On A Priori Estimates for Rough PDEs

Qi Feng and Samy Tindel

Dedicated to Rodrigo Bañuelos on occasion of his 60th birthday

Abstract In this note, we present a new and simple method which allows to get a priori bounds on rough partial differential equations. The technique is based on a weak formulation of the equation and a rough version of Gronwall's lemma. The method is presented on a simple linear example, but might be generalized to a wide number of situations.

Keywords A priori estimate • Rough Gonwall lemma • Rough paths • Stochastic PDEs

1 Introduction

This paper proposes to review a recent method allowing to get a priori estimates for rough partial differential equations, taken from [6]. Our aim here is to show how to implement the technique on a simple example. Namely, we shall consider the following noisy heat equation on an interval $[0, \tau] \times \mathbb{R}^d$ for $\tau > 0$ and a spatial dimension $d \geq 1$:

$$\partial_t u_t(x) = \frac{\Delta}{2} u_t(x) + \sum_{i=1}^{\infty} \beta_i u_t(x) e_i(x) dw_t^i, \tag{1.1}$$

where Δ stands for the Laplace operator, $\{e_i; \geq 1\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $\{\beta_i; \geq 1\}$ is a family of coefficients satisfying some summability conditions (see Hypothesis 2.4 below). In Eq. (1.1), $\{w_i; \geq 1\}$ is also a family of noises,

Q. Feng • S. Tindel (✉)
Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, USA
e-mail: stindel@purdue.edu

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interpreted as p-variation paths with $p < 3$, which can be lifted to a rough path $\mathbf{w}$ (see Hypothesis 2.3 for a more complete definition).

The recent activity on existence and uniqueness results for rough PDEs has been thriving. A lot of this activity concerns situations which require renormalization techniques and a way to handle pathwise products of distributions [10, 12, 13]. Here we are concerned with a different context, for which the noise is smooth enough in space, so that the solution of (1.1) is directly expected to be a function and the integrals with respect to w are usual rough paths integrals. This situation does not require the whole regularity structure machinery, and one advantage of this reduced setting is that more information on the solution is available. We are concerned in this paper about a priori estimates, which can be either seen as a crucial step in the proof of existence of solutions, or as a first piece of valuable information about the solution. Furthermore, we believe that a priori estimates exhibit the core of the pathwise methods for stochastic PDEs, even though many more technical steps have to be performed in order to get existence and uniqueness results.

Let us summarize some of the (unrelated) approaches leading to estimates of equations like (1.1).

1. The references [2, 11] handle stochastic PDEs by considering random flows (induced by a finite dimensional rough path) which change the stochastic PDE into a deterministic PDE with random coefficients. A priori bounds are then potentially obtained by composing bounds on deterministic PDEs and estimates on rough flows. This possibility has not been fully exploited yet, and might lead to nontrivial considerations.
2. In [5, 9], a variant of the rough paths theory is introduced in order to cope with PDEs of the form (1.1), considered in the mild sense. This involves some lengthy and intricate considerations on twisted increments of the form $\hat{\delta} f_{ts} = f_t - S_{t-s} f_s$, where S designates the heat semi-group and f is a generic $L^2(\mathbb{R}^d)$-valued function. However, this formalism yields a priori estimates for (1.1), especially when one considers related numerical schemes as in [4].
3. For linear equations like (1.1), Feynman-Kac representations for the solution are available. This gives raise to explicit moment computations for $u_t(x)$, for a fixed couple $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Many cases of Gaussian noises have been examined in this context, and we refer to [3] for a situation which is close to ours, namely a rough noise in time which is smooth in space.

Let us highlight again the fact that we only recall here results concerning smooth noises in space. In cases like [10, 12, 13] where renormalization is needed, the mere existence of moments for the renormalized solution is still an open problem (to the best of our knowledge).

With these preliminary considerations in mind, the main point of the current paper is to show that the variational approach to rough PDEs, introduced in [1, 6], provides a handy way to obtain $L^2(\mathbb{R}^d)$ (and more generally $L^\alpha(\mathbb{R}^d)$) estimates on the solution. The main advantages of this new setting are the following:

1. The variational formulation is convenient at an algebraic and analytic level, when compared with the other methods mentioned above.

2. Unlike Feynman-Kac representations, the variational approach is not restricted to linear equations (though generalizations require a nontrivial extra work).

We shall illustrate this point of view with the simple model (1.1), for which we shall deduce L^α-estimates in a detailed way. It should be noticed that variational methods have been considered previously in [15] for pathwise PDEs driven by a fractional Brownian motion. With respect to this reference, our computations are restricted to linear cases. However, [15] only considers fBm's with a Hurst parameter $H > \frac{1}{2}$, while we are concerned with a true rough case (corresponding to $\frac{1}{3} < H \leq \frac{1}{2}$ for fBm).

Our article is structured as follows: in Sect. 2 we introduce some notations and the variational method framework, and we also present our first a priori estimate in Proposition 2.8. This estimate (adapted from [6, Theorem 2.5]) is valid for general linear equations, and will be suitable for our stochastic heat equation with multidimensional noise. Then in Sect. 3 we prove our main a priori bounds, namely Theorems 3.5 and 3.9 for the solution of Eq. (1.1), both in $L^2(\mathbb{R}^d)$ and $L^\alpha(\mathbb{R}^d)$ norms. Finally, Sect. 4 is devoted to the application of our abstract results to equations driven by fractional Brownian motion. A first example concerns a bounded domain, which enables us to compare our result with those of [15], while a second example deals with the whole space $\mathbb{R}^d$.

2 Rough Variational Framework

As mentioned above, our framework relies on a variational formulation of the heat equation, which is algebraically quite convenient. In this section we first recall some basic vocabulary about algebraic integration, then we give the main general results needed for the rough heat equation (1.1).

2.1 Notions of Algebraic Integration

First of all, let us recall the definition of the increment operator, denoted by δ. If g is a path defined on $[0, T]$ and $s, t \in [0, T]$ then we set $\delta g_{st} := g_t - g_s$. Whenever g is a 2-index map defined on $[0, T]^2$, we define $\delta g_{sut} := g_{st} - g_{su} - g_{ut}$. The norm of the element g in the Banach space E will be written as $\mathcal{N}[g; E]$. For two quantities a and b the relation $a \lesssim_x b$ means $a \leq c_x b$, for a constant c_x depending on a multidimensional parameter x.

In the sequel, given an interval I we call *control on I* (and denote it by ω) any continuous superadditive map on $\Delta_I := \{(s, t) \in I^2 : s \leq t\}$, that is, any continuous map $\omega : \Delta_I \to [0,\infty)$ such that, for all $s \le u \le t$,

$$\omega(s,u) + \omega(u,t) \le \omega(s,t).$$

Given a control ω on an interval $I = [a,b]$, we will use the notation $\omega(I) := \omega(a,b)$. For a fixed time interval I, a parameter $p > 0$, a Banach space E and any continuous function $g : I \to E$ we define the norm

$$\mathcal{N}[g; V_1^p(I;E)] := \sup_{(t_i)\in\mathcal{P}(I)} \left(\sum_i |\delta g_{t_i t_{i+1}}|^p \right)^{\frac{1}{p}},$$

where $\mathcal{P}(I)$ denotes the set of all partitions of the interval I. In this case,

$$\omega_g(s,t) = \mathcal{N}[g; V_1^p([s,t];E)]^p$$

defines a control on I. We denote by $V_2^p(I;E)$ the set of continuous two-index maps $g : I \times I \to E$ for which there exists a control ω such that

$$|g_{st}| \le \omega(s,t)^{\frac{1}{p}}$$

for all $s,t \in I$. We also define the space $V_{2,\mathrm{loc}}^p(I;E)$ of maps $g : I \times I \to E$ such that there exists a countable covering $\{I_k\}_k$ of I satisfying $g \in V_2^p(I_k;E)$ for any k.

The following result is often referred to as *sewing lemma* in the literature, and is at the core of our approach to generalized integration.

Lemma 2.1 *Fix an interval I, a Banach space E and a parameter $\zeta > 1$. Consider a function $h : I^3 \to E$ such that $h \in \mathrm{Im}\,\delta$ and for every $s < u < t \in I$,*

$$|h_{sut}| \le \omega(s,t)^{\zeta}, \tag{2.1}$$

for some control ω on I. Then there exists a unique element $\Lambda h \in V_2^{\frac{1}{\zeta}}(I;E)$ such that $\delta(\Lambda h) = h$ and for every $s < t \in I$,

$$|(\Lambda h)_{st}| \le C_\zeta\, \omega(s,t)^{\zeta}, \tag{2.2}$$

for some universal constant C_ζ.

Our computations also hinge on the following rough version of Gronwall's lemma, borrowed from [6, Lemma 2.7].

Lemma 2.2 *Fix a time horizon $T > 0$ and let $Q : [0,T] \to [0,\infty)$ be a path such that for some constants $C, L > 0$, $\kappa \ge 1$ and some controls ω_1, ω_2 on $[0,T]$, one has*

$$\delta Q_{st} \le C\Big(\sup_{0\le r\le t} Q_r\Big)\,\omega_1(s,t)^{\frac{1}{\kappa}} + \omega_2(s,t), \tag{2.3}$$

for every $s < t \in [0, T]$ *satisfying* $\omega_1(s,t) \le L$. *Then it holds*

$$\sup_{0\le t\le T} Q_t \le 2\exp\left(c_{\kappa,L}\,\omega_1(0,T)\right)\cdot\Big\{Q_0 + \sup_{0\le t\le T}\Big(\omega_2(0,t)\,\exp\left(-c_{\kappa,L}\,\omega_1(0,t)\right)\Big)\Big\},$$

for a strictly positive constant $c_{\kappa,L}$.

2.2 Linear Equations with Distributional Drifts

In this section we shall first generalize Eq. (1.1), and consider the following:

$$dg_t = \mu(dt) + \sum_{i=1}^{\infty} \beta_i g_t e_i dw_t^i, \tag{2.4}$$

where μ is a distributional-valued measure. Before we give a rigorous meaning to this equation, let us label our hypothesis on the coefficients. We start by a rough path assumption for each couple of components of the driving noise w:

Hypothesis 2.3 *Let* $p \in [2,3)$ *be given. We assume that the family* $\{w^i;\ i \ge 1\}$ *is such that there exist increments* $\mathbf{w}^{\mathbf{1},i}, \mathbf{w}^{\mathbf{2},ij}$ *satisfying the two following properties:*

(i) Algebraic condition: *For each* $i,j \ge 1$ *and* $0 \le s \le u \le t \le \tau$, *Chen's relation holds true:*

$$\delta\mathbf{w}^{\mathbf{1},i}_{st} = 0, \quad \textit{and} \quad \delta\mathbf{w}^{\mathbf{2},ij}_{sut} = \mathbf{w}^{\mathbf{1},i}_{su}\,\mathbf{w}^{\mathbf{1},j}_{ut}. \tag{2.5}$$

(ii) Analytic condition: *For all* $i,j \ge 1$, *we have*

$$\mathcal{N}[\mathbf{w}^{\mathbf{1},i}; V_2^p([s,t])] < \infty, \quad \textit{and} \quad \mathcal{N}[\mathbf{w}^{\mathbf{2},ij}; V_2^{p/2}([s,t])] < \infty.$$

The rough variational setting introduced in [1, 6] uses the concept of scale. A scale is defined as a sequence $\left(E_n, \|\cdot\|_n\right)_{n\in\mathbb{N}_0}$ of Banach spaces such that E_{n+1} is continuously embedded into E_n. Besides, for $n \in \mathbb{N}_0$ we denote by E_{-n} the topological dual of E_n. For the heat equation (1.1), we will consider the scale $E_n = W^{n,\infty}$.

Having the concept of scale in mind, the noise w should also fulfill the following hypothesis as an infinite dimensional object:

Hypothesis 2.4 *Recall that the scale* E_n *is given by* $E_n = W^{n,\infty}$. *We assume that* $\{\beta_i;\ \ge 1\}$ *is a family of positive coefficients satisfying* $\sum_{i\ge 1}\beta_i < \infty$. *Consider an orthonormal basis* $\{e_i;\ \ge 1\}$ *of* $L^2(\mathbb{R}^d)$, *composed of bounded functions. The noise* w *is such that* $\{w_i;\ \ge 1\}$ *is a family of p-variation paths with* $p < 3$, *whose first and* *second order increments* $\mathbf{w}^{\mathbf{1},i}$, $\mathbf{w}^{\mathbf{2},ij}$ *are such that* $\omega_{\mathbf{w}^1}$ *and* $\omega_{\mathbf{w}^2}$ *below are two controls on* $[0,\tau]$*:*

$$\omega_{\mathbf{w}^1}(s,t) \equiv \left(\sum_{i=1}^{\infty} \beta_i \left(1 + |e_i|_{E_1}\right) \mathcal{N}[\mathbf{w}^{\mathbf{1},i}; V_2^p([s,t])] \right)^p \tag{2.6}$$

$$\omega_{\mathbf{w}^2}(s,t) \equiv \left(\sum_{i,j=1}^{\infty} \beta_i \beta_j |e_i|_{E_1} |e_j|_{E_1} \mathcal{N}[\mathbf{w}^{\mathbf{2},ij}; V_2^{p/2}([s,t])] \right)^{p/2}. \tag{2.7}$$

We can now give a more formal definition of solution to our Eq. (2.4), in terms of expansions of the increments up to a regularity order greater than 1:

Definition 2.5 Let $p \in [2,3)$ and fix an interval $I \subseteq [0,\tau]$. Let μ be a distributional-valued measure lying in $V_1^1(I; E_{-1})$. A path $g : I \to E_{-0}$ is called solution (on I) of Eq. (2.4) provided there exists $q < 3$ and $g^{\natural} \in V_{2,\mathrm{loc}}^{\frac{q}{3}}(I, E_{-1})$ such that we have:

$$\delta g_{st}(\varphi) = \sum_{i=1}^{\infty} \beta_i g_s(e_i \varphi) \mathbf{w}_{st}^{\mathbf{1},i} + \delta\mu_{st}(\varphi) + \sum_{i,j=1}^{\infty} \beta_i \beta_j g_s(e_i e_j \varphi) \mathbf{w}_{st}^{\mathbf{2},ij} + g_{st}^{\natural}(\varphi), \tag{2.8}$$

for every $s,t \in I$ satisfying $s < t$ and every $\varphi \in E_1$.

Remark 2.6 On top of (2.5), we will use the following expressions for δg_{st}:

$$\delta g_{st}(\varphi) = \sum_{i=1}^{\infty} \beta_i g_s(e_i \varphi) \mathbf{w}_{st}^{\mathbf{1},i} + g^{\sharp}(\varphi), \tag{2.9}$$

where $g^{\sharp}$ is a $V_2^{\frac{p}{2}}(E_{-1})$ increment satisfying:

$$g_{st}^{\sharp}(\varphi) = \delta g_{st}(\varphi) - \sum_{i=1}^{\infty} \beta_i g_s(e_i \varphi) \mathbf{w}_{st}^{\mathbf{1},i} = \delta\mu_{st}(\varphi) + \sum_{i,j=1}^{\infty} \beta_i \beta_j g_s(e_i e_j \varphi) \mathbf{w}_{st}^{\mathbf{2},ij} + g_{st}^{\natural}(\varphi). \tag{2.10}$$

Remark 2.7 Equation (2.8) is expressed as an expansion along the increments of w^i. However, according to [7, Theorem 4.10], a solution u of (2.8) also solves the following integral equation (which has to be interpreted in the rough paths sense in time and weak sense in space):

$$\delta g_{st} = \mu([s,t)) + \sum_{i=1}^{\infty} \beta_i e_i \int_s^t g_r dw_r^i. \tag{2.11}$$

Furthermore, a change of variable formula (see [7, Proposition 5.6]) holds for g verifying (2.11). Namely, for $h \in \mathcal{C}^3(\mathbb{R})$ we have (still in the weak rough paths sense):

$$\delta h(g)_{st} = \int_s^t h'(g_r)\,\mu(dr) + \sum_{i=1}^{\infty} \beta_i e_i \int_s^t h'(g_r)\, g_r\, dw_r^i. \tag{2.12}$$

2.3 A General Estimate for Linear Equations

The following proposition gives our first a priori estimate for the solution to Eq. (2.4). It should be seen as an adaptation of [6, Theorem 2.5] to our current context.

Proposition 2.8 *Let $p \in [2,3)$ and fix an interval $I \subseteq [0,T]$. Let $\mathbf{w}$ be a rough path verifying Hypothesis 2.3 and 2.4. Consider a path $\mu \in V_1^1(I; E_{-1})$ such that for every $\varphi \in E_1$, there exists a control ω_μ verifying*

$$|\delta\mu_{st}(\varphi)| \le \omega_\mu(s,t)\,\|\varphi\|_{E_1}. \tag{2.13}$$

Let g be a solution on I of Eq. (2.4), with the following additional hypothesis: g is controlled over the whole interval I, that is we have $g^\natural \in V_2^{\frac{q}{3}}(I; E_{-1})$ for $q < 3$. Moreover let $S_t^g = \sup_{s\le t}\|g_s\|_{E_{-0}}$, and consider the following control:

$$\omega_I(s,t) \equiv \omega_\mu(s,t)\left(\omega_{\mathbf{w}^1}^{1/p}(s,t) + \omega_{\mathbf{w}^2}^{2/p}(s,t)\right) + S_t^g\left(2\omega_{\mathbf{w}^1}^{1/p}(s,t)\omega_{\mathbf{w}^2}^{2/p}(s,t) + \omega_{\mathbf{w}^2}^{4/p}(s,t)\right). \tag{2.14}$$

Then there exists a constant $L = L_p > 0$ (independent of I) such that if

$$\omega_{\mathbf{w}^1}(s,t) + \omega_{\mathbf{w}^2}^2(s,t) \le L,$$

then for all $s,t \in I$ such that $s < t$, we have:

$$\|g_{st}^\natural\|_{E_{-1}} \lesssim_p \omega_I(s,t). \tag{2.15}$$

Proof Let $\omega_\natural(s,t)$ be a regular control such that $\|g_{st}^\natural\|_{E_{-1}} \le \omega_\natural(s,t)^{\frac{3}{q}}$ for any $s,t \in I$ such that $s < t$. We divide this proof in several steps.

Step 1: An Algebraic Identity Let $\varphi \in E_1$ be such that $\|\varphi\|_{E_3} \le 1$. We first show that

$$\delta g_{sut}^\natural(\varphi) = \sum_{i=1}^{\infty} \beta_i g_{su}^\sharp(e_i\varphi)\mathbf{w}_{ut}^{\mathbf{1},i} + \sum_{i,j=1}^{\infty} \beta_i\beta_j \delta g_{su}(e_ie_j\varphi)\mathbf{w}_{ut}^{\mathbf{2},ij} \equiv K_{sut}^1 + K_{sut}^2, \tag{2.16}$$

where $g^\sharp$ was defined in (2.10). Indeed, owing to (2.8), we have

$$g^\natural_{st} = \delta g_{st}(\varphi) - \sum_{i=1}^{\infty} \beta_i g_s(e_i\varphi)\mathbf{w}^{\mathbf{1},i}_{st} - \delta\mu_{st}(\varphi) - \sum_{i,j=1}^{\infty} \beta_i\beta_j g_s(e_ie_j\varphi)\mathbf{w}^{\mathbf{2},ij}_{st}.$$

Applying δ on both sides of this identity and recalling Chen's relations (2.5) as well as the fact that $\delta\delta = 0$ we thus get

$$\delta g^\natural_{sut}(\varphi) = \sum_{i=1}^{\infty} \beta_i \delta g_{su}(e_i\varphi)\mathbf{w}^{\mathbf{1},i}_{ut} + \sum_{i,j=1}^{\infty} \beta_i\beta_j \delta g_{su}(e_ie_j\varphi)\mathbf{w}^{\mathbf{2},ij}_{ut} - \sum_{i,j=1}^{\infty} \beta_i\beta_j g_s(e_ie_j\varphi)\mathbf{w}^{\mathbf{1},i}_{su}\mathbf{w}^{\mathbf{1},j}_{ut}.$$

Plugging relation (2.10) again into this identity, we end up with our claim (2.16).

Step 2: Bound for K^1 In order to bound the term $g^\sharp_{su}(e_i\varphi)$ in K^1, we invoke decomposition (2.10), which yields:

$$g^\sharp_{su}(e_i\varphi) = \delta\mu_{su}(e_i\varphi) + \sum_{j,k=1}^{\infty} \beta_j\beta_k g_s(e_ie_je_k\varphi)\mathbf{w}^{\mathbf{2},kl}_{su} + g^\natural_{su}(e_i\varphi),$$

and hence:

$$\begin{aligned}&|g^\sharp_{su}(e_i\varphi)| \\ &\le \left[\omega_\mu(s,t)|e_i|_{E_1} + S^g_u \sum_{j,k=1}^{\infty} \beta_j\beta_k |e_i|_{E_0}|e_j|_{E_0}|e_k|_{E_0}\omega^{2/p}_{\mathbf{w},jk}(s,u) + \omega^{3/p}_\natural(s,u)|e_i|_{E_1}\right]|\varphi|_{E_1}.\end{aligned}$$

Therefore, thanks to our assumption (2.7), we have:

$$|g^\sharp_{su}(e_i\varphi)| \le \left[\omega_\mu(s,u) + S^g_u\,\omega^{2/p}_{\mathbf{w^2}}(s,u) + \omega^{3/p}_\natural(s,u)\right]|e_i|_{E_1}|\varphi|_{E_1}. \tag{2.17}$$

Plugging this identity into the definition of K^1, we have thus obtained:

$$\begin{aligned}|K^1_{sut}| &\le |\varphi|_{E_1}\left[\omega_\mu(s,u) + S^g_u\,\omega^{2/p}_{\mathbf{w^2}}(s,u) + \omega^{3/p}_\natural(s,u)\right]\sum_{i=1}^{\infty} \beta_i|e_i|_{E_1}\omega^{1/p}_{\mathbf{w^{1,i}}}(u,t) \\ &\le |\varphi|_{E_1}\left[\omega_\mu(s,u) + S^g_u\,\omega^{2/p}_{\mathbf{w^2}}(s,u) + \omega^{3/p}_\natural(s,u)\right]\omega^{1/p}_{\mathbf{w^1}}(u,t). \end{aligned} \tag{2.18}$$

Step 3: Bound for K^2 and $\delta g^\natural$ The main term to treat for K^2 is the increment δg_{su}. To this aim, we resort to decomposition (2.9). This yields:

$$K^2_{sut} = \sum_{i,j,k=1}^{\infty} \beta_i \beta_j \beta_k \, g_s(e_i e_j e_k \varphi) \, \mathbf{w}^{\mathbf{1},k}_{su} \, \mathbf{w}^{\mathbf{2},ij}_{ut} + \sum_{i,j=1}^{\infty} \beta_i \beta_j \, g^\sharp(e_i e_j \varphi) \, \mathbf{w}^{\mathbf{2},ij}_{ut} \equiv K^{21}_{sut} + K^{22}_{sut}.$$

Furthermore, we have:

$$\begin{aligned} |K^{21}_{sut}| &\le S^g_t |\varphi|_{E_0} \left(\sum_{k=1}^{\infty} \beta_k |e_k|_{E_0} \omega^{1/p}_{\mathbf{w}^{\mathbf{1},k}}(s,u) \right) \left(\sum_{i,j=1}^{\infty} \beta_i \beta_j |e_i|_{E_0} |e_j|_{E_0} \omega^{2/p}_{\mathbf{w}^{\mathbf{2},ij}}(u,t) \right) \\ &\le S^g_t |\varphi|_{E_0} \omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(s,u) \, \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(u,t). \end{aligned}$$

In order to handle K^{22}, we elaborate slightly on our estimate (2.17) in order to get:

$$\begin{aligned} |K^{22}_{sut}| &\le |\varphi|_{E_1} \left[\omega_\mu(s,u) + S^g_u \, \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,u) + \omega^{3/p}_\natural(s,u) \right] \left(\sum_{i,j=1}^{\infty} \beta_i \beta_j |e_i|_{E_1} |e_j|_{E_1} \omega^{2/p}_{\mathbf{w}^{\mathbf{2},ij}}(u,t) \right) \\ &\le |\varphi|_{E_1} \left[\omega_\mu(s,u) + S^g_u \, \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,u) + \omega^{3/p}_\natural(s,u) \right] \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(u,t). \end{aligned}$$

Hence, gathering our estimates on K^{21} and K^{22} we end up with:

$$|K^2_{sut}| \le |\varphi|_{E_1} \left[S^g_t \left(\omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(s,u) + \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,u) \right) + \omega_\mu(s,u) + \omega^{3/p}_\natural(s,u) \right] \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(u,t). \tag{2.19}$$

We can now easily conclude for the increment $\delta g^\natural$: plugging (2.18) and (2.19) into (2.16), we get:

$$\begin{aligned} \left| \delta g^\natural_{sut}(\varphi) \right| \le |\varphi|_{E_1} \Big\{ &\left(\omega_\mu(s,u) + S^g_u \, \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,u) \right) \, \omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(u,t) \\ &+ \left(\omega_\mu(s,u) + S^g_t \left(\omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(s,u) + \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,u) \right) \right) \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(u,t) \\ &+ \omega^{3/p}_\natural(s,u) \left(\omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(u,t) + \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(u,t) \right) \Big\}. \end{aligned}$$

Otherwise stated, with our definition (2.14) in mind, we have obtained:

$$\left| \delta g^\natural_{sut}(\varphi) \right| \le |\varphi|_{E_1} \left\{ \omega_I(s,t) + \omega^{3/p}_\natural(s,t) \left(\omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(s,t) + \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,t) \right) \right\}. \tag{2.20}$$

Step 4: Conclusion It is readily checked, thanks to the fact that ω_μ, $\omega_{\mathbf{w}^{\mathbf{1}}}$, $\omega_{\mathbf{w}^{\mathbf{2}}}$ and $\omega_\natural$ are controls, plus [8, Exercise 1.9], that ω_I is a control as well as $\omega^{3/p}_\natural(s,t)(\omega^{1/p}_{\mathbf{w}^{\mathbf{1}}}(s,t) + \omega^{2/p}_{\mathbf{w}^{\mathbf{2}}}(s,t))$. One can thus apply Lemma 2.1 to relation (2.20) and get:

$$\left|g^{\natural}_{st}(\varphi)\right| \le c_p\,|\varphi|_{E_1}\left\{\omega_I(s,t)+\omega_{\natural}^{3/p}(s,t)\left(\omega_{\mathbf{w}^1}^{1/p}(s,t)+\omega_{\mathbf{w}^2}^{2/p}(s,t)\right)\right\}.$$

We now take I such that $c_p\,(\omega_{\mathbf{w}^1}^{1/p}(s,t)+\omega_{\mathbf{w}^2}^{2/p}(s,t)) \le \frac{1}{2}$. We obtain:

$$\|g^{\natural}_{st}\|_{E_{-1}} \le 2c_p\omega_I(s,t),$$

which ends our proof. □

Remark 2.9 In order to apply Proposition 2.8 to the heat equation (1.1), we shall consider a measure μ defined by $\mu([0,t]) = \int_0^t \Delta u_s\,ds$. It is worth noting that for a noisy equation like (1.1), we cannot assume that Δu_s is properly defined. This is why we consider $\mu([0,t])$ as an element of E_{-1} and perform our computations with distributional increments.

3 L^2 and L^α Type Estimates

Let us now go back to Eq. (1.1), for which we will derive some a priori estimates in $L^2(\mathbb{R}^d)$ and $L^\alpha(\mathbb{R}^d)$. We start by giving some basic properties of our linear heat equation.

3.1 Preliminary Considerations

Let us begin by giving a precise meaning to Eq. (1.1), as a particular case of rough PDE in the weak sense.

Definition 3.1 Let $\mathbf{w}$ be a rough path satisfying Hypothesis 2.3 and 2.4. Consider the following equation:

$$du_t(x) = \frac{1}{2}\Delta u_t(x) + \sum_{i=1}^{\infty} \beta_i u_t(x) e_i dw_t^i. \tag{3.1}$$

We interpret this system as in Definition 2.5, with a measure μ given by

$$\mu([s,t)) = \int_s^t \Delta u_r\,dr.$$

As mentioned in the introduction, we are only focusing here on a priori estimates for the heat equation, which are representative of the methods at stake without being too technical. To this aim, we label the following assumption, which prevails until the end of the article:

Hypothesis 3.2 *One can construct a path u on* $[0, \tau]$ *which solves* (3.1) *according to Definition 3.1. In addition, u can be obtained as a limit of a sequence of functions* u^ε*, where* u^ε *solves:*

$$du_t^\varepsilon(x) = \frac{1}{2}\Delta u_t^\varepsilon(x) + \sum_{i=1}^{\infty} \beta_i u_t^\varepsilon(x) e_i dw_t^{\varepsilon,i}. \tag{3.2}$$

In (3.2)*, the family* $\{w_t^{\varepsilon,i};\ \varepsilon > 0, i \geq 1\}$ *is a sequence of smooth functions converging to w. Recalling our notations* (2.6) *and* (2.7)*, we also assume that:*

$$\lim_{\varepsilon \to 0} \omega_{\mathbf{w}^{1} - \mathbf{w}^{1,\varepsilon}}(0, \tau) + \omega_{\mathbf{w}^{2} - \mathbf{w}^{2,\varepsilon}}(0, \tau) = 0.$$

Remark 3.3 Since we assume that u is obtained as a limit of smoothed paths u^ε (see Hypothesis 3.2), all the remaining computations have to be understood as follows: we first derive our relations for u^ε, and we then take limits as $\varepsilon \to 0$. This step will often be implicit for sake of conciseness.

With Hypothesis 3.2 in hand, we now derive the equation followed by the path u^2 as a first step towards L^2 estimates.

Proposition 3.4 *Let u be the solution of Eq.* (3.1) *alluded to in Hypothesis 3.2. We also set*

$$f_t = \|u_t\|_{L^2}^2 + \int_0^t \|\nabla u_r\|_{L^2} dr, \quad \textit{and} \quad S_t^f = \sup_{s \leq t} f_s. \tag{3.3}$$

Then the following holds true:

(i) *Let* μ^2 *be the* E_{-1}*-valued measure defined as:*

$$\delta\mu_{st}^2(\psi) = -\int_s^t |\nabla u|^2(\psi) dr - \int_s^t (u_r \nabla u_r)(\nabla \psi) dr. \tag{3.4}$$

Then we have:

$$\omega_{\mu^2}(s,t) \leq \frac{3}{2}\int_s^t \|\nabla u\|_{L^2}^2 dr + \frac{1}{2}\int_s^t \|u\|_{L^2}^2 dr \leq \frac{3}{2}\int_s^t \|\nabla u\|_{L^2}^2 dr + \frac{(t-s)S_t^f}{2}, \tag{3.5}$$

provided the quantity above is finite.

(ii) *The squared path* u^2 *admits the following representation:*

$$\delta u_{st}^2(\psi) = \delta\mu_{st}^2(\psi) + \sum_{i=1}^{\infty} 2\beta_i u_s^2(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 4u_s^2(\psi e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{2,\natural}(\psi), \tag{3.6}$$

where ψ is a generic test function, and where $u^{2,\natural}$ is an element of $V_2^{\frac{q}{3}}$ for a certain $q < 3$.

(iii) *The increment f satisfies the following relation: for $0 \le s < t \le \tau$ we have*

$$\delta f_{st} = 2\sum_{i=1}^{\infty} u_s^2(e_i)\beta_i \mathbf{w}_{st}^{\mathbf{1},i} + 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_s^2(e_ie_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{2,\natural}(\mathbf{1}), \tag{3.7}$$

where $\mathbf{1}$ designates the function defined on $\mathbb{R}^d$ and identically equal to 1.

Proof With Remark 3.3 in mind, let us divide our proof in several steps.

Proof of (i) Similarly to [6, Remark 2.6], and working in the scale $E_n = W^{n,\infty}(\mathbb{R}^d)$, we have

$$|(\delta\mu^2)_{st}(\psi)| \le \int_s^t \|\nabla u\|_{L^2}^2 dr \|\psi\|_{L^\infty} + \left(\int_s^t \|\nabla u\|_{L^2}^2 dr\right)^{\frac{1}{2}} \left(\int_s^t \|u\|_{L^2}^2 dr\right)^{\frac{1}{2}} \|\psi\|_{W^{1,\infty}}, \tag{3.8}$$

Invoking now Young's inequality (namely $AB \le \frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta}$ for two positive numbers A, B with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) we get our claim (3.5).

Proof of (ii) According to Definitions 2.5 and 3.1, the solution of Eq. (3.1) can be decomposed as:

$$\delta u_{st}(\psi) = \sum_{i=1}^{\infty} \beta_i u_s(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{i,j=1}^{\infty} \beta_i\beta_i u_s(e_ie_j\psi)\mathbf{w}_{st}^{\mathbf{2},ij} + \delta\mu_{st}(\psi) + u_{st}^{\natural}(\psi). \tag{3.9}$$

As mentioned in Remark 2.7, u can also be seen as a solution to the integral equation (2.11), for which the change of variable formula (2.12) holds true. Applying this relation (written in its weak form) to $h(z) = z^2$, we obtain:

$$\delta u_{st}^2(\psi) = 2\int_s^t \Delta u_r(u_r\psi)\, dr + 2\sum_{i=1}^{\infty} \beta_i \int_s^t u_r^2(e_i\psi) dw_r^i,$$

so that an integration by parts in the first integral above yields:

$$\delta u_{st}^2(\psi) = -2\int_s^t |\nabla u|^2(\psi)\, dr - 2\int_s^t (u_r\nabla u_r)(\nabla\psi)\, dr + 2\sum_{i=1}^{\infty} \beta_i \int_s^t u_r^2(e_i\psi)\, dw_r^i. \tag{3.10}$$

We now expand the rough integral in (3.10) along the increments of w. We end up with relation (3.6), for a certain remainder $u^{2,\natural} \in V_2^{\frac{q}{3}}(E_{-1})$.

Proof of (ii) Relation (3.7) is simply obtained from (3.6) by considering a sequence of test functions $\{\psi_n; n \geq 1\}$ such that $\lim_{n\to\infty} \psi_n = \mathbf{1}$ and $\lim_{n\to\infty} \nabla\psi_n = 0$. □

3.2 A Priori Estimate in L^2

With Proposition 3.4 in hand, we can now derive the main estimate of this section.

Theorem 3.5 *Suppose w fulfills Hypothesis 2.3 and 2.4, and let u be the solution of Eq. (3.1) given in Hypothesis 3.2. For* $0 \leq s < t \leq \tau$, *set:*

$$\omega_1(s,t) = \omega_{\mathbf{w}^1}(s,t) + \omega^2_{\mathbf{w}^2}(s,t) + \omega_{\mathbf{w}^1}(s,t)\,\omega^2_{\mathbf{w}^2}(s,t) + \omega^4_{\mathbf{w}^2}(s,t). \tag{3.11}$$

Then the following L^2 *norm estimate for the solution u holds true:*

$$S^f_\tau = \sup_{0\leq t\leq\tau}\left(\|u_r\|^2_{L^2} + \int_0^t \|\nabla u_r\|^2_{L^2} dr\right) \leq 2\exp\left(c_p\omega_1(0,\tau)\right)\|u_0\|^2_{L^2}, \tag{3.12}$$

where c_p *is a strictly positive constant.*

Remark 3.6 Notice that $\|u_r\|^2_{L^2}$ and $\int_0^t \|\nabla u_r\|^2_{L^2} dr$ are positive. Therefore relation (3.12) implies that both terms are bounded from above.

Proof of Theorem 3.5 Recall that we have obtained the following decomposition in Proposition 3.4:

$$\delta u^2_{st}(\psi) = \delta\mu^2_{st}(\psi) + \sum_{i=1}^{\infty} 2\beta_i u^2_s(e_i\psi)\mathbf{w}^{\mathbf{1},i}_{st} + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 4u^2_s(\psi e_i e_j)\beta_i\beta_j\mathbf{w}^{\mathbf{2},ij}_{st} + u^{2,\natural}_{st}(\psi), \tag{3.13}$$

If we now set $g = u^2$ and $\mu^g = \mu^2$, we can recast (3.13) as:

$$\delta g_{st}(\psi) = \delta\mu^g_{st}(\psi) + \sum_{i=1}^{\infty} 2\beta_i\, g_s(e_i\psi)\mathbf{w}^{\mathbf{1},i}_{st} + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 4\, g_s(\psi e_i e_j)\beta_i\beta_j\mathbf{w}^{\mathbf{2},ij}_{st} + g^\natural_{st}(\psi).$$

This equation is of the same form as (2.8), and thus we can apply Proposition 2.8 directly. We get the following bound for $g^\natural_{st}$, which is valid whenever $\omega_1(s,t) + \omega^2_2(s,t) \leq L_p$ (recall that p is the regularity index of $\mathbf{w}$):

$$\|g^\natural_{st}\|_{E_{-1}} \leq c_p\omega_I(s,t), \quad \text{or equivalently} \quad \|u^{2,\natural}_{st}\|_{E_{-1}} \leq c_p\,\omega_I(s,t), \tag{3.14}$$

where the control ω_I is defined by:

$$\omega_I(s,t) \equiv \omega_{\mu^2}(s,t)\left(\omega_{\mathbf{w}^1}^{1/p}(s,t)+\omega_{\mathbf{w}^2}^{2/p}(s,t)\right)+S_t^{u^2}\left(2\omega_{\mathbf{w}^1}^{1/p}(s,t)\omega_{\mathbf{w}^2}^{2/p}(s,t)+\omega_{\mathbf{w}^2}^{4/p}(s,t)\right), \tag{3.15}$$

and where we recall that we have set:

$$S_t^{u^2}=\sup_{s\le t}|u_s^2|_{E_{-0}}=\sup_{s\le t}|u_s|_{L^2}^2.$$

Let us now go back to (3.13), and apply this relation to $\psi=\mathbf{1}$ (notice that the function $\mathbf{1}$ obviously sits in E_1). It is readily checked from (3.4) that:

$$\delta\mu_{st}^2(\mathbf{1})=-\int_s^t\|\nabla u\|_{L^2}^2dr,$$

and thus, with our notation (3.3) in mind, relation (3.13) becomes:

$$\delta f_{st}=\sum_{i=1}^{\infty}2\beta_i u_s^2(e_i)\mathbf{w}_{st}^{\mathbf{1},i}+\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}4u_s^2(e_ie_j)\beta_i\beta_j\mathbf{w}_{st}^{\mathbf{2},ij}+u_{st}^{2,\natural}(\mathbf{1})$$

Therefore, bounding $\|u_s^2\|_{E_{-0}}$ by S_t^f and invoking (3.14) in order to estimate $u_{st}^{2,\natural}(\mathbf{1})$, we obtain:

$$|\delta f_{st}|\le\left[2\omega_{\mathbf{w}^1}^{1/p}(s,t)+4\omega_{\mathbf{w}^2}^{2/p}(s,t)\right]S_t^f+c_p\,\omega_I(s,t), \tag{3.16}$$

where ω_I is given by (3.15). In order to close this expression, let us further bound the term ω_{μ^2} in the definition of ω_I. Namely, according to (3.5), we have

$$\omega_{\mu^2}(s,t)\le\frac{3}{2}\int_s^t\|\nabla u\|_{L^2}^2dr+\frac{(t-s)S_t^f}{2}\le c_\tau S_t^f, \tag{3.17}$$

where we recall that we are working on a time interval $[0,\tau]$. Plugging this inequality into the definition of ω_I, we end up with:

$$\omega_I(s,t)\le c_\tau S_t^f\left(\omega_{\mathbf{w}^1}^{1/p}(s,t)+\omega_{\mathbf{w}^2}^{2/p}(s,t)+\omega_{\mathbf{w}^1}^{1/p}(s,t)\omega_{\mathbf{w}^2}^{2/p}(s,t)+\omega_{\mathbf{w}^2}^{4/p}(s,t)\right).$$

Reporting the relation above into (3.16), we get

$$\begin{aligned}|\delta f_{st}|&\le c_\tau S_t^f\left(\omega_{\mathbf{w}^1}^{1/p}(s,t)+\omega_{\mathbf{w}^2}^{2/p}(s,t)+\omega_{\mathbf{w}^1}^{1/p}(s,t)\,\omega_{\mathbf{w}^2}^{2/p}(s,t)+\omega_{\mathbf{w}^2}^{4/p}(s,t)\right)\\&\le c_{\tau,p}S_t^f\,\omega_1(s,t),\end{aligned} \tag{3.18}$$

where ω_1 is the control introduced in (3.11). Recall again that inequality (3.18) is valid when $\omega_1(s,t) + \omega_2^2(s,t) \le L_p$. It is thus also satisfied when $\omega_1(s,t) \le L_p$.

We are now in a position to directly apply our rough Gronwall Lemma 2.2 to (3.18), with $Q = f$, $\kappa = 1/p$ and $\omega_2 = 0$. It is readily checked that ω_1 is a control, and hence:

$$S_t^f \le 2\exp\Big(c_p\,\omega_1(0,\tau)\Big)f_0 = 2\exp\Big(c_p\,\omega_1(0,\tau)\Big)\|u_0\|_{L^2}^2, \tag{3.19}$$

which ends our proof. □

3.3 L^α Type Estimates

In this part, we are going to derive some L^α estimates for the solution of Eq. (3.1), generalizing the case $\alpha = 2$. As the reader will notice, the method is the same as for the L^2 case, but we include some computational details for convenience.

Remark 3.7 We will handle the case of L^α estimates for an even integer α, in order to have $u^\alpha \ge 0$ and $u^{\alpha-2} \ge 0$ in the computations below. However, notice that other values of α can then be reached by simple interpolation methods.

We start this section with an analogue of Proposition 3.4.

Proposition 3.8 *Let u be the solution of Eq. (3.1) alluded to in Hypothesis 3.2, and consider an even integer α. We also set*

$$\ell_t = \|u_t\|_{L^\alpha}^\alpha + \int_0^t u_r^{\alpha-2}\|\nabla u_r\|^2 dr, \quad \textit{and} \quad S_t^\ell = \sup_{s\le t}\ell_s.$$

Then the following holds true:

(i) *Let μ^α be the E_{-1}-valued measure defined as:*

$$\delta\mu_{st}^\alpha(\psi) = -\frac{\alpha(\alpha-1)}{2}\int_s^t u_r^{\alpha-2}|\nabla u|^2(\psi)dr - \frac{\alpha}{2}\int_s^t (u_r^{\alpha-1}\nabla u_r)(\nabla\psi)dr \tag{3.20}$$

Then we have:

$$\omega_{\mu^\alpha}(s,t) \le \frac{\alpha(\alpha-1)}{4}\int_s^t u_r^{\alpha-2}|\nabla u_r|^2 dr + \frac{\alpha(t-s)S_t^\ell}{4}, \tag{3.21}$$

provided the quantity above is finite.

(ii) *The path u^α admits the following representation :*

$$\delta u_{st}^\alpha(\psi) = \delta\mu_{st}^\alpha(\psi) + \sum_{i=1}^\infty \alpha\beta_i u_s^\alpha(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^\infty\sum_{i=1}^\infty \alpha^2 u_s^\alpha(\psi e_i e_j)\beta_i\beta_j\mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{\alpha,\natural}(\psi) \tag{3.22}$$

where ψ is a generic test function, and where $u^{\alpha,\natural}$ is an element of $V_2^{\frac{q}{3}}$ for a certain $q < 3$.

(iii) *The increment ℓ satisfies the following relation: for $0 \le s < t \le \tau$ we have*

$$\delta\ell_{st} = \alpha \sum_{i=1}^{\infty} u_s^{\alpha}(e_i)\beta_i \mathbf{w}_{st}^{\mathbf{1},i} + \alpha^2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_s^{\alpha}(e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{\alpha,\natural}(\mathbf{1}), \tag{3.23}$$

where $\mathbf{1}$ designates the function defined on $\mathbb{R}^d$ and identically equal to 1.

Proof With Remark 3.3 in mind and μ^{α} defined in (3.21), it is readily checked that:

$$\begin{aligned} |(\delta\mu^{\alpha})_{st}(\psi)| \le{}& \frac{\alpha(\alpha-1)}{2} \int_s^t u_r^{\alpha-2}|\nabla u_r|^2 dr \|\psi\|_{L^\infty} \\ &+ \frac{\alpha}{2}\left(\int_s^t u_r^{\alpha-2}|\nabla u_r|^2 dr\right)^{1/2} \left(\int_s^t \|u_r\|_{L^\alpha}^{\alpha} dr\right)^{1/2} \|\psi\|_{W^{1,\infty}}. \end{aligned} \tag{3.24}$$

Invoking now Young's inequality as we did in the previous L^2 case, we get our claim (3.21).

The proof of (3.22) is similar to the L^2 case, except that we apply the change of variable formula and relation (3.9) to $h(z) = z^{\alpha}$. We obtain:

$$\delta u_{st}^{\alpha}(\psi) = \alpha \int_s^t \Delta u_r(u_r^{\alpha-1}\psi)\, dr + \alpha \sum_{i=1}^{\infty} \beta_i \int_s^t u_r^{\alpha}(e_i\psi) dw_r^i,$$

so that an integration by parts in the first integral above yields:

$$\begin{aligned} \delta u_{st}^{\alpha}(\psi) = -\alpha(\alpha-1) \int_s^t u_r^{\alpha-2}|\nabla u|^2(\psi)\, dr - \alpha \int_s^t (u_r^{\alpha-1}\nabla u_r)(\nabla\psi)\, dr \\ + \alpha \sum_{i=1}^{\infty} \beta_i \int_s^t u_r^{\alpha}(e_i\psi)\, dw_r^i. \end{aligned} \tag{3.25}$$

We now expand the rough integral in (3.10) along the increments of w. We end up with relation (3.22), for a certain remainder $u^{\alpha,\natural} \in V_2^{q/3}(E_{-1})$.

As in the L^2 case, relation (3.23) is simply obtained from (3.22) by considering a sequence of test functions $\{\psi_n; n \ge 1\}$ such that $\lim_{n\to\infty} \psi_n = 1$. □

With Proposition 3.8 in hand, we can now derive the announced estimate in L^{α} type spaces.

Theorem 3.9 *Suppose w fulfills Hypothesis 2.3 and 2.4, and let u be the solution of Eq. (3.1) given in Hypothesis 3.2. For $0 \le s < t \le \tau$, set:*

$$\omega_1(s,t) = \omega_{\mathbf{w}^1}(s,t) + \omega_{\mathbf{w}^2}^2(s,t) + \omega_{\mathbf{w}^1}(s,t)\,\omega_{\mathbf{w}^2}^2(s,t) + \omega_{\mathbf{w}^2}^4(s,t). \tag{3.26}$$

Then for any even integer α, the following L^α norm estimate for the solution u holds true:

$$\sup_{0\le t\le\tau}\left(\|u_r\|_{L^\alpha}^\alpha+\int_0^t u_r^{\alpha-2}|\nabla u_r|^2dr\right)\le 2\exp\left(c_p\omega_1(0,\tau)\right)\|u_0\|_{L^\alpha}^\alpha, \tag{3.27}$$

where c_p is a strictly positive constant.

Proof Recall that we have obtained the following decomposition in Proposition 3.8:

$$\delta u_{st}^\alpha(\psi)=\delta\mu_{st}^\alpha(\psi)+\sum_{i=1}^\infty\alpha\beta_i u_s^\alpha(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i}+\sum_{j=1}^\infty\sum_{i=1}^\infty\alpha^2u_s^\alpha(\psi e_ie_j)\beta_i\beta_j\mathbf{w}_{st}^{\mathbf{2},ij}+u_{st}^{\alpha,\natural}(\psi). \tag{3.28}$$

If we now set $g=u^\alpha$ and $\mu^g=\mu^\alpha$, we can proceed as in Theorem 3.5 and recast (3.13) as:

$$\delta g_{st}(\psi)=\sum_{i=1}^\infty\alpha\beta_i g_s(e_i\varphi)\mathbf{w}_{st}^{\mathbf{1},i}+\delta\mu_{st}^g(\varphi)+\sum_{i,j=1}^\infty\alpha^2\beta_i\beta_j g_s(e_ie_j\varphi)\mathbf{w}_{st}^{\mathbf{2},ij}+g_{st}^\natural(\varphi),$$

This equation is of the same form as (2.8), and thus we can apply Proposition 2.8 directly. We get the following bound for $g_{st}^\natural$, which is valid whenever $\omega_1(s,t)+\omega_2^2(s,t)\le L_{p,\alpha}$:

$$\|g_{st}^\natural\|_{E_{-1}}\le c_p\omega_I(s,t),\quad\text{or equivalently}\quad\|u_{st}^{\alpha,\natural}\|_{E_{-1}}\le c_p\omega_I(s,t), \tag{3.29}$$

where the control ω_I is defined by:

$$\omega_I(s,t)\equiv\omega_{\mu^\alpha}(s,t)\left(\omega_{\mathbf{w^1}}^{1/p}(s,t)+\omega_{\mathbf{w^2}}^{2/p}(s,t)\right)+S_t^{u^\alpha}\left(2\omega_{\mathbf{w^1}}^{1/p}(s,t)\omega_{\mathbf{w^2}}^{2/p}(s,t)+\omega_{\mathbf{w^2}}^{4/p}(s,t)\right). \tag{3.30}$$

and where we recall that we have

$$S_t^{u^\alpha}=\sup_{s\le t}|u_s^\alpha|_{E_{-0}}=\sup_{s\le t}|u_s|_{L^\alpha}^\alpha.$$

Let us now go back to (3.13), and apply this relation to $\psi=\mathbf{1}$ (notice that the function $\mathbf{1}$ obviously sits in E_1). It is readily checked from (3.20) that:

$$\delta\mu_{st}^\alpha(\mathbf{1})=-\frac{\alpha(\alpha-1)}{2}\int_s^t u_r^{\alpha-2}|\nabla u|^2dr,$$

and thus (3.13) becomes:

$$\delta \ell_{st} = \sum_{i=1}^{\infty} \alpha \beta_i u_s^{\alpha}(e_i) \mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \alpha^2 u_s^{\alpha}(e_i e_j) \beta_i \beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{\alpha,\natural}(\mathbf{1})$$

Therefore, bounding $\|u_s^{\alpha}\|_{E_{-0}}$ by S_t^{ℓ} and invoking (3.29) in order to estimate $u_{st}^{\alpha,\natural}(\mathbf{1})$, we obtain:

$$|\delta \ell_{st}| \leq \left[2\omega_{\mathbf{w}^1}^{1/p}(s,t) + 4\omega_{\mathbf{w}^2}^{2/p}(s,t) \right] S_t^{\ell} + \omega_I(s,t), \tag{3.31}$$

where ω_I is given by (3.15). In order to close this expression, let us further bound the term $\omega_{\mu^{\alpha}}$ in the definition of ω_I. Namely, according to (3.21), we have

$$\omega_{\mu^{\alpha}}(s,t) \leq \frac{\alpha(\alpha-1)}{4} \int_s^t u_r^{\alpha-2} |\nabla u_r|^2 dr + \frac{\alpha(t-s) S_t^{\ell}}{4} \leq c_{\tau,p} S_t^{\ell},$$

which is the equivalent of relation (3.17) in our context. Starting from this point, we can conclude exactly as in Theorem 3.5. □

4 Application to Fractional Brownian Motion

This section is devoted to the application of our abstract results of Sect. 3 to some more concrete examples of heat equations driven by an infinite dimensional fractional Brownian motion. Though our general analysis was focused on equations in $\mathbb{R}^d$, we shall treat the case of both bounded and unbounded domains.

4.1 Equations in Bounded Domains

We first consider the case of an equation in a bounded domain D. This will enable us to compare our hypothesis with the assumptions contained in [15] for similar situations. Let us first label the conditions on our domain.

Hypothesis 4.1 *In this section, we consider an open, bounded domain D with smooth boundary ∂D, and satisfying the cone property.*

On such a domain D, we wish to give conditions which are close enough to the ones produced in [15]. This is why we consider an operator C given as follows:

Hypothesis 4.2 *In the remainder of the section, C will stand for a linear, self-adjoint, positive trace-class operator on $L^2(D)$. This operator admits an orthonormal basis $(e_i)_{i\in\mathbb{N}_+}$ of eigenfunctions, with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}_+}$. It* *also admits an integral representation, whose generating kernel is denoted as κ. Summarizing, for all $i \geq 0$ and for almost every $x \in D$ we have:*

$$Ce_i(x) = \int_D \kappa(x, y)e_i(y)\,dy = \lambda_i e_i(x). \tag{4.1}$$

We can now formulate our a priori estimate in this context:

Proposition 4.3 *Let $D \subset \mathbb{R}^d$ be a domain fulfilling Hypothesis 4.1, together with an operator C as in Hypothesis 4.2. On D, we consider the following equation:*

$$du_t(x) = \frac{1}{2}\Delta u_t(x) + \sum_{i=1}^{\infty} \lambda_i^{\nu} u_t(x)e_i(x)dB_t^i, \tag{4.2}$$

where $(B_t^i)_{t\in\mathbb{R}^+})_{i\in N^+}$ is a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter $H \in (\frac{1}{3}, 1)$, and $\nu \geq 0$ is a positive parameter. For the definition of e_i and λ_i, we refer to Hypothesis 4.2. In addition, we suppose that our operator C and its kernel κ satisfy the following conditions:

$$A_\kappa \equiv \sup_{x\in D} \|\kappa(x,\cdot)\|_{L^2(D)} + \|\nabla\kappa(x,\cdot)\|_{L^2(D)} < \infty, \quad \textit{and} \quad \sum_{i\geq 0} \lambda_i^{\nu-1} < \infty. \tag{4.3}$$

Then the results from Theorems 3.5 and 3.9 apply.

Proof It is well known (see e.g. [8, Chap. 15]) that any finite dimensional fractional Brownian motion $(B^i)_{i\leq N}$ can be lifted as a rough path. It is thus enough to prove conditions (2.6) and (2.7). We shall focus on condition (2.6), the other one being checked with the same kind of arguments.

In order to verify (2.6), similarly to [15], we start by recasting (4.1) as:

$$e_i(x) = \lambda_i^{-1} \int_D \kappa(x, y)e_i(y)\,dy, \quad \text{and} \quad \nabla e_i(x) = \lambda_i^{-1} \int_D \nabla\kappa(x, y)e_i(y)\,dy.$$

Hence, invoking Cauchy-Schwarz' inequality and relation (4.3), we obtain:

$$|e_i|_{E_1} \leq \lambda_i^{-1} A_\kappa \|e_i\|_{L^2(D)} = \lambda_i^{-1} A_\kappa. \tag{4.4}$$

Now notice that (2.6) is ensured by the condition $\mathbb{E}[\omega_{\mathbf{w}^1}^{1/p}(0, \tau)] < \infty$, where τ is our time horizon. Furthermore,

$$\mathbb{E}\left[\omega_{\mathbf{w}^1}^{1/p}(0, \tau)\right] = \sum_{i=1}^{\infty} \lambda_i^{\nu} \left(1 + |e_i|_{E_1}\right) \mathbb{E}\left[\mathcal{N}[\mathbf{w}^{1,i}; V_2^p([s, t])]\right],$$

and since $\mathbb{E}[\mathcal{N}[\mathbf{w}^{\mathbf{1},i}; V_2^p([s,t])]]$ is uniformly bounded in i, we end up with

$$\mathbb{E}\left[\omega_{\mathbf{w}^{\mathbf{1}}}^{1/p}(0,\tau)\right] \le c_{\kappa,\mathbf{w}} \sum_{i=1}^{\infty} \lambda_i^{\nu-1},$$

which is a finite quantity according to our assumption (4.3). In conclusion, Hypothesis 2.3 and 2.4 are satisfied, and Theorems 3.5 and 3.9 hold true. □

Remark 4.4 With respect to [15], we have added here the assumption

$$\sup_{x\in D} \|\nabla\kappa(x,\cdot)\|_{L^2(D)} < \infty,$$

which is an artifact of our variational approach. This being said, let us recall that our method applies to rough situations (compared to the case $H > 1/2$ dealt with in [15]). We also believe that our method extends to non linear equations, with a noisy term of the form $\sum_{i=1}^{\infty} \lambda_i^{\nu}\sigma(u_t(x))e_i(x)dB_t^i$ for a smooth coefficient σ.

4.2 Equations in $\mathbb{R}^d$

On the whole space $\mathbb{R}^d$, choices of orthonormal basis of L^2 are wide. For sake of concreteness, we will stick here to a wavelet basis based on Shannon's wavelet, though a much more general setting can be found e.g. in [14].

Let us start by defining the L^2 basis alluded to above (we refer again to [14] for proofs of general facts on wavelets).

Lemma 4.5 *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined as*

$$\psi(x) = \frac{\sin 2\pi(x-1/2)}{2\pi(x-1/2)} - \frac{\sin \pi(x-1/2)}{\pi(x-1/2)}.$$

Then $\psi \in L^2(\mathbb{R})$, and the following holds true:

(i) *Let us introduce a family of scaled functions $\{\psi_{j,k};\, j \ge 0,\, k \in \mathbb{Z}\}$ by:*

$$\psi_{j,k}(x) = 2^{-\frac{j}{2}}\psi\left(\frac{x-2^j k}{2^j}\right). \tag{4.5}$$

This family is an orthonormal basis of $L^2(\mathbb{R})$.

(ii) *One can obtain an orthonormal basis of $L^2(\mathbb{R}^d)$ by tensorizing the previous basis of $L^2(\mathbb{R})$. Namely, for all $j \ge 0$ and for $n = (n_1, \cdots, n_d)$, we denote*

$$\psi_{j,n}(x) = 2^{-dj/2}\psi\left(\frac{x_1-2^j n_1}{2^j}, \cdots, \frac{x_d-2^j n_d}{2^j}\right).$$

Then $\{\psi_{j,n}(x)\}_{(j,n)\in\mathbb{Z}^{d+1}}$ *is an orthonormal basis of* $L^2(\mathbb{R}^d)$*. In addition, it is readily checked that:*

$$|\psi_{j,k}|_{E_1} \leq 2^{\frac{jd}{2}}, \tag{4.6}$$

where we recall that we work in the scale $E_n = W^{n,\infty}(\mathbb{R})$*.*

Remark 4.6 A completely correct version of Lemma 4.5 should include a so-called father wavelet ϕ. We omit this step for notational sake.

Under the setting of Lemma 4.5, here is our example of stochastic heat equation on $\mathbb{R}^d$:

Proposition 4.7 *Consider the equation*

$$du_t(x) = \frac{1}{2}\Delta u_t(x) + \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^d} \beta_{j,n} u_t(x)\psi_{j,n}(x)dB_t^{j,n},$$

where $\{B^{j,n}; j \geq 0, n \in \mathbb{Z}^d\}$ *is a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter* $H \in (\frac{1}{3}, 1)$*, and* $\{\beta_{j,n}; j \geq 0, n \in \mathbb{Z}^d\}$ *is a family of positive coefficients. We assume that*

$$A_\beta \equiv \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^d} 2^{\frac{dj}{2}}\beta_{j,n} < \infty. \tag{4.7}$$

Then the results of Theorems 3.5 and 3.9 apply.

Proof We proceed as for Proposition 4.3, and we are easily reduced to show that $\mathbb{E}[\omega_{\mathbf{w^1}}^{1/p}(0,\tau)]$ is a finite quantity. In our case, we have

$$\mathbb{E}\left[\omega_{\mathbf{w^1}}^{1/p}(0,\tau)\right] = \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^d} \beta_{j,n}\left(1 + |\psi_{j,n}|_{E_1}\right)\mathbb{E}\left[\mathcal{N}[\mathbf{w^1}^{j,n}; V_2^p([s,t])]\right].$$

Moreover, the coefficients $\mathbb{E}[\mathcal{N}[\mathbf{w^1}^{j,n}; V_2^p([s,t])]]$ are uniformly bounded in j, n. Hence, owing to relation (4.6), we get:

$$\mathbb{E}\left[\omega_{\mathbf{w^1}}^{1/p}(0,\tau)\right] \leq c_{\psi,\mathbf{w}}\sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^d} 2^{\frac{dj}{2}}\beta_{j,n} = c_{\psi,\mathbf{w}} A_\beta,$$

where A_β is introduced in condition (4.7). This concludes our proof in a straightforward way. □

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