) - \mathbf{\Gamma}^{\mathbf{2},\eta}_{ts}(1,2)|^{2}\right]}{|t-s|^{4\mu p+4}}\, dsdt.$$

As in Lemma 3.2, we are now reduced to an estimate of the form

$$\mathbf{E}[|(\mathbf{\Gamma}^{\mathbf{2},\varepsilon} - \mathbf{\Gamma}^{\mathbf{2},\eta})_{ts}|^2] \le c|t-s|^{2\hat{\mu}}|\varepsilon-\eta|^{\beta},$$

for a certain $\beta > 0$ and $\mu < \hat{\mu} < \alpha$. But this estimate stems directly from Lemma 3.6, and gathering our estimates on A and B, we have thus proved our claim:

$$\mathbf{E}\left[\mathcal{N}[\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(1,2) - \mathbf{\Gamma}^{\mathbf{2},\eta}(1,2); \mathcal{C}_2^{2\mu}]\right] \lesssim |\varepsilon-\eta|^{\beta}.$$

□

Remark 3.8 As in Sect. 3.1, the L^p-convergence of $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(i,j)$ in $\mathcal{C}_2^{2\mu}(\Omega)$ can also be obtained here for $i, j \in \{1,2\}$, by slightly adapting our computations for the L^1-convergence.

Putting together the results we have obtained so far, we can now state the following existence result for a rough path of order 2 based on Γ, for any value of the Hurst parameter $\alpha \in (0, 1/2)$:

Theorem 3.9 *Let Γ be an analytic fractional Brownian motion with Hurst parameter $\alpha \in (0,1/2)$, and Γ^{ε} its regular approximation. Let also $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}$ be the regularized Lévy area given by formula (40), and consider $0 < \mu < \alpha$. Then*:

(1) *For any $p \ge 1$, the couple $(\Gamma^{\varepsilon}, \mathbf{\Gamma}^{\mathbf{2},\varepsilon})$ converges in $L^p(\mathcal{U}; \mathcal{C}_1^{\mu}(\Omega;\mathbb{C}^d) \times \mathcal{C}_2^{2\mu}(\Omega;\mathbb{C}^{d,d}))$ to a couple $(\Gamma, \Gamma^{\mathbf{2}})$, where Γ is the analytic fractional Brownian motion mentioned above.*

(2) *The increment $\Gamma^{\mathbf{2}}$ satisfies the multiplicative and geometric algebraic relations prescribed in Definition 3.1, namely:*

$$\delta\Gamma^{\mathbf{2}}(i,j) = \delta\Gamma(i)\delta\Gamma(j), \quad \textit{and} \quad \Gamma^{\mathbf{2}}(i,j) + \Gamma^{\mathbf{2}}(j,i) = \delta\Gamma(i)\circ\delta\Gamma(j),$$

for $i, j = 1, \ldots, d$.

Proof The first part of our assertion is trivially deduced from Propositions 3.2, 3.4 and 3.7.

As far as the second part of our claim is concerned, it is sufficient to notice that, since $\mathbf{\Gamma}^{\mathbf{2},\varepsilon}$ is a smooth process, the relation

$$\delta\mathbf{\Gamma}^{\mathbf{2},\varepsilon}(i,j) = \delta\Gamma^{\varepsilon}(i)\delta\Gamma^{\varepsilon}(j), \quad \text{and} \quad \mathbf{\Gamma}^{\mathbf{2},\varepsilon}(i,j) + \mathbf{\Gamma}^{\mathbf{2},\varepsilon}(j,i) = \delta\Gamma^{\varepsilon}(i)\circ\delta\Gamma^{\varepsilon}(j),$$

is automatically satisfied, by some algebraic manipulations involving only usual Riemann integrals. The desired result is then obtained by taking limits on both sides of the identity above, and taking into account that Γ^{ε} converges in any $L^q(\mathcal{U}; \mathcal{C}_1^{\mu}(\Omega; \mathbb{C}^d))$, for $q \geq 1$. $\square$

3.3 Multidimensional estimates

Let $\Gamma = (\Gamma(1), \ldots, \Gamma(d))$ be an d-dimensional analytic fractional Brownian motion. This section is a generalization of the previous one to the case of multiply iterated integrals of any order. As a result, we finally obtain a rough-path lying above Γ. Let us consider then our fixed bounded neighborhood Ω of 0 and the analytic approximation Γ^{ε} of Γ. For $s, t \in \Omega$, $n \leq N = \lfloor 1/\alpha \rfloor$, and any tuple $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$, the natural approximation of $\mathbf{\Gamma}^{\mathbf{n}}_{ts}(i_1, \ldots, i_n)$ is given by the Riemann iterated integral:

$$\mathbf{\Gamma}^{\mathbf{n},\varepsilon}_{ts}(i_1, \ldots, i_n) := \int_{[s,t]} d\Gamma^{\varepsilon}_{u_1}(i_1) \int_{[s,u_1]} d\Gamma^{\varepsilon}_{u_2}(i_2) \cdots \int_{[s,u_{n-1}]} d\Gamma^{\varepsilon}_{u_n}(i_n). \tag{41}$$

In particular, we shall denote by $\mathcal{V}^{\varepsilon}_{ts}$ the following Lévy 'hypervolume':

$$\mathcal{V}^{\varepsilon}_{ts} = \mathbf{\Gamma}^{\mathbf{n},\varepsilon}_{ts}(1, \ldots, n).$$

As in the case of Sect. 3.2, an important preliminary step in order to obtain the convergence of $\mathbf{\Gamma}^{\mathbf{n},\varepsilon}$ is the following bound:

Lemma 3.10 *Let Ω be a fixed bounded neighborhood of* 0 *in $\bar{\Pi}^+$, and $p \geq 1$. For every $\rho \in (0, 2\alpha)$, there exists a constant C_ρ such that for every $n \geq 3$ and any n-uple $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$, we have:*

$$\mathbf{E}\left[\left|\mathbf{\Gamma}^{\mathbf{n},\varepsilon}_{ts}(i_1, \ldots, i_n) - \mathbf{\Gamma}^{\mathbf{n},\eta}_{ts}(i_1, \ldots, i_n)\right|^{2p}\right] \leq C_\rho |\varepsilon - \eta|^{p\rho} |t - s|^{p(2n\alpha - \rho)}, \quad s, t \in \Omega. \tag{42}$$

Proof First of all, since we are dealing with random variables in the nth chaos of the Gaussian process Γ, it is enough to prove inequality (42) for $p = 1$. Next, the following lines prove that it is enough to estimate $\mathbf{E}[|\mathcal{V}^{\varepsilon}_{ts} - \mathcal{V}^{\eta}_{ts}|^2]$. Namely, suppose that some of the indices $(i_1, \ldots, i_n)$ coincide, and let $\Sigma_I \subset \Sigma_d$ be the subgroup of permutations $\sigma \in \Sigma_d$ such that $i_{\sigma(j)} = i_j$ for all $j = 1, \ldots, n$. Then

$$\begin{aligned}
&\mathbf{E}\left[\left|\mathbf{\Gamma}^{\mathbf{n},\varepsilon}_{ts}(i_1, \ldots, i_n) - \mathbf{\Gamma}^{\mathbf{n},\eta}_{ts}(i_1, \ldots, i_n)\right|^2\right] \\
&\quad = \sum_{\sigma \in \Sigma_I} \mathbf{E}\Big[\left(\mathbf{\Gamma}^{\mathbf{n},\varepsilon}_{ts}(1, \ldots, n) - \mathbf{\Gamma}^{\mathbf{n},\eta}_{ts}(1, \ldots, n)\right) \\
&\qquad \times \overline{\left(\mathbf{\Gamma}^{\mathbf{n},\varepsilon}_{ts}(\sigma(1), \ldots, \sigma(n)) - \mathbf{\Gamma}^{\mathbf{n},\eta}_{ts}(\sigma(1), \ldots, \sigma(n))\right)}\Big],
\end{aligned} \tag{43}$$

and it is easily seen by the Cauchy–Schwarz inequality that this last term is bounded by $|\Sigma_I|\, \mathbf{E}[|\mathcal{V}^{\varepsilon}_{ts} - \mathcal{V}^{\eta}_{ts}|^2]$. In order to justify Eq. 43, let us just take the example $n = 3$ and $(i_1, i_2, i_3) = (1, 1, 2)$. Then the computation of $\mathbf{E}[|\mathbf{\Gamma}^{\mathbf{3},\varepsilon}_{ts}(1, 1, 2) - \mathbf{\Gamma}^{\mathbf{3},\eta}_{ts}(1, 1, 2)|^2]$ involves products of the form:

$$\mathbf{E}\left[\left(\Gamma'_{z_1}(1)\Gamma'_{z_2}(1)\Gamma'_{z_3}(2)\right)\left(\bar{\Gamma}'_{w_1}(1)\bar{\Gamma}'_{w_2}(1)\bar{\Gamma}'_{w_3}(2)\right)\right],$$

which, invoking Wick's formula, are equal to

$$\begin{aligned}
&\mathbf{E}\left[\Gamma'_{z_1}(1)\bar{\Gamma}'_{w_1}(1)\right]\mathbf{E}\left[\Gamma'_{z_2}(1)\bar{\Gamma}'_{w_2}(1)\right]\mathbf{E}\left[\Gamma'_{z_3}(2)\bar{\Gamma}'_{w_3}(2)\right] \\
&\quad + \mathbf{E}\left[\Gamma'_{z_1}(1)\bar{\Gamma}'_{w_2}(1)\right]\mathbf{E}\left[\Gamma'_{z_2}(1)\bar{\Gamma}'_{w_1}(1)\right]\mathbf{E}\left[\Gamma'_{z_3}(2)\bar{\Gamma}'_{w_3}(2)\right].
\end{aligned} \tag{44}$$

These two terms correspond to $\sigma = \mathrm{Id}$ or $\sigma = \tau_{12}$ in the right-hand side of (43), and one can check that expression (44) is equal to

$$\mathbf{E}\left[\left(\Gamma'_{z_1}(1)\Gamma'_{z_2}(2)\Gamma'_{z_3}(3)\right)\left(\bar{\Gamma}'_{w_1}(1)\bar{\Gamma}'_{w_2}(2)\bar{\Gamma}'_{w_3}(3)+\bar{\Gamma}'_{w_1}(2)\bar{\Gamma}'_{w_2}(1)\bar{\Gamma}'_{w_3}(3)\right)\right].$$

The general case can be treated along the same lines, up to some cumbersome notations.

Hence all we need is to obtain a bound of the form:

$$\mathbf{E}\left[|\mathcal{V}^{\varepsilon}_{ts}-\mathcal{V}^{\eta}_{ts}|^2\right] \le C_\rho|\varepsilon-\eta|^\rho|t-s|^{2n\alpha-\rho}, \quad s,t\in\Omega. \tag{45}$$

As in the Lévy area case of the previous section (see Lemma 3.6), a straightforward application of identity (17) yields

$$\mathbf{E}\left[|\mathcal{V}^{\varepsilon}_{ts}-\mathcal{V}^{\eta}_{ts}|^2\right] = \left(\int_{[s,t]} du_1 \int_{[s,u_1]} du_2 \cdots \int_{[s,u_{n-1}]} du_n\right) \times \left(\int_{[\bar{s},\bar{t}]} d\bar{v}_1 \int_{[\bar{s},\bar{v}_1]} d\bar{v}_2 \cdots \int_{[\bar{s},\bar{u}_{n-1}]} d\bar{v}_n\right) F^{(n)}_{\varepsilon,\eta}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n),$$

where, recalling Notation 3.5, the function $F^{(d)}_{\varepsilon,\eta}$ is defined by

$$F^{(n)}_{\varepsilon,\eta}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n) = \prod_{j=1}^n F_{\varepsilon,\varepsilon}(u_j,\bar{v}_j) + \prod_{j=1}^n F_{\eta,\eta}(u_j,\bar{v}_j) - 2\prod_{j=1}^n F_{\varepsilon,\eta}(u_j,\bar{v}_j).$$

Observe that the latter function can be further decomposed into:

$$F^{(n)}_{\varepsilon,\eta}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n) = \sum_{j=1}^d G^j_{\varepsilon,\eta}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n) + G^j_{\eta,\varepsilon}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n),$$

where the functions G^j are defined by:

$$G^j_{\varepsilon,\eta}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n) = F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)\cdots F_{\varepsilon,\varepsilon}(u_{j-1},\bar{v}_{j-1})[F_{\varepsilon,\varepsilon}(u_j,\bar{v}_j)-F_{\varepsilon,\eta}(u_j,\bar{v}_j)] \\ \times F_{\varepsilon,\eta}(u_{j+1},\bar{v}_{j+1})\cdots F_{\varepsilon,\eta}(u_n,\bar{v}_n)$$

We have thus proved that $\mathbf{E}\left[|\mathcal{V}^{\varepsilon}_{ts}-\mathcal{V}^{\eta}_{ts}|^2\right] = \sum_{j=1}^d I^j_{\varepsilon,\eta} + I^j_{\eta,\varepsilon}$, where

$$I^j_{\varepsilon,\eta} = \left(\int_{[s,t]} du_1 \int_{[s,u_1]} du_2 \cdots \int_{[s,u_{n-1}]} du_n\right) \times \left(\int_{[\bar{s},\bar{t}]} d\bar{v}_1 \int_{[\bar{s},\bar{v}_1]} d\bar{v}_2 \cdots \int_{[\bar{s},\bar{u}_{n-1}]} d\bar{v}_n\right) G^j_{\varepsilon,\eta}(u_1,\bar{v}_1,\ldots,u_n,\bar{v}_n),$$

In order to show relation (45), it is thus sufficient to prove that, for all $j = 1,\ldots,n$, we have $|I^j_{\varepsilon,\eta}| \le C_\rho|\varepsilon-\eta|^\rho|t-s|^{2n\alpha-\rho}$. Observe now that we may cast the term $I^j_{\varepsilon,\eta}$ into the following form:

$$I^j_{\varepsilon,\eta} = \left(\int\limits_{[s,t]} du_1 \int\limits_{[s,u_1]} du_2 \cdots \int\limits_{[s,u_{j-1}]} du_j \right) \left(\int\limits_{[\bar{s},\bar{t}]} d\bar{v}_1 \int\limits_{[\bar{s},\bar{v}_1]} d\bar{v}_2 \cdots \int\limits_{[\bar{s},\bar{u}_{j-1}]} d\bar{v}_j \right)$$
$$\times F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)\cdots F_{\varepsilon,\varepsilon}(u_{j-1},\bar{v}_{j-1}) \left[F_{\varepsilon,\varepsilon}(u_j,\bar{v}_j) - F_{\varepsilon,\eta}(u_j,\bar{v}_j) \right] \phi(u_j,\bar{v}_j;s),$$

where

$$\phi(u_j,\bar{v}_j;s) = \left(\int\limits_{[s,u_j]} du_{j+1} \cdots \int\limits_{[s,u_{n-1}]} du_n \right) \left(\int\limits_{[\bar{s},\bar{v}_j]} d\bar{v}_{j+1} \cdots \int\limits_{[\bar{s},\bar{v}_{n-1}]} d\bar{v}_n \right)$$
$$\times F_{\varepsilon,\eta}(u_{j+1},\bar{v}_{j+1})\cdots F_{\varepsilon,\eta}(u_d,\bar{v}_d).$$

It is thus readily checked that the function $\phi(u_j,\bar{v}_j;s)$ is an analytic iterated integral in the sense of Lemma 2.6, bounded by a constant times $|t-s|^{2\alpha(n-j)}$. Hence it satisfies the hypothesis of Lemma 2.7. As in the proof of the latter result, let $\gamma := [s, s+\imath|t-s|] \cup [s+\imath|t-s|, t+\imath|t-s|] \cup [t+\imath|t-s|, t]$ be a complex deformation of the contour $[s,t]$, and, if $z \in \gamma$, let $\gamma(z)$ be the section of the path γ comprised between s and z. Then

$$I^j_{\varepsilon,\eta} = \left(\int\limits_{\gamma} du_1 \int\limits_{\gamma(u_1)} du_2 \ldots \int\limits_{\gamma(u_{j-1})} du_j \right) \left(\int\limits_{\bar{\gamma}} d\bar{v}_1 \int\limits_{\overline{\gamma(v_1)}} d\bar{v}_2 \ldots \int\limits_{\overline{\gamma(v_{j-1})}} d\bar{v}_j \right)$$
$$\times F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)\cdots F_{\varepsilon,\varepsilon}(u_{j-1},\bar{v}_{j-1}) \left[F_{\varepsilon,\varepsilon}(u_j,\bar{v}_j) - F_{\varepsilon,\eta}(u_j,\bar{v}_j) \right] \phi(u_j,\bar{v}_j;s),$$

and thus

$$|I^j_{\varepsilon,\eta}| \le \left(\int\limits_{\gamma} |du_1| \int\limits_{\gamma} |du_2| \ldots \int\limits_{\gamma} |du_j| \right) \left(\int\limits_{\bar{\gamma}} |d\bar{v}_1| \int\limits_{\bar{\gamma}} |d\bar{v}_2| \ldots \int\limits_{\bar{\gamma}} |d\bar{v}_j| \right)$$
$$|F_{\varepsilon,\varepsilon}(u_1,\bar{v}_1)|\cdots|F_{\varepsilon,\varepsilon}(u_{j-1},\bar{v}_{j-1})||F_{\varepsilon,\varepsilon}(u_j,\bar{v}_j) - F_{\varepsilon,\eta}(u_j,\bar{v}_j)||\phi(u_j,\bar{v}_j;s)|.$$

Now the multiple integral factorizes, and one is left with an expression of the form $A_1^{j-1}A_2$, where

$$A_1 = \int\limits_{\gamma} |du| \int\limits_{\bar{\gamma}} |d\bar{v}| \left|(-\imath(u-\bar{v})+2\varepsilon)^{2\alpha-2}\right| \lesssim |t-s|^{2\alpha} \tag{46}$$

by Lemma 2.5, and

$$A_2 = \int\limits_{\gamma} |du_j| \int\limits_{\bar{\gamma}} |d\bar{v}_j||\phi(u_j,\bar{v}_j;s)| \tag{47}$$

which is bounded as in Lemma 2.7 by a constant times $|t-s|^{2\alpha(n-j)+2\alpha-\rho}|\varepsilon-\eta|^{\rho}$ for any $\rho \in (0,2\alpha)$. The above estimates now yield easily the desired bound $|I^j_{\varepsilon,\eta}| \le C_\rho|\varepsilon-\eta|^{\rho}|t-s|^{2n\alpha-\rho}$. $\square$

The rough-path convergence of the multiplicative functional $(\Gamma^{\varepsilon},\ldots,\mathbf{\Gamma}^{\mathbf{n},\varepsilon})$ to order n is a consequence from the above computations and may be stated as follows:

Theorem 3.11 *Let Γ be an analytic fractional Brownian motion with Hurst parameter $\alpha \in (0, 1/2)$, and Γ^{ε} its regular approximation. Let also $n = \lfloor 1/\alpha \rfloor$, $\mathbf{\Gamma^{k,\varepsilon}}$, $k = 2, \ldots, n$ be the regularized iterated integrals given by formula (41), and consider $0 < \mu < \alpha$. Then*:

(1) *For any $p \geq 1$, the truncated multiplicative functional $(\Gamma^{\varepsilon}, \mathbf{\Gamma^{2,\varepsilon}}, \ldots, \mathbf{\Gamma^{n,\varepsilon}})$ converges in $L^p(\mathcal{U}; \mathcal{C}_1^{\mu}(\Omega; \mathbb{C}^d) \times \cdots \times \mathcal{C}_2^{n\mu}(\Omega; (\mathbb{C}^d)^{\otimes n}))$ to an n-tuple $(\Gamma, \mathbf{\Gamma^2}, \ldots, \mathbf{\Gamma^n})$, where Γ is the analytic fractional Brownian motion defined in Sect. 4.*

(2) *The truncated multiplicative functional $(\Gamma^{\varepsilon}, \mathbf{\Gamma^{2,\varepsilon}}, \ldots, \mathbf{\Gamma^{n,\varepsilon}})$ satisfies the multiplicative and geometric algebraic relations prescribed in Definition 3.1.*

Proof Similar to that of Theorem 3.9. □

4 Solutions of differential equations driven by Γ

We close this article by showing how to solve *pathwise* differential equations driven by our analytic fBm Γ. The power of complex analytic methods will allow us to dispense with the rough path apparatus constructed in the previous sections. Note however that our method of rough path construction also includes a natural way to obtain sharp moment estimates for the multiplicative functional $(\Gamma, \mathbf{\Gamma^2}, \ldots, \mathbf{\Gamma^n})$. This might be of interest in some typical rough paths expansions, where those moments enter into the picture (see e.g. the stochastic Taylor expansions in [2,11,14]).

Our estimates will essentially be an outcome of the following simple lemma:

Lemma 4.1 *Let Ω be a neighbourhood of 0 in $\bar{\Pi}^+$, and $\rho > 1 - \alpha$. Then there exists a positive, almost surely finite random variable $C_\rho = C_\rho(\omega)$ such that*

$$\sup_{z\in\Omega} |\Im z|^{\rho} |\Gamma'_z| = C_\rho(\omega).$$

Proof Let $Z_z := |\Im z|^{\rho} \Gamma'_z$ and $\epsilon = \rho - 1 + \alpha > 0$. Let also $M_y(x, x') := \mathbf{E}|Z_{x+\imath y} - Z_{x'+\imath y}|^2$, for $x, x' \in \mathbb{R}$ and $y > 0$. A simple application of (17) yields

$$M_y(x, x') = C \left(2^{2\alpha-2} y^{2\epsilon} - y^{2-2\alpha+2\epsilon} \Re(2y + \imath(x - x'))^{2\alpha-2}\right),$$

for some constant C. If $|x - x'| \gtrsim y$, we thus get $M_y(x, x') \lesssim y^{2\epsilon} \lesssim |x - x'|^{2\epsilon}$. If on the contrary $|x - x'| \lesssim y$ then (by Taylor expanding to order 2) $M_y(x, x') \lesssim y^{2\epsilon} \left(\frac{x-x'}{y}\right)^2 \lesssim |x - x'|^{2\epsilon}$ too.

Similarly, let, for $y_2 > y_1 > 0$,

$$\begin{aligned} M_x(y_1, y_2) &:= \mathbf{E}|Z_{x+\imath y_1} - Z_{x+\imath y_2}|^2 \\ &= C \left[2^{2\alpha-2}(y_1^{2\epsilon} + y_2^{2\epsilon}) - 2(y_1 y_2)^{1-\alpha+\epsilon}(y_1 + y_2)^{2\alpha-2}\right] \end{aligned}$$

for some constant $C > 0$. Let $\delta := y_2 - y_1 > 0$. If $\delta \ll y_1$ then a Taylor expansion to order 1 yields $M_x(y_1, y_2) = y_1^{2\epsilon} O(\delta/y_1) \lesssim \delta^{2\epsilon}$. If on the other hand δ is comparable or much larger than y_1, then simply $M_x(y_1, y_2) \lesssim y_2^{2\epsilon} \lesssim \delta^{2\epsilon}$.

Hence, putting together our estimates on $M_y(x, x')$ and $M_x(y_1, y_2)$, and letting $y_2 > y_1 > 0$, we obtain

$$\begin{aligned} \mathbf{E}|Z_{x'+\imath y_2} - Z_{x+\imath y_1}|^2 &\leq 2 \left(\mathbf{E}|Z_{x'+\imath y_2} - Z_{x+\imath y_2}|^2 + \mathbf{E}|Z_{x+\imath y_2} - Z_{x+\imath y_1}|^2\right) \\ &\lesssim |x' - x|^{2\epsilon} + |y_1 - y_2|^{2\epsilon}. \end{aligned}$$

This, together with the classical Kolmogorov-Centsov lemma, implies the a.s. continuity of $(Z_z, z \in \bar{\Pi}^+)$, and in particular its boundedness on every compact. □

A new space of functions, which shall be called $\mathcal{H}_{\rho,r}$, emerges naturally from the previous lemma:

Definition 4.2 Fix $\rho \in (0, 1)$ and $r > 0$. Let, for f analytic on a complex neighbourhood of $\mathcal{B}(0, r) \cap \Pi^+$,

$$\|f\|_{\rho,r} := |f_0| + \sup_{z \in \mathcal{B}(0,r) \cap \Pi^+} |\Im z|^{\rho} |f'_z|.$$

Let also $\mathcal{H}_{\rho,r}$ be the space of functions f, analytic on a complex neighbourhood of $\mathcal{B}(0, r) \cap \Pi^+$, such that $\|f\|_{\rho,r} < \infty$.

By the classical Weierstrass theorem [1], $\mathcal{H}_{\rho,r}$ is a Banach space. Furthermore, it is included in a space of Hölder functions:

Lemma 4.3 *Let $f \in \mathcal{H}_{\rho,r}$. Then f is in the Hölder space $\mathcal{C}^{1-\rho}$ and*

$$\sup_{z \in \Pi^+, |z| < r} |f_z| \leq |f_0| + Cr^{1-\rho} \|f - f_0\|_{\rho,r}$$

for some universal constant C.

Proof Let $z_1, z_2 \in \mathcal{B}(0, r) \cap \Pi^+$. We shall prove that $|f_{z_1} - f_{z_2}| \leq C\|f\|_{\rho,r}|z_1 - z_2|^{1-\rho}$, which shows both assertions simultaneously. Let $\delta = |z_1 - z_2|$, and let γ be the path $[z_1, z_1 + \imath\delta|] \cup [z_1 + \imath\delta, z_2 + \imath\delta] \cup [z_2 + \imath\delta, z_2]$, so that $f_{z_1} - f_{z_2} = \int_\gamma f'_\zeta d\zeta$ (at least if the path is included in $\mathcal{B}(0, r)$, otherwise one may replace $z_1 + \imath\delta$, resp. $z_2 + \imath\delta$ by the last point in the segment $[z_1, z_1 + \imath\delta]$, resp. $[z_2, z_2 + \imath\delta]$ contained in $\mathcal{B}(0, r)$; unessential details are left to the reader). One has

$$\begin{aligned} \left| \int_{z_1}^{z_1+\imath\delta} f'_\zeta d\zeta \right| &\leq \|f\|_{\rho,r} \int_{z_1}^{z_1+\imath\delta} |\Im z|^{-\rho} d|z| \\ &= \|f\|_{\rho,r} \int_{\Im z_1}^{\Im z_1+\delta} y^{-\rho} dy \\ &\leq C\|f\|_{\rho,r} \delta^{1-\rho} \end{aligned}$$

for some constant $C = C(\rho)$, as the reader may check (whether $\frac{\Im z_1}{\delta}$ small or large). The same estimate holds for the integral along $[z_2 + \imath\delta, z_2]$, while

$$\left| \int_{z_1+\imath\delta}^{z_2+\imath\delta} f'_\zeta d\zeta \right| \leq \|f\|_{\rho,r} \delta^{-\rho} \int_{z_1+\imath\delta}^{z_2+\imath\delta} |d\zeta| = \|f\|_{\rho,r} \delta^{1-\rho}.$$

□

The previous observations imply the stability of $\mathcal{H}_{\rho,r}$ by integration:

Proposition 4.4 *Let f, g be two elements of $\mathcal{H}_{\rho,r}$. For $z \in \mathcal{B}(0,r) \cap \Pi^+$, set*

$$\mathcal{J}_{z0}(dgf) = \lim_{\varepsilon \to 0} \int_{\imath\varepsilon}^{z+\imath\varepsilon} g'_\zeta f_\zeta,$$

where the right hand side is understood in the Riemann sense. Then $z \mapsto \mathcal{J}_{z0}(dgf)$ is well defined as an element of $\mathcal{H}_{\rho,r}$.

Proof The existence of the limit is obviously proved as in Lemma 4.3. In order to show that $z \mapsto \mathcal{J}_{z0}(dgf)$ lies in $\mathcal{H}_{\rho,r}$, it is enough to observe that, if $z \in \mathcal{B}(0,r) \cap \Pi^+$,

$$|\Im z|^\rho |f_z||g'_z| \le \sup_{\zeta \in \mathcal{B}(0,r)\cap\Pi^+} |f_\zeta| \cdot \|g - g_0\|_{\rho,r}.$$

□

Since we have seen that the analytic fBm is an element of $\mathcal{H}_{\rho,r}$, the proposition above is a clear indication that one should be able to solve differential equations driven by Γ in the space $\mathcal{H}_{\rho,r}$. This is indeed the case:

Theorem 4.5 *Let $\Gamma : \bar{\Pi}^+ \to \mathbb{C}$ be a function in the space $\mathcal{H}_{\rho,r}$ for some r (for instance, Γ may be taken to be almost any path of afbm, and r may be chosen arbitrarily large). Let $du_z = V(u_z)d\Gamma_z$ be a differential equation driven by Γ, with initial condition $u_0 = a$; the function V is supposed to be analytic on a complex neighbourhood of a. Then the differential equation has a unique solution in $\mathcal{H}_{\rho,r'}$ if $r' \le r$ is small enough.*

Proof Let $\mathcal{H}^a_{\rho,r} = \{f \in \mathcal{H}_{\rho,r};\ f_0 = a\}$, and

$$\Theta : \mathcal{H}^a_{\rho,r} \to \mathcal{H}^a_{\rho,r}, \qquad \Theta : u \to [\Theta(u)]_z := a + \int_0^z V(u_\zeta)d\Gamma_\zeta.$$

Then, letting $r' \le r$ be small enough (depending on $\|u\|_{\rho,r}$)

$$\begin{aligned}\|\Theta(u)\|_{\rho,r'} &= \sup_{z\in\Pi^+, |z|\le r'} |\Im z|^\rho |V(u_z)\Gamma'_z| \\ &\le \|\Gamma\|_{\rho,r'} \left(|V(a)| + C|V'(a)|\ \|u - a\|_{\rho,r'} (r')^{1-\rho}\right) \end{aligned} \tag{48}$$

for some universal constant C. This allows to prove that Θ leaves invariant a ball of the form $\mathcal{B}_M := \{u \in \mathcal{H}^a_{\rho,r'};\ \|u - a\|_{\rho,r'} \le M\}$ for r' small enough and M depending on V and Γ. Indeed, set $M = 2\|\Gamma\|_{\rho,1}|V(a)|$, and then

(i) choose $r_0 = r_0(M)$ small enough so that $\Theta(u)$ is defined on $\mathcal{B}_M$, seen as a subset of $\mathcal{H}^a_{\rho,r_0}$;
(ii) then inequality (48) easily yields the invariance of $\mathcal{B}_M \subset \mathcal{H}^a_{\rho,r'}$ by Θ whenever $(r')^{1-\rho} \lesssim \left(V'(a)\|\Gamma\|_{\rho,1}\right)^{-1}$.

Once that ball $\mathcal{B}_M \subset \mathcal{H}^a_{\rho,r'}$ is shown to be invariant by Θ, one can also prove that Θ is a contraction of $\mathcal{H}^a_{\rho,\hat{r}}$ for some $\hat{r} < r'$. This stems from the fact that, whenever $\hat{r} < r'$,

$$\|\Theta(u) - \Theta(\hat{u})\|_{\rho,\hat{r}} \le C_{r'}|V'(a)|.\|u - \hat{u}\|_{\rho,\hat{r}}(\hat{r})^{1-\rho},$$

which can be proven as for Eq. 48. Choose then $\hat{r}$ small enough so that $C_{r'}|V'(a)|(\hat{r})^{1-\rho} \le \frac{1}{2}$, and Θ becomes a contraction of $\mathcal{H}^a_{\rho,\hat{r}}$. The proof now follows from the usual fixed point theorem on Banach spaces. □

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