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Discretizing the fractional Lévy area☆

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Abstract

In this article, we give sharp bounds for the Euler discretization of the Lévy area associated to a d-dimensional fractional Brownian motion. We show that there are three different regimes for the exact root mean square convergence rate of the Euler scheme, depending on the Hurst parameter $H \in (1/4, 1)$. For $H < 3/4$ the exact convergence rate is $n^{-2H+1/2}$, where n denotes the number of the discretization subintervals, while for $H = 3/4$ it is $n^{-1}\sqrt{\log(n)}$ and for $H > 3/4$ the exact rate is n^{-1}. Moreover, we also show that a trapezoidal scheme converges (at least) with the rate $n^{-2H+1/2}$. Finally, we derive the asymptotic error distribution of the Euler scheme. For $H \leq 3/4$ one obtains a Gaussian limit, while for $H > 3/4$ the limit distribution is of Rosenblatt type.

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1. Introduction and main results

Let $B = (B^{(1)}, \ldots, B^{(d)})$ be a d-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (1/4, 1)$ indexed by $\mathbb{R}$, i.e. B is composed of d independent centered Gaussian processes with continuous sample paths and covariance function given by

$$R_H(s,t) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \quad s,t \in \mathbb{R}.$$

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For an arbitrary $T > 0$, a typical stochastic differential equation (SDE) on $[0, T]$ driven by B can be written as

$$Y_t = a + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s, \quad t \in [0, T], \tag{1}$$

where $a \in \mathbb{R}^m$ is a given initial condition and $\sigma : \mathbb{R}^m \to \mathbb{R}^{m,d}$ is sufficiently smooth. During the last years, the rough paths theory has allowed to handle several aspects of differential equations like (1), ranging from existence and uniqueness results (see [8,16] for equations of type (1) and [4,11,21] for extensions to other kinds of systems) to density estimates [5] or ergodic theorems [12].

It is also important, and in fact at the very core of the rough path analysis, to derive good numerical approximations for fractional differential equations like (1). This problem has so far been considered in three types of situations: (i) When $H > 1/2$, it is proved independently in [7] and [18] that the Euler scheme associated to Eq. (1), based on the grid $\{iT/n;\ i \le n\}$, converges with the rate $n^{-(2H-1)+\varepsilon}$ for arbitrarily small $\varepsilon > 0$. The exact rate of convergence of the Euler scheme is computed in [20] in the particular case of a one-dimensional equation. (ii) In the Brownian case $H = 1/2$, there exists a huge amount of literature on approximation schemes for SDEs, and we just send the interested reader to the references [14,17] for an overview of the topic. (iii) For $1/3 < H < 1/2$, the rough path strategy in order to solve Eq. (1), see e.g. [8,9,16], tells us that one should use at least a Milstein-type scheme in order to approximate its solution. Moreover, it can be easily seen that for $H < 1/2$ the standard Euler scheme does not converge for stepsizes going to zero, even in the one-dimensional case. Indeed, consider the one-dimensional SDE

$$\mathrm{d}X_t = X_t\,\mathrm{d}B_t, \quad X_0 = 1,$$

whose exact solution is given by $X_t = \exp(B_t)$. The Euler approximation of this equation at $t = 1$ is given by

$$X_1^{(n)} = \prod_{k=0}^{n-1} (1 + (B_{(k+1)/n} - B_{k/n})).$$

So, for $n \in \mathbb{N}$ sufficiently large and using a Taylor expansion, we have

$$\begin{aligned} X_1 - X_1^{(n)} &= \exp(B_1) - \exp\Big(\sum_{k=0}^{n-1} \log(1 + (B_{(k+1)/n} - B_{k/n}))\Big) \\ &= \exp(B_1) - \exp\Big(B_1 - \frac{1}{2}\sum_{k=0}^{n-1} |B_{(k+1)/n} - B_{k/n}|^2 + \rho_n\Big), \end{aligned}$$

where $\rho_n \xrightarrow{\text{a.s.}} 0$ for $n \to \infty$ for $H > 1/3$. Now, it is well known that

$$\sum_{k=0}^{n-1} |B_{(k+1)/n} - B_{k/n}|^2 \xrightarrow{\text{a.s.}} \infty$$

for $H < 1/2$, so we have $X_1^{(n)} \xrightarrow{\text{a.s.}} 0$. However, Milstein-type schemes are known to be convergent for such a one-dimensional equation; see [10].

For general multi-dimensional equations of type (1), a Milstein-type scheme is studied in [7]: set $\overline{Y}_0 = a$, and for a grid given by $t_k = kT/n$, $k = 0, \dots, n-1$, let

$$\overline{Y}_{t_{k+1}} = \overline{Y}_{t_k} + \sum_{i=1}^{d} \sigma^{(i)}(\overline{Y}_{t_k})(B^{(i)}_{t_{k+1}} - B^{(i)}_{t_k}) + \sum_{i,j=1}^{d} \mathcal{D}^{(i)}\sigma^{(j)}(\overline{Y}_{t_k}) \int_{t_k}^{t_{k+1}} (B^{(i)}_s - B^{(i)}_{t_k})\, \mathrm{d}B^{(j)}_s, \quad (2)$$

for $k = 0, \ldots, n-1$, where $\mathcal{D}^{(i)}$ is the differential operator $\sum_{l=1}^{m} \sigma_l^{(i)} \partial_{x_l}$. Davie then proves that this scheme has convergence rate $n^{-(3H-1)+\varepsilon}$, and this result has been extended in [8] in an abstract setting to higher order schemes for a rough path with a given regularity.

The above Milstein-type scheme (2) requires knowledge of the iterated integrals

$$X_t^{(i,j)} = \int_0^t B_s^{(i)}\, \mathrm{d}B_s^{(j)}, \quad t \in [0, T], i, j = 1, \ldots, d, \quad (3)$$

whose explicit distribution is unknown for $i \neq j$. Thus discretization procedures for (3) are crucial for an implementation of this numerical method. This has already been addressed in [6], where dyadic linear approximations of the fBm B are used in order to define a Wong–Zakai-type approximation $\widehat{X}^n$ of X. In the last reference, the process $\widehat{X}^n$ is shown to converge almost surely in p-variation distance, and the (non-sharp) error bound

$$\mathbf{E}|\widehat{X}_T^n - X_T|^2 \leq C \cdot 2^{-n(4H-1)/2}$$

is also determined. The current article takes up this kind of program, and we consider the approximation of

$$X_T = \int_0^T B_s^{(1)}\, \mathrm{d}B_s^{(2)} \quad (4)$$

by the Euler and a trapezoidal scheme based on equidistant discretizations.

For the approximation of (4) the standard Euler method has the explicit expression

$$X_T^n = \sum_{i=0}^{n-1} B_{iT/n}^{(1)} \left(B_{(i+1)T/n}^{(2)} - B_{iT/n}^{(2)} \right). \quad (5)$$

The results we obtain for the Euler scheme are then of two kinds. First, we determine the exact L^2-convergence rate.

Theorem 1. *Let X_T defined by (4) and its Euler approximation X_T^n given by expression (5). Let*

$$\alpha_1(H) = c_0 + 2\sum_{k=1}^{\infty} c_k \quad \textit{and} \quad \alpha_2(H) = \frac{H^2(2H-1)}{4(4H-3)},$$

where the constants c_0, c_k will be defined below by (23) and (24). Then $\sum_{k=1}^{\infty} c_k$ is a convergent series for $H \in (1/4, 3/4)$ and

$$\mathbf{E}|X_T - X_T^n|^2 = \begin{cases} \alpha_1(H) \cdot T^{4H} \cdot n^{-4H+1} + o(n^{-4H+1}) & \text{for } H \in (1/4, 3/4), \\ \dfrac{9}{128} \cdot T^3 \cdot \log(n) n^{-2} + o(\log(n) n^{-2}) & \text{for } H = 3/4, \\ \alpha_2(H) \cdot T^{4H} \cdot n^{-2} + o(n^{-2}) & \text{for } H \in (3/4, 1). \end{cases}$$

In the case $H = 1/2$, i.e. for the approximation of the Wiener Lévy area, the above result is well known and can be obtained by straightforward computations. In particular, one can easily check that $\alpha_1(1/2) = \frac{1}{2}$, which means that we recover the classical result for Brownian motion.

The convergence rate breaks up into several regimes which are reminiscent of the cases obtained in [23] concerning weighted quadratic variations of the one-dimensional fBm. In particular, the convergence rate does not improve for $H > 3/4$, i.e. is equal to n^{-1} independently of H. Finally, note that our study starts obviously at $H = 1/4^+$, since the Lévy area is not even defined for $H \leq 1/4$.

Using a trapezoidal rule for the approximation of the integral leads to the following scheme, which coincides with the Wong–Zakai approximation used in [6]:

$$\widehat{X}_T^n = \frac{1}{2} \sum_{i=0}^{n-1} \left(B^{(1)}_{iT/n} + B^{(1)}_{(i+1)T/n} \right) \left(B^{(2)}_{(i+1)T/n} - B^{(2)}_{iT/n} \right). \qquad (6)$$

This trapezoidal scheme avoids the "breakdown" of the convergence rate of the Euler scheme for $H \geq 3/4$.

Theorem 2. *Let $H > 1/4$. Then there exists a constant $C(H) > 0$ such that*

$$\mathbf{E}|X_T - \widehat{X}_T^n|^2 \leq C(H) \cdot T^{4H} \cdot n^{-4H+1}.$$

We strongly suppose that the trapezoidal scheme has exact root mean square convergence rate $n^{-2H+1/2}$. Moreover, we suppose that this rate is the best possible. In other words, we conjecture that the conditional expectation of X_T given $B_{T/n}, B_{2T/n}, \ldots, B_T$ satisfies

$$\mathbf{E}\left|X_T - \mathbf{E}(X_T \mid B_{T/n}, B_{2T/n}, \ldots, B_T)\right|^2 \geq c(H) \cdot T^{4H} \cdot n^{-4H+1},$$

where $c(H) > 0$.

The third result in this article is a refinement of Theorem 1, meaning that we obtain a limit theorem for the asymptotic error distribution of the Euler scheme.

Theorem 3. *Let X_T, X_T^n and $\alpha_1(H), \alpha_2(H)$ defined as above. Moreover, let Z be a standard normal random variable. Then:*

(1) *Case $1/4 < H \leq 3/4$. Here the following central limit theorems hold:*

$$\lim_{n\to\infty} n^{2H-1/2}\,(X_T - X_T^n) \stackrel{\mathcal{L}}{=} \sqrt{\alpha_1(H)}T^{2H} \cdot Z \quad \text{for } H \in (1/4, 3/4),$$

and

$$\lim_{n\to\infty} n(\log(n))^{-1/2}\,(X_T - X_T^n) \stackrel{\mathcal{L}}{=} \frac{3}{4\sqrt{8}} T^{3/2} \cdot Z$$

for $H = 3/4$.

(2) *Case $H > 3/4$. Let R_1 and R_2 be two independent Rosenblatt processes (see Section 5 for a definition). Then it holds*

$$\lim_{n\to\infty} n\,(X_T - X_T^n) \stackrel{\mathcal{L}}{=} \sqrt{2\alpha_2(H)}T^{2H} \cdot (R_1 - R_2).$$

Let us say a few words about the methodology we have adopted in order to prove Theorem 3. It should be mentioned first that we have used the analytic approximations introduced in [26] in order to define the Lévy area X, which allows us to use some elegant complex analysis methods for moments estimates in this context. Then, for $H \in (1/4, 3/4)$, the central limit type results are obtained through the criterion introduced in [24] for random variables in a fixed chaos. For this we control the fourth moments of X with the help of (Feynman) diagrams. For the case $H \geq 3/4$ we proceed in a different way. Here the trapezoidal approximation of X_T performs better than the Euler method. Expressing the difference between both schemes as the difference of quadratic variations for two independent one-dimensional fBms, thanks to a simple geometrical trick given in [22], one obtains the limit theorems for $H \geq 3/4$ using the limit results for quadratic variations of fBm; see e.g. [3,23] and the references therein. (We use a similar trick also to compute the mean square convergence rate of the Euler scheme for $H \geq 3/4$). In particular, this leads to the Rosenblatt-type limit distribution for $H > 3/4$. For the trapezoidal scheme, whose error seems to behave like the second order quadratic variations of fBm, see e.g. [2], a central limit theorem could be also derived using the criterion in [24], but we omit this here for the sake of conciseness.

The remainder of this article is structured as follows. Integrals with respect to the fractional Brownian motion will always be understood as limits of analytic integrals as in [26]. We thus recall the definition of the analytic fBm, as well as some preliminaries in Section 2. Section 3 contains the proofs of Theorems 1 and 2. The proof of Theorem 3 is given in Sections 4 and 5.

2. Definition of the analytic FBM and preliminaries

This section is devoted to recall the definition of the fractional Brownian motion introduced in [26], and to state some of the properties of this process which will be used in what follows. All the random variables introduced here will be defined on a complete probability space $(\mathcal{U}, \mathcal{F}, \mathbf{P})$, without any further mention (notice the unusual notation $\mathcal{U}$ for our probability space, due to the fact that the letter Ω will serve for the complex domains we consider in what follows). The following kernels will also be essential for our future computations. Here and in what follows, $\Re$ and $\Im$ stand respectively for the real and imaginary part of a complex number.

Definition 4 (*η-Regularized Power Functions*). For $\beta \in \mathbb{R} \setminus \mathbb{Z}$ and $\eta > 0$ let

$$[x]_\eta^{\pm,\beta} = (\pm \mathrm{i}x + \eta)^\beta \quad \text{and} \quad [x]_\eta^\beta = 2\Re[x]_\eta^{\pm,\beta} = [x]_\eta^{+,\beta} + [x]_\eta^{-,\beta}.$$

Then, for $\eta > 0$ and $x, y \in \mathbb{R}$, define $K'^{,\pm}(\eta; x, y)$ as

$$K'^{,\pm}(\eta; x, y) = \frac{H(1-2H)}{2\cos \pi H}(\pm \mathrm{i}(x-y) + \eta)^{2H-2} = \frac{H(1-2H)}{2\cos \pi H}[x-y]_\eta^{\pm,2H-2}.$$

Set also

$$K'(\eta; x, y) := 2\Re K'^{,\pm}(\eta; x, y) = K'^{,+}(\eta; x, y) + K'^{,-}(\eta; x, y).$$

Note that the above kernels are well defined on our prescribed domain $\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}$.

2.1. Definition of the analytic fBm

The article [26] introduces the fractional Brownian motion as the real part of the trace on $\mathbb{R}$ of an analytic process Γ (called: *analytic fractional Brownian motion* [25]) defined on the complex upper-half plane $\Pi^+ = \{z \in \mathbb{C};\ \Im(z) > 0\}$. This is achieved by first noticing that the kernel $K'(\eta)$ is positive definite and represents (for every fixed $\eta > 0$) the covariance of a real-analytic centered Gaussian process with real time parameter t. The easiest way to see this is to make use of the following explicit series expansion: for $k \geq 0$ and $z \in \Pi^+$, set

$$f_k(z) = 2^{H-1} \sqrt{\frac{H(1-2H)}{2\cos \pi H}} \sqrt{\frac{\boldsymbol{\Gamma}(2-2H+k)}{\boldsymbol{\Gamma}(2-2H)k!}} \left(\frac{z+\mathrm{i}}{2\mathrm{i}}\right)^{2H-2} \left(\frac{z-\mathrm{i}}{z+\mathrm{i}}\right)^{k}, \tag{7}$$

where $\boldsymbol{\Gamma}$ stands for the usual Gamma function. Then these functions are well defined on Π^+, and it can be checked that one has

$$\sum_{k\geq 0} f_k\left(x+\mathrm{i}\frac{\eta_1}{2}\right) \overline{f_k\left(y+\mathrm{i}\frac{\eta_2}{2}\right)} = K'^{,-}\left(\frac{1}{2}(\eta_1+\eta_2); x, y\right).$$

Define more generally a Gaussian process with time parameter $z \in \Pi^+$ as follows:

$$\Gamma'(z) = \sum_{k\geq 0} f_k(z)\xi_k \tag{8}$$

where $(\xi_k)_{k\geq 0}$ are independent standard complex Gaussian variables, i.e. $\mathbf{E}[\xi_j\xi_k] = 0$, $\mathbf{E}[\xi_j\bar{\xi}_k] = \delta_{j,k}$. The Cayley transform $z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$ maps Π^+ to $\mathcal{D}$, where $\mathcal{D}$ stands for the unit disk of the complex plane. This allows us to prove trivially that the series defining Γ' is a random entire series – i.e. a series of the form $\sum_{k\geq 0} a_k z^k \xi_k$, see [13] – which may be shown to be analytic on the unit disk. Hence the process Γ' is analytic on Π^+. Furthermore note that, restricting to the horizontal line $\mathbb{R} + \mathrm{i}\frac{\eta}{2}$, the following identity holds true:

$$\mathbf{E}[\Gamma'(x+\mathrm{i}\eta/2)\overline{\Gamma'(y+\mathrm{i}\eta/2)}] = K'^{,-}(\eta; x, y).$$

One may now integrate the process Γ' over any path $\gamma : (0,1) \to \Pi^+$ with endpoints $\gamma(0) = 0$ and $\gamma(1) = z \in \Pi^+ \cup \mathbb{R}$ (the result does not depend on the particular path but only on the endpoint z). The result is a process Γ which is still analytic on Π^+, and let us stress the (obvious) fact that Γ' is the derivative of Γ on Π^+. Furthermore, one may retrieve the fractional Brownian motion by considering the real part of the boundary value of Γ on $\mathbb{R}$. Another way to look at it is to define $\Gamma_t(\eta) := \Gamma(t+\mathrm{i}\eta)$ as a regular process living on $\mathbb{R}$, and to remark that the real part of $\Gamma(\eta)$ converges when $\eta \to 0$ to fBm. In the following Proposition, we give precise statements which summarize what has been said until now:

Proposition 5 (*See [26,25]*). *Let Γ' be the process defined on Π^+ by relation (8).*

(1) *Let $\gamma : (0,1) \to \Pi^+$ be a continuous path with endpoints $\gamma(0) = 0$ and $\gamma(1) = z$, and set $\Gamma_z = \int_\gamma \Gamma'_u \, \mathrm{d}u$. Then Γ is an analytic process on Π^+. Furthermore, as z runs along any path in Π^+ going to $t \in \mathbb{R}$, the random variables Γ_z converge almost surely to a random variable called again Γ_t.*
(2) *The family $\{\Gamma_t;\ t \in \mathbb{R}\}$ defines a centered Gaussian complex-valued process whose paths are almost surely κ-Hölder continuous for any $\kappa < H$. Its real part $B_t := 2\Re\Gamma_t$ has the same law as fBm.*
(3) *The family of centered Gaussian real-valued processes $B_t(\eta) := 2\Re\Gamma_{t+\mathrm{i}\eta}$ converges a.s. to B_t in α-Hölder norm for any $\alpha < H$, on any interval of the form $[0,T]$ for an arbitrary constant $T > 0$. Its infinitesimal covariance kernel $\mathbf{E}B'_x(\eta)B'_y(\eta)$ is $K'(\eta; x, y)$.*

2.2. Definition of the Lévy area

Let us describe a natural possible definition of the Lévy area associated to Γ. Since the process $B_t(\eta) := 2\Re\Gamma_{t+\mathrm{i}\eta}$ is a smooth one, one can define the following integral in the Riemann sense for all $0 \le s < t$ and $\eta > 0$:

$$\mathcal{A}_{st}(\eta) = \int_s^t \mathrm{d}B_{u_1}^{(2)}(\eta) \int_s^{u_1} \mathrm{d}B_{u_2}^{(1)}(\eta). \tag{9}$$

It turns out that $\mathcal{A}(\eta)$ converges in some Hölder spaces, in a sense which can be specified as follows. Let T be an arbitrary positive constant, $\mathcal{C}_j$ be the set of continuous complex-valued functions defined on $[0, T]^j$, and for $\mu > 0$, define a space $\mathcal{C}_2^\mu$ of μ-Hölder functions on $[0, T]^2$ by

$$\|f\|_\mu := \sup_{s,t\in[0,T]} \frac{|f_{st}|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \left\{f \in \mathcal{C}_2(\Omega; V);\ \|f\|_\mu < \infty\right\}. \tag{10}$$

The μ-Hölder semi-norm for a function $g \in \mathcal{C}_1$ is then defined by setting $h_{st} = g_t - g_s$ as an element of $\mathcal{C}_2$, and $\|g\|_\mu := \|h\|_\mu$ in the sense given by (10).

According to [26,25], the Lévy area $\mathcal{A}$ of B can then be defined in the following way:

Proposition 6. *Let $T > 0$ be an arbitrary constant, and for $s, t \in [0, T]^2$, $\eta > 0$, define $\mathcal{A}_{st}(\eta)$ as in Eq. (9). Consider also $0 < \gamma < H$. Then:*

(1)*For any $p \ge 1$, the couple $(B(\eta), \mathcal{A}(\eta))$ converges when $\eta \to 0$ in $L^p(\Omega; \mathcal{C}_1^\gamma([0, T]; \mathbb{R}) \times \mathcal{C}_2^{2\gamma}([0, T]^2; \mathbb{R}))$ to a couple $(B, \mathcal{A})$, where B is a fractional Brownian motion.*

(2)*The increment $\mathcal{A}$ satisfies the following algebraic relation:*

$$\mathcal{A}_{st} - \mathcal{A}_{su} - \mathcal{A}_{ut} = \left(B_t^{(2)} - B_u^{(2)}\right)\left(B_u^{(1)} - B_s^{(1)}\right),$$

for $s, u, t \in [0, T]$.

Notice that the algebraic property (2) in Proposition 6 is the one which qualifies $\mathcal{A}$ to be a reasonable definition of the Lévy area of B.

It will be essential for us to estimate the moments of $\mathcal{A}$. For this we will use the following definition:

Definition 7. For $\eta > 0$ and $a_1, a_2 \in \mathbb{R}$, let us define the function $K_{a_1,a_2}(\eta; \cdot, \cdot)$ on $\mathbb{R} \times \mathbb{R}$ by

$$K_{a_1,a_2}(\eta; x_1, x_2) = \int_{a_1}^{x_1} \mathrm{d}y_1 \int_{a_2}^{x_2} \mathrm{d}y_2 K'(\eta; y_1, y_2). \tag{11}$$

Notice then that, invoking the conventions of Definition 4, we have

$$K_{a_1,a_2}(\eta; x_1, x_2) = \frac{1}{4\cos(\pi H)}\big([x_1 - x_2]_\eta^{2H} - [x_1 - a_2]_\eta^{2H} - [a_1 - x_2]_\eta^{2H} + [a_1 - a_2]_\eta^{2H}\big). \tag{12}$$

We also state the classical Wick lemma for further use.

Proposition 8. *Let $Z = (Z_1, \ldots, Z_{2N})$ be a centered Gaussian vector. Then*

$$\mathbf{E}[Z_1 \cdots Z_{2N}] = \sum_{(i_1,i_2),\ldots,(i_{2N-1},i_{2N})} \prod_{j=1}^{N} \mathbf{E}[Z_{i_{2j}} Z_{i_{2j+1}}] \tag{13}$$

where the sum ranges over the $(2N-1)!! = 1 \cdot 3 \cdot 5 \cdots (2N-1)$ couplings of the indices $1, \ldots, 2N$.

We can now give the announced expression for the moments of $\mathcal{A}(\eta)$ (recall that $\mathcal{A}(\eta)$ is defined by (9)):

Lemma 9. *Let $N \geq 1$ and $\{s_i, t_i;\ i \leq 2N\}$ be a family of real numbers satisfying $s_i < t_i$. Then*

$$\begin{aligned}\mathbf{E}\left[\prod_{j=1}^{2N} \mathcal{A}_{s_j,t_j}(\eta)\right] &= \int_{s_1}^{t_1} \mathrm{d}x_1 \cdots \int_{s_{2N}}^{t_{2N}} \mathrm{d}x_{2N} \\ &\quad \times \sum_{(i_1,i_2),\ldots,(i_{2N-1},i_{2N})} \ \sum_{(j_1,j_2),\ldots,(j_{2N-1},j_{2N})} \prod_{k=1}^{N} K'(\eta; x_{i_{2k-1}}, x_{i_{2k}}) \\ &\quad \times \prod_{k=1}^{N} K_{s_{j_{2k-1}},s_{j_{2k}}}(\eta; x_{j_{2k-1}}, x_{j_{2k}}).\end{aligned} \tag{14}$$

Proof. By definition of the approximation $\mathcal{A}(\eta)$, we have

$$\mathbf{E}\left[\prod_{j=1}^{2N} \mathcal{A}_{s_j,t_j}(\eta)\right] = \prod_{j=1}^{2N} \int_{s_j}^{t_j} \mathrm{d}x_j \int_{s_j}^{x_j} \mathrm{d}y_j \mathbf{E}\left[B_{x_1}^{\prime(1)}(\eta) B_{y_1}^{\prime(2)}(\eta) \cdots B_{x_{2N}}^{\prime(1)}(\eta) B_{y_{2N}}^{\prime(2)}(\eta)\right]. \tag{15}$$

Our claim follows then from a direct application of Proposition 5 point (3), Proposition 8 and Definition 7. □

Remark 10. As a side result of our Theorem 2, we also obtain that the fractional Lévy area $\mathcal{A}$ constructed by analytic approximation (see Proposition 6) coincides with the one constructed in [6] by linear interpolations of the fBm B, and thus also with the area obtained by Malliavin calculus techniques as in e.g. [21].

2.3. Analytic preliminaries

We gather here some elementary integral estimates which turn out to be essential for our computations. The first one concerns the behavior of the kernel K_{a_1,a_2} given in Definition 4 when $|a_1 - x_1|, |a_2 - x_2|$ are of order 1 and $|x_1 - x_2|$ is large.

Lemma 11. *Assume $\eta, |x_1 - a_1|, |x_2 - a_2| \leq 1$ and $|x_1 - x_2|, |a_1 - x_2|, |a_2 - x_1|, |a_1 - a_2|$ are bounded from below by a positive constant C. Then*

$$|K_{a_1,a_2}(\eta; x_1, x_2)| \leq C \left(\min(|x_1 - x_2|, |a_1 - x_2|, |a_2 - x_1|, |a_1 - a_2|)\right)^{2H-2}.$$

Proof. The proof is elementary using the integral expression (12) for K_{a_1,a_2}. □

We shall also need to estimate convolution integrals of the form $\int_0^t K'(\eta; z, u) f(u)\, \mathrm{d}u$ or $\int_0^t K_{a,b}(\eta; z, u) f(u)\, \mathrm{d}u$. The following lemma gives a precise answer when f is analytic on a neighborhood of (a, b) for two given constants $a, b \in \mathbb{R}$, and multivalued with a power behavior near a and b.

Lemma 12 (*See [27]*). *Fix two real constants a, b with $a < b$, and let f be a function in $L^1([a, b], \mathbb{C})$. Define another function ϕ by $\phi : z \mapsto \int_a^b (-\mathrm{i}(z-u))^\beta (u-a)^\gamma f(u)\,\mathrm{d}u$ with $\gamma > -1$ and $\beta + \gamma \in \mathbb{R} \setminus \mathbb{Z}$. Then:*

(1) *Assume f is analytic in a (complex) neighborhood of $s \in (a, b)$. Then ϕ has an analytic extension to a complex neighborhood of s.*
(2) *Assume f is analytic in a complex neighborhood of a. Then ϕ may be written on a small enough neighborhood of a as the multivalued function*

$$\phi(z) = (z-a)^{\beta+\gamma+1} F(z) + G(z) \tag{16}$$

where both F and G are analytic.
(3) *More precisely, the following continuity property holds: let Ω be a complex neighborhood of $[a, b]$ and $\varepsilon \in (0, 1/2)$. If f is analytic on a relatively compact domain $\tilde{\Omega}$ containing the closure $\bar{\Omega}$ of Ω, then ϕ extends analytically to the cut domain $\Omega_{\text{cut}} := \Omega \setminus ((a + \mathbb{R}_-) \cup (b + \mathbb{R}_+))$ and writes $(z-a)^{\beta+\gamma+1} F(z) + G(z)$ on $B(a, \varepsilon(b-a))$ (F, G analytic) with*

$$\sup_{\Omega_{\text{cut}} \setminus (B(0,\varepsilon(b-a)) \cup B(b,\varepsilon(b-a)))} |\phi|, \quad \sup_{B(a,\varepsilon(b-a))} |F|, \quad \sup_{B(a,\varepsilon(b-a))} |G| \le C \sup_{\tilde{\Omega}} |f| \tag{17}$$

for some constant C which does not depend on f.

Proof. Points (1) and (2) follow directly from [27], Lemmas 3.2 and 3.3. Point (3) may be shown very easily by following the proof of the above two lemmas step by step and using the analyticity of f. Note that (under the hypotheses of (3)) ϕ is analytic on the larger domain $\tilde{\Omega} \setminus ((a + \mathbb{R}_-) \cup (b + \mathbb{R}_+))$, but the method of contour deformation used in the proof gives a bound for $\phi(z)$ which goes to infinity when z comes closer and closer to the boundary of $\tilde{\Omega}$ (hence the need for the relatively compact inclusion of Ω into $\tilde{\Omega}$). □

We shall also need the following elementary lemma. Here and later on, we will write $x \lesssim y$ for $x, y \in \mathbb{R}$, if there exists a constant $C > 0$ such that $x \le C \cdot y$.

Lemma 13. *Let $\alpha, \beta > -1$ and $0 < a < b < 1$. Then:*

$$\int_0^1 |t-a|^\alpha |t-b|^\beta \,\mathrm{d}t \lesssim 1 + |a-b|^{\alpha+\beta+1}. \tag{18}$$

Proof. Let $\sigma_a(b) = \max(0, 2a-b)$ and $\sigma_b(a) = \min(1, 2b-a)$. Split the above integral into $\int_0^{\sigma_a(b)} + \int_{\sigma_a(b)}^a + \int_a^b + \int_b^{\sigma_b(a)} + \int_{\sigma_b(a)}^1$. We show that the integral over each subinterval is $\lesssim 1 + |a-b|^{\alpha+\beta+1}$ (by symmetry, it is sufficient to check this for the three first subintervals only). Now, a simple study of the function $t \mapsto |t-a|/|t-b|$ shows that $c < \frac{|t-a|}{|t-b|} < C$ on $[0, \sigma_a(b)]$, so

$$\int_0^{\sigma_a(b)} |t-a|^\alpha |t-b|^\beta \,\mathrm{d}t \lesssim \int_0^{\sigma_a(b)} (t-b)^{\alpha+\beta} \,\mathrm{d}t \lesssim (b-a)^{\alpha+\beta+1} + b^{\alpha+\beta+1}. \tag{19}$$

If $\alpha + \beta + 1 < 0$, resp. $\alpha + \beta + 1 > 0$, then this is $\lesssim (b-a)^{\alpha+\beta+1}$, resp. $\lesssim 1$. On $[\sigma_a(b), a]$, one has $c < \frac{|t-b|}{b-a} < C$ this time, so

$$\int_{\sigma_a(b)}^a |t-a|^\alpha |t-b|^\beta \,\mathrm{d}t \lesssim (b-a)^{\alpha+\beta+1}. \tag{20}$$

Finally, $\int_a^b |t-a|^\alpha |t-b|^\beta \, \mathrm{d}t = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}(b-a)^{\alpha+\beta+1}$, where Γ is the Gamma function. □

3. Mean square error computations

This section is devoted to prove Theorems 1 and 2. We will start with the error of the Euler scheme for $H \in (1/4, 3/4)$, then we will consider the trapezoidal scheme and we will conclude with the error of the Euler scheme for $H \in (3/4, 1)$. Throughout this section and the remainder of this article, we will use extensively the self-similarity or scaling property of fBm, i.e. for any $c > 0$ the process

$$\widetilde{B}^{(i)}_{\cdot} = c^H B^{(i)}_{\cdot/c} \tag{21}$$

is again an fBm, and the stationarity property, that is for any $h \in \mathbb{R}$ the process

$$\widetilde{B}^{(i)}_{\cdot} = B^{(i)}_{\cdot+h} - B^{(i)}_h \tag{22}$$

is an fBm. As a consequence we have e.g. that

$$\int_s^t (B^{(1)}_u - B^{(1)}_s)\,\mathrm{d}B^{(2)}_s \overset{\mathcal{L}}{=} \int_0^{t-s} B^{(1)}_u\,\mathrm{d}B^{(2)}_s \overset{\mathcal{L}}{=} (t-s)^{2H}\int_0^1 B^{(1)}_u\,\mathrm{d}B^{(2)}_s, \quad 0 \le s \le t \le T.$$

3.1. Some moment estimates

The preliminary results we need for the proof of Theorem 1 are summarized in the following Lemma:

Lemma 14. *Let $\mathcal{A}_{k,k+1} = \int_k^{k+1} (B^{(1)}_s - B^{(1)}_k)\,\mathrm{d}B^{(2)}_s$, $k \in \mathbb{N}$, be the iterated integrals obtained by applying Proposition 6. Define*

$$c_0 = \mathbf{E}\left[\left|\mathcal{A}_{0,1}\right|^2\right] \quad \textit{and} \quad c_k = \mathbf{E}\left[\mathcal{A}_{0,1}\mathcal{A}_{k,k+1}\right].$$

Then we have

$$c_0 = \frac{H}{2}\left(\beta(2H,2H) + \frac{1}{4H-1}\right), \tag{23}$$

and

$$\begin{aligned} c_k = {} & \frac{H}{4(4H-1)}((k+1)^{4H} - 2k^{4H} + (k-1)^{4H}) \\ & - \frac{1}{4}k^{2H}((k+1)^{2H} - 2k^{2H} + (k-1)^{2H}) \\ & + \frac{H}{2}\int_k^{k+1} (|x-1|^{2H}|x|^{2H-1} - |x|^{2H}|x-1|^{2H-1})\,\mathrm{d}x. \end{aligned} \tag{24}$$

Proof. Both identities are obtained thanks to the same kind of considerations. Furthermore, relation (23) is obtained in [1, Theorem 34] or [26]. We thus focus on identity (24).

Recall that c_k can be obtained as a limit of $c_k(\eta)$ when $\eta \to 0$, where $c_k(\eta)$ is given by:

$$\begin{aligned} c_k(\eta) &:= \mathbf{E}\left[\int_0^1 B_s^{(1)}(\eta)\, \mathrm{d}B_s^{(2)}(\eta) \int_k^{k+1} \left(B_s^{(1)}(\eta) - B_k^{(1)}(\eta)\right) \mathrm{d}B_s^{(2)}(\eta)\right] \\ &= \mathbf{E}\left[\mathcal{A}_{0,1}(\eta)\, \mathcal{A}_{k,k+1}(\eta)\right]. \end{aligned}$$

We can thus apply identity (14) with $N = 1$, $s_1 = 0$, $t_1 = 1$, $s_2 = k$, $t_2 = k + 1$, use expression (12) for the kernel K, and let $\eta \to 0$ in order to obtain:

$$\begin{aligned} c_k &= \frac{1}{2}\gamma_H \int_k^{k+1} \int_0^1 |s_1 - s_2|^{2H-2} \left(s_1^{2H} - k^{2H} - |s_1 - s_2|^{2H} + |s_2 - k|^{2H}\right) \mathrm{d}s_2\, \mathrm{d}s_1 \\ &:= c_{k,1} + c_{k,2} + c_{k,3} + c_{k,4}, \end{aligned}$$

with $\gamma_H := H(2H - 1)$. It should be noticed here that, since we are integrating on the rectangle $[0, 1] \times [k, k + 1]$ with $k \geq 1$, the limits as $\eta \to 0$ can be taken without much care about singularities of our kernels $[x]_\eta^\beta$ for negative β's. Moreover, direct calculations yield

$$\begin{aligned} c_{k,1} &= \frac{1}{2}\gamma_H \int_k^{k+1} \int_0^1 s_1^{2H} |s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= \frac{H}{2} \int_k^{k+1} s_1^{2H} (s_1^{2H-1} - |s_1 - 1|^{2H-1})\, \mathrm{d}s_1 \\ &= \frac{1}{8}((k+1)^{4H} - k^{4H}) - \frac{H}{2} \int_k^{k+1} x^{2H} |x - 1|^{2H-1}\, \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} c_{k,2} &= -\frac{1}{2}\gamma_H k^{2H} \int_k^{k+1} \int_0^1 |s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= -\frac{1}{2} k^{2H} \mathbf{E}\left[(B_{k+1}^{(1)} - B_k^{(1)}) B_1^{(1)}\right] \\ &= -\frac{1}{4} k^{2H} ((k+1)^{2H} - 2k^{2H} + (k-1)^{2H}). \end{aligned}$$

Finally, we have

$$\begin{aligned} c_{k,3} &= -\frac{1}{2}\gamma_H \int_k^{k+1} \int_0^1 |s_1 - s_2|^{4H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= -\frac{\gamma_H}{2(4H-1)} \int_k^{k+1} (s_1^{4H-1} - |s_1 - 1|^{4H-1})\, \mathrm{d}s_1 \\ &= -\frac{2H-1}{8(4H-1)} ((k+1)^{4H} - 2k^{4H} - (k-1)^{4H}) \end{aligned}$$

and

$$\begin{aligned} c_{k,4} &= \frac{1}{2}\gamma_H \int_k^{k+1} \int_0^1 |s_2 - k|^{2H} |s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= \frac{H}{2} \int_0^1 |s_2 - k|^{2H} (|k + 1 - s_2|^{2H-1} - |k - s_2|^{2H-1})\, \mathrm{d}s_2 \\ &= \frac{1}{8}(-k^{4H} + (k-1)^{4H}) + \frac{H}{2} \int_k^{k+1} |x - 1|^{2H} |x|^{2H-1}\, \mathrm{d}x. \end{aligned}$$

Hence, putting together our elementary calculations for $c_{k,1}, \dots, c_{k,4}$, expression (24) follows easily. □

3.2. *Proof of* Theorem 1 *for* $H \in (1/4, 3/4)$

The case $H = 1/2$ is well known, thus we omit this case. Note that we can decompose the error of the Euler scheme as $X_T - X_T^n = \sum_{i=1}^n J_i^n$, with random variables J_i^n defined by

$$J_i^n = \int_{iT/n}^{(i+1)T/n} (B_s^{(1)} - B_{i/n}^{(1)}) \, \mathrm{d}B_s^{(2)} = \mathcal{A}_{(iT)/n,(i+1)T/n}, \quad i = 0, \dots, n-1,$$

where $\mathcal{A}_{st}$ is obtained as the L^2-limit of $\mathcal{A}_{st}(\eta)$ according to Proposition 6. In particular, $\mathbf{E}[|X_T - X_T^n|^2] = \sum_{i,j} \mathbf{E}[J_i^n J_j^n]$, which means that we are first reduced to study the quantities $\mathbf{E}[J_i^n J_j^n]$ in terms of i, j and n. Towards this aim, one can first remark that, since fBm is self-similar, we have

$$\mathbf{E}[J_i^n J_j^n] = T^{4H} n^{-4H} \mathbf{E}[\mathcal{A}_{i,i+1} \mathcal{A}_{j,j+1}]. \tag{25}$$

Moreover, as in the proof of Lemma 9, for $|i - j| > 1$ it holds:

$$\mathbf{E}\left[\mathcal{A}_{i,i+1}(\eta)\mathcal{A}_{j,j+1}(\eta)\right] = \int_i^{i+1} \int_j^{j+1} \int_i^{s_1} \int_j^{s_2} K'(\eta; s_1, s_2) \times K'(\eta; u_1, u_2) \, \mathrm{d}u_2 \, \mathrm{d}u_1 \, \mathrm{d}s_2 \, \mathrm{d}s_1.$$

Since we are away from the diagonal, one can take safely the limit $\eta \to 0$ in the expression above, which gives:

$$\mathbf{E}\left[\mathcal{A}_{i,i+1} \, \mathcal{A}_{j,j+1}\right] = H^2(2H-1)^2 \times \int_i^{i+1} \int_j^{j+1} \int_i^{s_1} \int_j^{s_2} |u_1 - u_2|^{2H-2} |s_1 - s_2|^{2H-2} \, \mathrm{d}u_2 \, \mathrm{d}u_1 \, \mathrm{d}s_2 \, \mathrm{d}s_1. \tag{26}$$

Note that the above expression implies that

$$\mathbf{E}\left[\mathcal{A}_{i,i+1} \, \mathcal{A}_{j,j+1}\right] > 0.$$

Step 1: *Diagonal terms.* For the diagonal terms, i.e. for $i = j$, we have by stationarity of the increments that

$$\mathbf{E}\left[\left|\mathcal{A}_{i,i+1}\right|^2\right] = \mathbf{E}\left[\left|\mathcal{A}_{0,1}\right|^2\right].$$

So (25) gives

$$\sum_{i=0}^{n-1} \mathbf{E}\left[|J_i^n|^2\right] = T^{4H} \cdot \mathbf{E}\left[\left|\mathcal{A}_{0,1}\right|^2\right] \cdot n^{-4H+1}. \tag{27}$$

Step 2: *Secondary diagonal terms.* For the secondary diagonal terms, we have again by stationarity of the increments of fBm that

$$\mathbf{E}\left[\mathcal{A}_{i,i+1}\mathcal{A}_{i+1,i+2}\right] = \mathbf{E}\left[\mathcal{A}_{0,1} \, \mathcal{A}_{1,2}\right].$$

Hence it follows

$$\sum_{i,j=0,|i-j|=1}^{n-1} \mathbf{E}\left[J_i^n J_{i+1}^n\right] = 2T^{4H} \cdot \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{1,2}\right] \cdot (n-1)n^{-4H}. \tag{28}$$

Step 3: *Close to diagonal terms.* Now we consider the terms, which are in a "log(n)-vicinity" of the diagonal terms, i.e.

$$\sum_{\substack{0\le i,j\le n-1\\ 1<|i-j|\le \log(n)}} \mathbf{E}\left[J_i^n J_{i+1}^n\right] = T^{4H} \cdot n^{-4H} \cdot \sum_{\substack{0\le i,j\le n-1\\ 1<|i-j|\le \log(n)}} \mathbf{E}\left[\mathcal{A}_{i,i+1}\,\mathcal{A}_{j,j+1}\right].$$

Using the stationarity of the fBm we have

$$\begin{aligned}\sum_{\substack{0\le i,j\le n-1\\ 1<|i-j|\le \log(n)}} \mathbf{E}\left[\mathcal{A}_{i,i+1}\,\mathcal{A}_{j,j+1}\right] &= 2 \sum_{\substack{0\le i,j\le n-1\\ 1<i-j\le \log(n)}} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{i-j,i-j+1}\right]\\ &= \sum_{k=2}^{\lfloor \log(n)\rfloor} (n-k)\mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right].\end{aligned}$$

Thus

$$\begin{aligned}\sum_{\substack{0\le i,j\le n-1\\ 1<|i-j|\le \log(n)}} \mathbf{E}\left[J_i^n J_{i+1}^n\right] &= T^{4H} \cdot n^{-4H+1} \cdot \sum_{k=2}^{\lfloor \log(n)\rfloor} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right]\\ &\quad - T^{4H} \cdot n^{-4H+1} \cdot \sum_{k=2}^{\lfloor \log(n)\rfloor} \frac{k}{n}\mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right].\end{aligned}$$

From Eq. (26) we have that

$$\begin{aligned}&\mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right]\\ &\quad = H^2(2H-1)^2 \int_0^1 \int_k^{k+1} \int_0^{s_1} \int_k^{s_2} |u_1-u_2|^{2H-2}|s_1-s_2|^{2H-2}\,\mathrm{d}u_2\,\mathrm{d}u_1\,\mathrm{d}s_2\,\mathrm{d}s_1.\end{aligned}$$

An application of the mean value theorem for $k > 1$ gives

$$\frac{1}{4}H^2(2H-1)^2|k+1|^{4H-4} \le \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right] \le \frac{1}{4}H^2(2H-1)^2|k-1|^{4H-4}.$$

Consequently, we have

$$\begin{aligned}\frac{1}{4}H^2(2H-1)^2 \sum_{k=3}^{\lfloor \log(n)\rfloor+1} |k|^{4H-4} &\le \sum_{k=2}^{\lfloor \log(n)\rfloor} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right]\\ &\le \frac{1}{4}H^2(2H-1)^2 \sum_{k=1}^{\lfloor \log(n)\rfloor-1} |k|^{4H-4},\end{aligned}$$

and notice that $\sum_{k=1}^{\infty} |k|^{4H-4} < \infty$ since $H < 3/4$. So, $\sum_{k=2}^{\lfloor \log(n)\rfloor} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right]$ converges as $n \to \infty$, and setting $S = \sum_{k=2}^{\infty} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right] = \sum_{k=2}^{\infty} c_k$ we end up with

$$\sum_{k=2}^{\lfloor \log(n)\rfloor} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right] = S + o(1).$$

Moreover, we have

$$0 \le \sum_{k=2}^{\lfloor \log(n) \rfloor} \frac{k}{n} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right] \le \frac{\log(n)}{n}\, S = O(\log(n) n^{-1}).$$

Consequently, we have derived that

$$\sum_{\substack{0 \le i,j \le n-1 \\ 1<|i-j|\le \log(n)}} \mathbf{E}\left[J_i^n J_j^n\right] = 2T^{4H} \cdot n^{-4H+1} \cdot \sum_{k=2}^{\infty} \mathbf{E}\left[\mathcal{A}_{0,1}\,\mathcal{A}_{k,k+1}\right] + o(n^{-4H+1}). \tag{29}$$

Step 4: *Off-diagonal terms.* Now, it remains to consider the off-diagonal terms, i.e. the terms with $|i-j| > \log(n)$. We show that these terms are asymptotically negligible for $H < 3/4$. Proceeding as in the previous steps, we have

$$\sum_{\substack{0 \le i,j \le n-1 \\ |i-j|> \log(n)}} \mathbf{E}\left[J_i^n J_{i+1}^n\right] = T^{4H} \cdot n^{-4H} \cdot \sum_{\substack{0 \le i,j \le n-1 \\ |i-j|> \log(n)}} \mathbf{E}\left[\mathcal{A}_{i,i+1}\,\mathcal{A}_{j,j+1}\right]$$

and

$$0 \le \mathbf{E}\left[\mathcal{A}_{i,i+1}\,\mathcal{A}_{j,j+1}\right] \le H^2(2H-1)^2 \left(\int_i^{i+1}\int_j^{j+1} |s_1-s_2|^{2H-2}\,\mathrm{d}s_1\,\mathrm{d}s_2\right)^2.$$

Since $|i-j| > 1$, the mean value theorem gives

$$0 \le \mathbf{E}\left[\mathcal{A}_{i,i+1}\,\mathcal{A}_{j,j+1}\right] \le H^2(2H-1)^2 ||i-j|-1|^{4H-4},$$

and we obtain

$$\begin{aligned}\sum_{\substack{0 \le i,j \le n-1 \\ |i-j|> \log(n)}} ||i-j|-1|^{4H-4} &= 2\sum_{i=0}^{n-1}\sum_{j=0}^{i-\lceil \log(n)\rceil} |i-j-1|^{4H-4} \\ &= 2\sum_{i=0}^{n-1}\sum_{j=\lceil \log(n)\rceil -1}^{i-1} j^{4H-4} \\ &\le 2n \sum_{j=\lfloor \log(n)\rfloor}^{n-1} j^{4H-4}.\end{aligned}$$

Since $H < 3/4$, we have

$$\sum_{j=\lfloor \log(n)\rfloor}^{n-1} j^{4H-4} \le \sum_{j=\lfloor \log(n)\rfloor}^{\infty} j^{4H-4} = O(\log(n)^{4H-3}).$$

Hence it follows

$$\sum_{\substack{0 \le i,j \le n-1 \\ |i-j|> \log(n)}} \mathbf{E}\left[J_i^n J_j^n\right] = o(n^{-4H+1}). \tag{30}$$

Step 5: *The asymptotic constant.* Combining (27)–(30) and using Lemma 14 now gives

$$\mathbf{E}[|X_T - X_T^n|^2] = T^{4H} \cdot n^{-4H+1} \left(c_0 + 2\sum_{k=1}^{\infty} c_k \right) + o(n^{-4H+1}),$$

which finishes the proof of Theorem 1 for $H \in (1/4, 3/4)$.

3.3. *Proof of* Theorem 2

Here we omit again the case $H = 1/2$, which is well known. Recall that the trapezoidal scheme is given by

$$\widehat{X}_T^n = \frac{1}{2} \sum_{i=0}^{n-1} \big(B^{(1)}_{iT/n} + B^{(1)}_{(i+1)T/n}\big)\big(B^{(2)}_{(i+1)T/n} - B^{(2)}_{iT/n}\big).$$

Thus, applying again the scaling property of fBm, the mean square error of the trapezoidal scheme is given by

$$\mathbf{E}|X_T - \widehat{X}_T^n|^2 = T^{4H} \cdot n^{-4H} \cdot \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \mathbf{E}\widetilde{\mathcal{A}}_{i,i+1}\widetilde{\mathcal{A}}_{j,j+1}$$

with

$$\widetilde{\mathcal{A}}_{i,i+1} = \frac{1}{2} \int_i^{i+1} (B^{(1)}_{i+1} + B^{(1)}_i - 2B^{(1)}_s)\, \mathrm{d}B^{(2)}_s$$

for $i, j = 0, 1, \dots, n-1$. We will use a similar decomposition as for the Euler scheme, i.e.

$$\mathbf{E}|X_T - \widehat{X}_T^n|^2 = T^{4H} \cdot n^{-4H} \,(I_1(n) + I_2(n) + I_3(n))$$

with

$$I_1(n) = \sum_{i=0}^{n-1} \mathbf{E}\left[|\widetilde{\mathcal{A}}_{i,i+1}|^2\right],$$

$$I_2(n) = \sum_{i=0, j=0, |i-j|=1}^{n-1} \mathbf{E}\left[\widetilde{\mathcal{A}}_{i,i+1}\widetilde{\mathcal{A}}_{j,j+1}\right],$$

$$I_3(n) = \sum_{i=0, j=0, |i-j|>1}^{n-1} \mathbf{E}\left[\widetilde{\mathcal{A}}_{i,i+1}\widetilde{\mathcal{A}}_{j,j+1}\right].$$

Moreover, note that as in the proof of Lemma 14, using Lemma 9 and moreover letting $\eta \to 0$ and applying dominated convergence, we obtain for $|i - j| > 1$ that

$$\mathbf{E}\left[\widetilde{\mathcal{A}}_{i,i+1}\widetilde{\mathcal{A}}_{j,j+1}\right] = \gamma_H \int_i^{i+1} \int_j^{j+1} \theta_{i,j}(s_1, s_2)|s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 \tag{31}$$

with $\gamma_H = H(2H-1)$ and

$$\theta_{i,j}(s_1, s_2) = \frac{1}{4}\mathbf{E}(2B^{(1)}_{s_1} - B^{(1)}_i - B^{(1)}_{i+1})(2B^{(1)}_{s_2} - B^{(1)}_j - B^{(1)}_{j+1})$$

for $s_1 \in [i, i+1]$, $s_2 \in [j, j+1]$.

Step 1: *Diagonal terms.* For the diagonal terms, i.e. for $i = j$, we have by stationarity of the increments of fBm that

$$I_1(n) = n \cdot \mathbf{E}\left[|\widetilde{\mathcal{A}}_{0,1}|^2\right] = O(n). \tag{32}$$

Step 2: *Secondary diagonal terms.* For the secondary diagonal terms it holds

$$I_2(n) = 2(n-1) \cdot \mathbf{E}\left[\widetilde{\mathcal{A}}_{0,1}\widetilde{\mathcal{A}}_{1,2}\right] = O(n). \tag{33}$$

Step 3: *Off-diagonal terms.* Again stationarity gives

$$\begin{aligned} I_3(n) &= \sum_{i=0,\, j=0,\, |i-j|>1}^{n-1} \mathbf{E}\left[\widetilde{\mathcal{A}}_{i,i+1}\, \widetilde{\mathcal{A}}_{j,j+1}\right] \\ &= 2\sum_{i=0}^{n-1}\sum_{j=0}^{i-2} \mathbf{E}\left[\widetilde{\mathcal{A}}_{0,1}\, \widetilde{\mathcal{A}}_{i-j,i-j+1}\right] \\ &= 2\sum_{k=2}^{n}(n-k)\mathbf{E}\left[\widetilde{\mathcal{A}}_{0,1}\, \widetilde{\mathcal{A}}_{k,k+1}\right]. \end{aligned} \tag{34}$$

Now (31) gives

$$\mathbf{E}\left[\widetilde{\mathcal{A}}_{0,1}\widetilde{\mathcal{A}}_{k,k+1}\right] = \gamma_H \int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)|s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1.$$

Setting

$$R^1_{0,k} = \{s_1, s_2 \in [0,1] \times [k, k+1] : \theta_{0,k}(s_1, s_2) \geq 0\},$$
$$R^2_{0,k} = \{s_1, s_2 \in [0,1] \times [k, k+1] : \theta_{0,k}(s_1, s_2) < 0\},$$

an application of the mean value theorem yields

$$\begin{aligned} &\int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)|s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &\quad = |k+\xi_1|^{2H-2} \int\int_{R^1_{0,k}} \theta_{0,k}(s_1, s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1 + |k+\xi_2|^{2H-2} \int\int_{R^2_{0,k}} \theta_{0,k}(s_1, s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1 \end{aligned}$$

with $\xi_1, \xi_2 \in (-1, 1)$. Thus

$$\int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)|s_1 - s_2|^{2H-2}\, \mathrm{d}s_2\, \mathrm{d}s_1 = |k|^{2H-2} \int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1 + \rho_k,$$

where

$$|\rho_k| \leq |2H-2| \cdot |k-1|^{2H-3} \cdot \int_0^1 \int_k^{k+1} |\theta_{0,k}(s_1, s_2)|\, \mathrm{d}s_2\, \mathrm{d}s_1.$$

Inserting this into (34) gives

$$I_3(n) = 2\gamma_H \sum_{k=2}^{n}(n-k)|k|^{2H-2} \int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1 + 2\gamma_H \sum_{k=2}^{n}(n-k)\rho_k.$$

Now note that $\theta_{i,j}$ can be bounded as follows: We have

$$\begin{aligned}\left|\theta_{i,j}(s_1,s_2)\right| &\le \frac{1}{4}\left(\mathbf{E}|(B_{s_1}-B_i)+(B_{s_1}-B_{i+1})|^2\right)^{1/2}\\ &\quad\times\left(\mathbf{E}|(B_{s_2}-B_j)+(B_{s_2}-B_{j+1})|^2\right)^{1/2}\\ &\le \frac{1}{4}\left(\left(\mathbf{E}|B_{s_1}-B_i|^2\right)^{1/2}+\left(\mathbf{E}|B_{s_1}-B_{i+1}|^2\right)^{1/2}\right)\\ &\quad\times\left(\left(\mathbf{E}|B_{s_2}-B_j|^2\right)^{1/2}+\left(\mathbf{E}|B_{s_2}-B_{j+1}|^2\right)^{1/2}\right)\end{aligned}$$

and therefore

$$\begin{aligned}&\sup_{s_1\in[i,i+1],s_2\in[j,j+1]}\left|\theta_{i,j}(s_1,s_2)\right|\\ &\quad\le \sup_{s_1\in[i,i+1],s_2\in[j,j+1]}\frac{1}{4}\left(|s_1-i|^H+|s_1-i-1|^H\right)\left(|s_2-j|^H+|s_2-j-1|^H\right)\le 1\end{aligned}$$

for all $i, j \in \mathbb{N}$. Hence

$$|\rho_k| \le 2|k-1|^{2H-3}$$

and

$$\left|2\gamma_H\sum_{k=2}^{n}(n-k)\rho_k\right| \le 4n\sum_{k=1}^{\infty}k^{2H-3}.$$

Since $\sum_{k=1}^{\infty}k^{2H-3}<\infty$, it follows

$$2\gamma_H\sum_{k=2}^{n}(n-k)\rho_k = O(n). \tag{35}$$

It remains to consider

$$2\gamma_H\sum_{k=2}^{n}(n-k)|k|^{2H-2}\int_0^1\int_k^{k+1}\theta_{0,k}(s_1,s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{36}$$

From [19] (see Appendix A) we know that

$$\begin{aligned}\int_0^1\int_k^{k+1}\theta_{0,k}(s_1,s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1 &= -\frac{1}{8}\cdot\left(2|k|^{2H}+|k+1|^{2H}+|k-1|^{2H}\right)\\ &\quad+\frac{1}{2(2H+1)}\cdot\left(|k+1|^{2H+1}-|k-1|^{2H+1}\right)\\ &\quad+\frac{1}{2(2H+1)(2H+2)}\cdot\left(2|k|^{2H+2}-|k+1|^{2H+2}-|k-1|^{2H+2}\right).\end{aligned}$$

A Taylor expansion up to order 5 of the right-hand side of the above equation yields

$$\int_0^1\int_k^{k+1}\theta_{0,k}(s_1,s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1 = O(|k-1|^{2H-3})$$

and thus

$$\sum_{k=2}^{\infty} |k|^{2H-2} \left| \int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1 \right| < \infty.$$

This allows us to control the sum in (36), i.e.

$$\left| 2\gamma_H \sum_{k=2}^{n} (n-k)|k|^{2H-2} \int_0^1 \int_k^{k+1} \theta_{0,k}(s_1, s_2)\, \mathrm{d}s_2\, \mathrm{d}s_1 \right| = O(n). \tag{37}$$

So, combining this with (35) we have

$$I_3(n) = O(n). \tag{38}$$

Step 4: *The error bound.* Combining (32), (33) and (38) now gives

$$\mathbf{E}[|X_T - \widehat{X}_T^n|^2] \le C(H) \cdot T^{4H} \cdot n^{-4H+1},$$

which is the assertion of Theorem 2.

3.4. *Proof of* Theorem 1 *for* $H \in [3/4, 1)$

Here the following relation will be very helpful: For the difference between the Euler and the trapezoidal scheme we have

$$X_T^n - \widehat{X}_T^n = -\frac{1}{2} \sum_{i=0}^{n-1} (B^{(1)}_{(i+1)T/n} - B^{(1)}_{iT/n})(B^{(2)}_{(i+1)T/n} - B^{(2)}_{iT/n}).$$

Since

$$\mathbf{E}|X_T - \widehat{X}_T^n|^2 = O(n^{-4H+1})$$

due to Theorem 2, it follows

$$\left(\mathbf{E}|X_T - X_T^n|^2\right)^{1/2} = \left(\mathbf{E}|X_T^n - \widehat{X}_T^n|^2\right)^{1/2} + O(n^{-2H+1/2}). \tag{39}$$

By scaling we have

$$\mathbf{E}|X_T^n - \widehat{X}_T^n|^2 = \frac{T^{4H}}{4} \cdot n^{-4H} \cdot \mathbf{E}\left| \sum_{i=0}^{n-1} (B^{(1)}_{i+1} - B^{(1)}_{i})(B^{(2)}_{i+1} - B^{(2)}_{i}) \right|^2.$$

Moreover,

$$\mathbf{E}\left| \sum_{i=0}^{n-1} (B^{(1)}_{i+1} - B^{(1)}_{i})(B^{(2)}_{i+1} - B^{(2)}_{i}) \right|^2 = \frac{1}{4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(|i-j-1|^{2H} - 2|i-j|^{2H} + |i-j+1|^{2H} \right)^2.$$

So, we have to analyze the behavior of

$$n^{-4H} \cdot \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(|i-j-1|^{2H} - 2|i-j|^{2H} + |i-j+1|^{2H} \right)^2$$

in detail. We can rewrite this sum as

$$n^{-4H} \cdot \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(|i-j-1|^{2H} - 2|i-j|^{2H} + |i-j+1|^{2H} \right)^2$$
$$= 2n^{-4H+1} + 2n^{-4H} \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} \left(|i-j-1|^{2H} - 2|i-j|^{2H} + |i-j+1|^{2H} \right)^2$$
$$= O(n^{-4H+1}) + 2n^{-4H} \sum_{k=2}^{n-1} (n-k) \left(|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H} \right)^2 .$$

Hence,

$$\mathbf{E}|X_T^n - \widehat{X}_T^n|^2 = \frac{T^{4H}}{8} \cdot n^{-4H} \sum_{k=2}^{n-1} (n-k) \left(|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H} \right)^2 + O(n^{-4H+1}). \tag{40}$$

Moreover,

$$n^{-4H} \sum_{k=2}^{n-1} (n-k) \left(|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H} \right)^2$$
$$= n \sum_{k=2}^{n-1} \left(1 - \tfrac{k}{n}\right) \left(\left|\tfrac{k-1}{n}\right|^{2H} - 2\left|\tfrac{k}{n}\right|^{2H} + \left|\tfrac{k+1}{n}\right|^{2H} \right)^2 .$$

A Taylor expansion up to order 4 gives

$$\left|\frac{k-1}{n}\right|^{2H} - 2\left|\frac{k}{n}\right|^{2H} + \left|\frac{k+1}{n}\right|^{2H} = 2H(2H-1)\left|\frac{k}{n}\right|^{2H-2} \cdot n^{-2} + \rho_k \cdot n^{-2H}$$

with

$$|\rho_k| = O(|k-1|^{2H-4})$$

for $k > 1$. Therefore

$$n \sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left(\left|\frac{k-1}{n}\right|^{2H} - 2\left|\frac{k}{n}\right|^{2H} + \left|\frac{k+1}{n}\right|^{2H} \right)^2$$
$$= 4H^2(2H-1)^2 \cdot n^{-2} \sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left|\frac{k}{n}\right|^{4H-4} \cdot \frac{1}{n} + n^{-4H+1} \cdot \sum_{k=2}^{n-1} \widetilde{\rho}_k,$$

where

$$|\widetilde{\rho}_k| = O(|k-1|^{4H-6}).$$

We have

$$\sum_{k=2}^{\infty} |\widetilde{\rho}_k| < \infty$$

and thus

$$n \sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left(\left|\frac{k-1}{n}\right|^{2H} - 2\left|\frac{k}{n}\right|^{2H} + \left|\frac{k+1}{n}\right|^{2H} \right)^2$$

$$= 4H^2(2H-1)^2 \cdot n^{-2} \sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left|\frac{k}{n}\right|^{4H-4} \cdot \frac{1}{n} + O(n^{-4H+1}).$$

Combining this with (40) gives

$$\mathbf{E}|X_T^n - \widehat{X}_T^n|^2 = \frac{T^{4H}}{2} H^2(2H-1)^2 \cdot n^{-2} \sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left|\frac{k}{n}\right|^{4H-4} \cdot \frac{1}{n} + O(n^{-4H+1}). \quad (41)$$

Now assume that $H > 3/4$. It holds

$$\sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left|\frac{k}{n}\right|^{4H-4} \cdot \frac{1}{n} = \int_0^1 (1-x)\, x^{4H-4}\, \mathrm{d}x + o(1).$$

Since

$$\int_0^1 (1-x)\, x^{4H-4}\, \mathrm{d}x = \frac{1}{(4H-2)(4H-3)},$$

the assertion of the theorem for $H > 3/4$ follows from (39). It remains to consider the case $H = 3/4$. Here we have

$$\frac{T^{4H}}{2} H^2(2H-1)^2 \cdot n^{-2} \sum_{k=2}^{n-1} \left(1 - \frac{k}{n}\right) \left|\frac{k}{n}\right|^{4H-4} \cdot \frac{1}{n} = \frac{9}{128} T^3 \cdot n^{-2} \sum_{k=2}^{n-1} \frac{1}{k} + O(n^{-2}).$$

Since

$$\sum_{k=2}^{n-1} \frac{1}{k} = c - 1 + \log(n) + o(1),$$

where c stands for the Euler–Mascheroni constant, we obtain

$$\mathbf{E}|X_T^n - \widehat{X}_T^n|^2 = \frac{9}{128} T^3 \cdot n^{-2} \log(n) + O(n^{-2}),$$

which together with (39) completes the proof.

4. Asymptotic error distribution of the Euler scheme: $H < 3/4$

Let us first explain the strategy we have adopted in order to obtain our central limit theorem for the difference $X_T - X_T^n$ of the Lévy area and its Euler approximation in the case $H < 3/4$. First, recall that the random variable $X_T - X_T^n$ can be expressed as

$$X_T - X_T^n = \sum_{i=1}^{n} J_i^n, \quad \text{with } J_i^n \triangleq \int_{iT/n}^{(i+1)T/n} (B_s^{(1)} - B_{i/n}^{(1)})\, \mathrm{d}B_s^{(2)}.$$

With this expression in hand, it can be seen in particular that $X_T - X_T^n$ is still an element of the second chaos of our underlying fBm B.

Let us then recall the following limit theorem for random variables in a fixed finite Gaussian chaos, which can be found in [24, Theorem 1]:

Proposition 15. *Fix $p \geq 1$. Let $\{Z_n;\ n \geq 1\}$ be a sequence of centered random variables belonging the pth chaos of a Gaussian process, and assume that*

$$\lim_{n\to\infty} \mathbf{E}[Z_n^2] = 1. \tag{42}$$

Then Z_n converges in distribution to a centered Gaussian random variable if and only if the following condition is met:

$$\lim_{n\to\infty} \mathbf{E}[Z_n^4] = 3. \tag{43}$$

This is the criterion we shall adopt in order to get our central limit theorem. The second order condition (42) is simply a normalization step, so that the essential point is to analyze the fourth order moments of $X_T - X_T^n$ in order to prove condition (43). It should be stressed at this point that [24, Theorem 1] contains in fact a series of equivalent statements for condition (43), based either on assumptions on the Malliavin derivatives of the random variables Z_n, or on purely deterministic criteria concerning the kernels defining the multiple integrals under consideration. These alternative criteria yield arguably some shorter computations, but we preferred to stick to the fourth order moment for two main reasons: (i) The computations we perform in this context are more intuitive, and in a sense, easier to follow. (ii) As we shall explain below, the fourth order computations lead to some visual representations in terms of graphs, and we will able to show easily that the CLT is equivalent to have *the sum of the connected diagrams tending to 0*. As we shall see, this latter criterion is really analogous to [24, Theorem 1, Condition (ii)].

In the remainder of this section, we check condition (43) for $X_T - X_T^n$, rescaled according to Theorem 1, in order to get a central limit theorem for our approximation. We shall first explain the basics of our diagrammatical method of computation and show how to reduce our problem to the analysis of connected diagrams. Then we split our study into regular and singular terms.

4.1. Reduction of the problem

Owing to Theorem 1, it is enough for our purposes to show that $\lim_{n\to\infty} \mathbf{E}[Z_n^4] = 3$, where

$$Z_n = n^{2H-1/2} T^{-2H} \left[\alpha_1(H)\right]^{-1/2} \sum_{i=1}^{n} J_i^n. \tag{44}$$

Furthermore, the self-similarity of fBm implies that

$$\mathbf{E}[Z_n^4] = (\alpha_1(H)n)^{-2}\, \mathbf{E}\left[\left(\sum_{i=1}^{n} I_i\right)^4\right],$$

where $I_i = \mathcal{A}_{i,i+1}$ is the Lévy area between i and $i+1$. Now, the most naive idea one can have in mind is to write Z_n as $\lim_{\eta\to 0} Z_n(\eta)$, where Z_n is obtained by considering regularized areas based on $B(\eta)$, and then expand $\mathbf{E}[Z_n^4(\eta)]$ as

$$\mathbf{E}[Z_n^4(\eta)] = (\alpha_1(H)n)^{-2} \sum_{i_1,\ldots,i_4=1}^{n} \mathbf{E}\left[\prod_{j=1}^{4} I_{i_j}(\eta)\right]$$

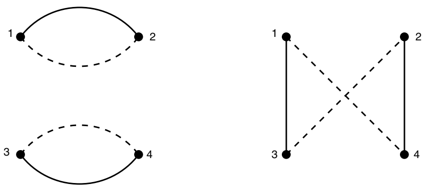

Fig. 1. Two examples of diagrams.

$$= (\alpha_1(H)n)^{-2} \prod_{j=1}^{4} \left(\int_{i_j}^{i_j+1} \mathrm{d}x_j \int_{i_j}^{x_j} \mathrm{d}y_j \right) \mathbf{E}\left[\prod_{j=1}^{4} B'^{(1)}_{x_j}(\eta) \right] \mathbf{E}\left[\prod_{j=1}^{4} B'^{(2)}_{y_j}(\eta) \right], \tag{45}$$

where we have used formula (15) with $N = 2$ in order to get the last equality.

We apply now Wick's formula (14) in order to get an expression for the expected values above, and this is where our diagrammatical representation can be useful. Indeed, $\mathbf{E}[\prod_{j=1}^4 B'^{(1)}_{x_j}(\eta)]\,\mathbf{E}[\prod_{j=1}^4 B'^{(2)}_{y_j}(\eta)]$ is the sum of 9 different terms, connecting the x_i's two by two according to formula (14), and also the y_i's two by two. Each term may be represented by a four-point diagram in the following way. Draw a simple line, resp. a dashed line between i and j if x_i and x_j, resp. y_i and y_j are connected. This procedure yields 9 different graphs, whose typical examples are given at Fig. 1. Moreover, the reader can then check easily that diagrams fall into two types: connected ones (6) and disconnected ones (3). Furthermore, up to permutations of the indices, there is only one disconnected diagram, namely the first diagram of Fig. 1. One checks immediately that the corresponding integral is $\mathbf{E}[I_{i_1}(\eta)I_{i_2}(\eta)]\mathbf{E}[I_{i_3}(\eta)I_{i_4}(\eta)]$. Write also the total contribution of the 6 *connected* diagrams as $\mathbf{E}[I_{i_1}(\eta)I_{i_2}(\eta)I_{i_3}(\eta)I_{i_4}(\eta)]_{(c)}$, where (c) stands for *connected*. Thanks to our graphical representation, it is then straightforward to prove the following: for arbitrary constants c_i, $i = 1, \ldots, n$, we have

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} c_i I_i(\eta) \right)^4 \right] - 3\,\mathbf{E}^2\left[\left(\sum_{i=1}^{n} c_i I_i(\eta) \right)^2 \right] = \mathbf{E}\left[\left(\sum_{i=1}^{n} c_i I_i(\eta) \right)^4 \right]_{(c)}. \tag{46}$$

Hence our condition (43) is satisfied for Z_n defined by (44) if and only if the right-hand side of Eq. (46) goes to zero for $c_i = n^{-1/2}$ (c_i is in fact independent of i). It should be stressed at that point that the latter condition (which is what we call *connected diagrams go to 0*) is an analogue of criterion (ii) in [24, Theorem 1], but is obtained here without Malliavin calculus tools. This terminology is inspired by the Feynman diagram analysis in the context of quantum field theory; see e.g. [15].

Let us set now $\tilde{Z}_n(\eta) = \sum_{i=1}^n I_i(\eta)$. With the above considerations in mind, we are reduced to show that

$$\lim_{n\to\infty} \lim_{\eta\to 0} \frac{1}{n^2} \mathbf{E}\left[\tilde{Z}_n^4(\eta) \right]_{(c)} = 0. \tag{47}$$

This relation will be first proved for $H \in (1/2, 3/4)$. In that case one may consider directly the situation where $\eta = 0$, that is the infinitesimal covariance kernel $(x, y) \mapsto H(2H-1)|x-y|^{2H-2}$, since it is locally integrable. The proof requires only a few lines of computations. Each diagram in $\mathbf{E}[\tilde{Z}_n^4(\eta)]_{(c)}$ splits into *regular terms* – which are also well defined for $H < \frac{1}{2}$ – and *singular terms* – which diverge when $H < \frac{1}{2}$. As we shall see, the bounds given for the non-singular terms also hold true for $H < \frac{1}{2}$. Then we shall see how to bound the singular terms for

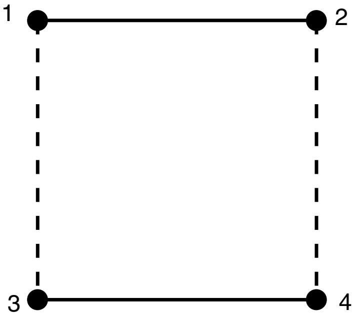

Fig. 2. Typical connected diagram.

arbitrary H by replacing the ill-defined kernel $H(2H-1)|x-y|^{2H-2}$ with its regularization $K'(\eta; x, y)$. This step is of course only needed in the case $H < \frac{1}{2}$, but computations are equally valid in the whole range $H \in (1/4, 3/4)$. In other words, the barrier $H = \frac{1}{2}$ is largely artificial (the proofs of the two cases are actually mixed, and one could also have written a general proof, at the price of some more technical calculations).

Before we enter into these computational details, let us reduce our problem a little bit more: recall again that we wish to prove relation (47) for $\tilde{Z}_n(\eta) = \sum_{i=1}^n I_i(\eta)$. As explained above, we evaluate $\mathbf{E}[\tilde{Z}_n^4(\eta)]_{(c)}$ with 6 different connected diagrams. Let us focus on the term, which will be called $\mathcal{T}$, corresponding to the diagram given at Fig. 2 (the other ones can be treated in a similar manner). Now, starting from expression (45), taking into account the fact that we are considering the particular diagram given at Fig. 2 and integrating over the internal variables y, we end up with $\mathcal{T} = n^{-2} \sum_{i_1,\ldots,i_4=1}^n I_{(i_1,\ldots,i_4)}$, where (recalling that the kernel K is defined by Eq. (11))

$$I_{(i_1,\ldots,i_4)} := \int_{i_1}^{i_1+1} \mathrm{d}x_1 \cdots \int_{i_4}^{i_4+1} \mathrm{d}x_4 \\ \times K'(\eta; x_1, x_2) K'(\eta; x_3, x_4) K_{i_1,i_3}(\eta; x_1, x_3) K_{i_2,i_4}(\eta; x_2, x_4). \tag{48}$$

The latter expression yields naturally a notion of *regular terms* and *singular terms*: split the set of indices $\{1, \ldots, n\}^4$ into $A_1 \cup A_2$, where

$$A_2 = \{(i_1, \ldots, i_4) \mid 1 \le i_1, \ldots, i_4 \le n,\ |i_1 - i_3|, |i_2 - i_4| \le 1\}, \\ A_1 = \{1, \ldots, n\}^4 \setminus A_2. \tag{49}$$

Regular terms, resp. singular terms are those for which $|i_1 - i_2|, |i_3 - i_4| \ge 2$, resp. $|i_1 - i_2| \le 1$ or $|i_3 - i_4| \le 1$. Split accordingly the sets of indices A_j, $j = 1, 2$ into $A_{j,\mathrm{reg}} \cup A_{j,\mathrm{sing}}$, and denote

$$\mathcal{T}_{j,\mathrm{reg}} = \sum_{(i_1,\ldots,i_4)\in A_j^{\mathrm{reg}}} I_{(i_1,\ldots,i_4)} \quad \text{and} \quad \mathcal{T}_{j,\mathrm{sing}} = \sum_{(i_1,\ldots,i_4)\in A_j^{\mathrm{sing}}} I_{(i_1,\ldots,i_4)}. \tag{50}$$

It remains to prove that $\mathcal{T}_{j,\mathrm{reg}} = o(n^2)$ and $\mathcal{T}_{j,\mathrm{sing}} = o(n^2)$, for $j = 1, 2$. These two steps will be performed respectively at Sections 4.2 and 4.3.

4.2. Regular terms and case $H > 1/2$

This section is devoted to the study of $\mathcal{T}_{j,\mathrm{reg}}$, and also of $\mathcal{T}_{j,\mathrm{sing}}$ for $H > 1/2$. In both cases, one is allowed to take limits as $\eta \to 0$ without much care, by a standard application of the dominated convergence theorem. We skip this elementary step, and consider directly our expressions for $\eta = 0$.

Let us start by $\mathcal{T}_{1,\mathrm{reg}}$, which is given by

$$\mathcal{T}_{1,\mathrm{reg}} = \sum_{|i_1-i_3|,|i_1-i_2|,|i_3-i_4|\geq 2} \int_{i_1}^{i_1+1} \mathrm{d}x_1 \cdots \int_{i_4}^{i_4+1} \mathrm{d}x_4\, K'(x_1,x_2)K'(x_3,x_4) \times K_{i_1,i_3}(x_1,x_3)K_{i_2,i_4}(x_2,x_4). \tag{51}$$

We shall bound this integral by different methods in the cases $H \in (1/2, 3/4)$ and $H < \frac{1}{2}$:

(i) Assume first $H \in (1/2, 3/4)$. Whenever $|s-i|, |t-j| \leq 1$, recall from Lemma 11 that $K_{i,j}(s,t) \lesssim |t-s|^{2H-2}$ if $|i-j| \geq 2$, and $s \in [i, i+1], t \in [j, j+1]$. In particular, the quantity $|K_{i_1,i_3}(x_1,x_3)|$ in Eq. (51) is bounded by $|x_1-x_3|^{2H-2}$. We also obviously have $|K'(x_1,x_2)| \lesssim |x_2-x_1|^{2H-2}$ and $|K'(x_3,x_4)| \lesssim |x_4-x_3|^{2H-2}$. As a consequence,

$$|\mathcal{T}_{1,\mathrm{reg}}| \leq 2C \sum_{|i_1-i_3|,|i_1-i_2|,|i_3-i_4|\geq 2} \int_{i_1}^{i_1+1} \mathrm{d}x_1 \cdots \int_{i_4}^{i_4+1} \mathrm{d}x_4\, |x_2-x_1|^{2H-2}|x_4-x_3|^{2H-2} \times |x_1-x_3|^{2H-2}|K_{i_2,i_4}(x_2,x_4)|.$$

Let us undo now the initial scaling by setting $t_j = x_j/n$. One gets

$$|\mathcal{T}_{1,\mathrm{reg}}| \lesssim n^{4+3(2H-2)} \int_0^1 \mathrm{d}t_1 \cdots \int_0^1 \mathrm{d}t_4\, |t_2-t_1|^{2H-2}|t_4-t_3|^{2H-2} \times |t_3-t_1|^{2H-2} K_{\lfloor nt_2\rfloor,\lfloor nt_4\rfloor}(nt_2,nt_4). \tag{52}$$

Applying Lemma 13 to the above expression (52) and integrating successively with respect to t_1 and t_3 yields

$$|\mathcal{T}_{1,\mathrm{reg}}| \lesssim n^{4+3(2H-2)} \int_0^1 \mathrm{d}t_2 \int_0^1 \mathrm{d}t_4 (1+|t_2-t_4|^{6H-4}) K_{\lfloor nt_2\rfloor,\lfloor nt_4\rfloor}(nt_2,nt_4). \tag{53}$$

Recall now that $|K_{\lfloor nt_2\rfloor,\lfloor nt_4\rfloor}(nt_2,nt_4)| \lesssim \min(1,(n|t_2-t_4|)^{2H-2})$. Hence, one can bound this kernel by 1 on $[0,1/n]$ and by $(nt)^{2H-2}$ on $[1/n,1]$, yielding

$$\int_0^1 \mathrm{d}t_4 K_{\lfloor nt_2\rfloor,\lfloor nt_4\rfloor}(nt_2,nt_4) \lesssim \int_0^{1/n} \mathrm{d}t + n^{2H-2}\int_{1/n}^1 t^{2H-2}\,\mathrm{d}t \lesssim n^{-1} + n^{2H-2}, \tag{54}$$

and also

$$\int_0^1 \mathrm{d}t_4 |t_2-t_4|^{6H-4} K_{\lfloor nt_2\rfloor,\lfloor nt_4\rfloor}(nt_2,nt_4) \lesssim \int_0^{1/n} t^{6H-4}\mathrm{d}t + n^{2H-2}\int_{1/n}^1 t^{8H-6}\mathrm{d}t \lesssim n^{3-6H} + n^{2H-2}.$$

Hence, multiplying by the prefactor $n^{4+3(2H-2)}$ from Eq. (53), one has found: $|\mathcal{T}_{1,\mathrm{reg}}| \lesssim n + n^{8H-4} + n^{6H-3} \lesssim n + n^{8H-4}$. In particular, if $H < 3/4$, then $|\mathcal{T}_{1,\mathrm{reg}}| = o(n^2)$.

(ii) Assume now $H < \frac{1}{2}$. In this case, the integrals we have been manipulating above are divergent, so that we will use series arguments instead. Let us observe then that, under the same conditions as in the case $H \in (1/2, 3/4)$, the bound $|K_{i_1,i_3}(x_1, x_3)| \lesssim |i_1 - i_3|^{2H-2}$ holds true. We also bound the factor $|K_{i_2,i_4}(x_2, x_4)|$ by a constant in order to get

$$
\begin{aligned}
|\mathcal{T}_{1,\mathrm{reg}}| &\lesssim \sum_{i_1,i_3:|i_1-i_3|\geq 2} |i_1 - i_3|^{2H-2} \left(\sum_{i_2:|i_2-i_1|\geq 2} |i_2 - i_1|^{2H-2} \right) \\
&\quad \times \left(\sum_{i_4:|i_4-i_3|\geq 2} |i_4 - i_3|^{2H-2} \right) \\
&\lesssim \sum_{i_1,i_3:|i_1-i_3|\geq 2} |i_1 - i_3|^{2H-2} = O(n).
\end{aligned}
$$

We now leave to the reader the task of checking, with the same kind of computations, that $|\mathcal{T}_{1,\mathrm{sing}}| = O(n)$ (*provided* $H > \frac{1}{2}$).

Turn now to the complementary set of indices, A_2: by simply bounding the kernels $K_{i,j}(x, y)$ by constants in (51), one gets

$$
\begin{aligned}
|\mathcal{T}_{2,\mathrm{reg}}| &\lesssim \sum_{|i_1-i_3|,|i_2-i_4|\leq 1;|i_1-i_2|,|i_3-i_4|\geq 2} \int_{i_1}^{i_1+1} \mathrm{d}x_1 \cdots \int_{i_4}^{i_4+1} \mathrm{d}x_4 |K'(x_1, x_2)||K'(x_3, x_4)| \\
&\lesssim \sum_{i_1,i_2:|i_1-i_2|\geq 2} |i_1 - i_2|^{2(2H-2)}. \qquad (55)
\end{aligned}
$$

Hence $|\mathcal{T}_{2,\mathrm{reg}}| = O(n^{4H-2}) = o(n^2)$ when $H < 3/4$, which is enough for our purposes.

Finally, *provided* $H > \frac{1}{2}$, some similar elementary considerations prove that

$$
|\mathcal{T}_{2,\mathrm{sing}}| \lesssim n \left(\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 |K'(x_1, x_2)| \right)^2 = O(n), \qquad (56)
$$

where we have used the fact that $|i_j - i_k| = O(1)$ for $j, k = 1, \ldots, 4$ if $(i_1, \ldots, i_4) \in A_{2,\mathrm{sing}}$.

4.3. Singular terms in the case $H < 1/2$

Let us reconsider the terms $\mathcal{T}_{1,\mathrm{sing}}$ and $\mathcal{T}_{2,\mathrm{sing}}$ in (50), taking now into account the fact that we deal with the regularized kernels $K'(\eta; x_1, x_2)$, $K'(\eta; x_3, x_4)$ instead of $K'(x_1, x_2)$, $K'(x_3, x_4)$.

In order to treat all the terms appearing in our sums in a systematic way, let us introduce a little of vocabulary: consider any multi-index $(i_1, \ldots, i_p)$, $p \geq 2$ (in our case $p = 4$). We shall say that $\{i_{j_1}, \ldots, i_{j_k}\}$, $j_1 \neq \cdots \neq j_k$ is a *maximal contiguity subset* of $(i_1, \ldots, i_p)$ if (up to a reordering) $i_{j_2} - i_{j_1} = \cdots = i_{j_k} - i_{j_{k-1}} = 1$ and $i_l \geq i_{j_k} + 2$ or $\leq i_{j_1} - 2$ if $l \neq j_1, \ldots, j_k$. Maximal contiguity subsets define a partition of the set $\{i_1, \ldots, i_p\}$. Then we shall write $(i_1, \ldots, i_p) \in J_{m_1,\ldots,m_q}$ if the lengths of the maximal contiguity subsets of $(i_1, \ldots, i_p)$ are $m_1 \geq \cdots \geq m_q$.

This terminology will help us classify the terms in $\mathcal{T}_{1,\mathrm{sing}} \cup \mathcal{T}_{2,\mathrm{sing}}$. Forgetting about the $O(n)$ multi-indices $(i_1, \ldots, i_4)$ in J_4 appearing in $\mathcal{T}_{2,\mathrm{sing}}$ (according to the fact that $\mathrm{Var}(\mathcal{A}_{st}(\eta))$ is uniformly bounded on $[0, T]$, proved in [26], this term contributes only $O(n)$ to the sum), the other singular terms are all in $\mathcal{T}_{1,\mathrm{sing}}$ and may be:

– either of type $J_{2,1,1}$, with maximal contiguity subsets $\{\{i_1, i_2\}, \{i_3\}, \{i_4\}\}$ or equivalently $\{\{i_3, i_4\}, \{i_1\}, \{i_2\}\}$;

– or of type $J_{2,2}$, with maximal contiguity subsets $\{\{i_1, i_2\}, \{i_3, i_4\}\}$;

– or of type $J_{3,1}$, with maximal contiguity subsets $\{\{i_1, i_2, i_3\}, \{i_4\}\}$ or equivalent possibilities.

Let us observe that, in our iterated multiple integrals, the most serious problems of singularity appear when the external variables x (represented by solid lines in our graphs) are contiguous. Indeed, the internal variables y are integrated, smoothing the kernels K' into $K_{a,b}$. However, one still has to cope with the highly singular kernel K' for the external variables. For instance, for the graph given at Fig. 2 (which is the one we are analyzing), this kind of problem appears for the terms of type $J_{2,1,1}$ (when the maximal contiguity subset is $\{\{i_1, i_2\}, \{i_3\}, \{i_4\}\}$) or $J_{2,2}$. But a simple Fubini-type argument allows us to get rid of these singularities. Indeed, when $\eta > 0$, the integral

$$\prod_{j=1}^{4} \int_{i_j}^{i_j+1} \mathrm{d}x_j \; K'(\eta; x_1, x_2) K'(\eta; x_3, x_4) \cdot \prod_{j=1}^{4} \int_{i_j}^{x_j} \mathrm{d}y_j \; K'(\eta; y_1, y_3) K'(\eta; y_2, y_4),$$

corresponding to the diagram of Fig. 2, is also equal to

$$\prod_{j=1}^{4} \int_{i_j}^{i_j+1} \mathrm{d}y_j \; K'(\eta; y_1, y_3) K'(\eta; y_2, y_4) \cdot \prod_{j=1}^{4} \int_{y_j}^{i_j+1} \mathrm{d}x_j \; K'(\eta; x_1, x_2) K'(\eta; x_3, x_4),$$

corresponding (up to time-reversal) to the reversed diagram obtained by exchanging full lines with dashed lines. The important point is that this *full-line dashed-line symmetry* maps the above singular diagrams of type $J_{2,1,1}$ or $J_{2,2}$ into regular diagrams, for which the external variables are separated. This situation can thus be handled along the same lines as in Section 4.2, and there only remains to estimate singular diagrams of type $J_{3,1}$.

For this latter class of diagram, assume for instance (without loss of generality) that $\{i_1, i_2, i_3\}$ is a maximal contiguity subset of $(i_1, \ldots, i_4)$. Then, owing to relation (12), the corresponding integral writes $E = E(i_1, \ldots, i_4)$, with

$$\begin{aligned} E = c_H \int_{i_3}^{i_3+1} \mathrm{d}x_3 \int_{i_1}^{i_1+1} \mathrm{d}x_1 \int_{i_2}^{i_2+1} \mathrm{d}x_2 \int_{i_4}^{i_4+1} \mathrm{d}x_4 \, [x_3 - x_4]_\eta^{2H-2} [x_1 - x_2]_\eta^{2H-2} \\ \times \left([x_3 - x_1]_\eta^{2H} + [i_3 - i_1]_\eta^{2H} - [x_3 - i_1]_\eta^{2H} - [x_1 - i_3]_\eta^{2H} \right) K_{i_2,i_4}(\eta; x_2, x_4), \quad (57) \end{aligned}$$

which is the sum of 4 terms, denoted in what follows by $E_1, \ldots, E_4$. The most complicated one is a priori E_1, obtained by choosing the contribution of $[x_3 - x_1]_\eta^{2H}$ to the integral. Let us first estimate this term.

Apply Lemma 12 with $f(x_4; u) = [u - x_4]_\eta^{2H-2}$, $z = x_1$ (x_4 is simply an additional parameter here, and f fulfills the analytic assumptions of Lemma 12 because i_3 and i_4 are not contiguous) and $\beta = 2H$, $\gamma = 0$: letting

$$\phi_1(x_4; x_1) := \int_{i_3}^{i_3+1} \mathrm{d}x_3 [x_1 - x_3]_\eta^{2H} [x_3 - x_4]_\eta^{2H-2},$$

we obtain that ϕ_1 is analytic in x_1 on a cut neighborhood Ω'_{cut} of $[i_1, i_1 + 1]$ excluding possibly i_3 and $i_3 + 1$ (depending on whether $i_3, i_3 + 1 \in \{i_1, i_1 + 1\}$ or not), and one can decompose ϕ_1 into

$$\phi_1(x_4; x_1) = [x_1 - i_3]_\eta^{2H+1} F_1(x_4; x_1) + G_1(x_4; x_1) \quad (58)$$

on a neighborhood of i_3 (and similarly around i_3+1), with F_1 possibly zero. The functions $\phi_1|_{\Omega'_{\text{cut}}}$, F_1 and G_1 are analytic and bounded by a constant times $|i_3-i_4|^{2H-2}$.

Apply once again Lemma 12 with $f(x_4;u)=\phi_1(x_4;u)$, $z=x_2$ and $\beta=2H-2$, $\gamma=0$ or (possibly) $2H+1$: letting

$$\phi_2(x_4;x_2)=\int_{i_1}^{i_1+1}\mathrm{d}x_1\,[x_2-x_1]_\eta^{2H-2}\phi_1(x_4;x_1), \tag{59}$$

ϕ_2 is analytic in x_2 on a cut neighborhood Ω''_{cut} of $[i_2,i_2+1]$ excluding possibly i_1 and i_1+1, and

$$\phi_2(x_4;x_2)=[x_2-i_1]_\eta^{2H-1}F_2(x_4;x_2)+[x_2-i_1]_\eta^{4H}F_3(x_4;x_2)+G_2(x_4;x_2) \tag{60}$$

on a neighborhood of i_1 (and similarly around i_1+1), with the same bounds as before for $\phi_2|_{\Omega''_{\text{cut}}}$, F_2, F_3 and G_2.

Finally, since ϕ_2 is integrable with respect to x_2 on $[i_2,i_2+1]$ and $K_{i_2,i_4}(\eta;x_2,x_4)$ is bounded by $C|i_3-i_4|^{2H-2}$ by Lemma 11, one gets

$$|E|\le C'\int_{i_4}^{i_4+1}\mathrm{d}x_4\,|i_3-i_4|^{4H-4}=C'|i_3-i_4|^{4H-4}. \tag{61}$$

There remain 3 'boundary' terms E_2, E_3, E_4 which are easier to cope with. Consider for instance E_3 defined as

$$E_3=\int_{i_4}^{i_4+1}\mathrm{d}x_4\int_{i_2}^{i_2+1}\mathrm{d}x_2\,K_{i_2,i_4}(\eta;x_2,x_4)$$
$$\times\int_{i_1}^{i_1+1}\mathrm{d}x_1\,[x_2-x_1]_\eta^{2H-2}\int_{i_3}^{i_3+1}\mathrm{d}x_3\,[x_3-i_1]_\eta^{2H}[x_3-x_4]_\eta^{2H-2}.$$

Applying again Lemma 12, we get

$$E_3=C\int_{i_4}^{i_4+1}\mathrm{d}x_4G_1(x_4;i_1)\int_{i_2}^{i_2+1}\mathrm{d}x_2K_{i_2,i_4}(\eta;x_2,x_4)$$
$$\times\left([x_2-i_1-1]_\eta^{2H-1}-[x_2-i_1]_\eta^{2H-1}\right),$$

where G_1 is as in Eq. (58). Since $x_2\mapsto[x_2-i_1-1]_\eta^{2H-1}$ and $x_2\mapsto[x_2-i_1]_\eta^{2H-1}$ are integrable and G_1, resp. K_{i_2,i_4} is bounded by a constant times $|i_3-i_4|^{2H-2}$, one easily gets an upper bound as the same form as before, namely, $|E_3|\le C|i_3-i_4|^{4H-4}$.

Thus we have that $E(i_1,\dots,i_4)$ defined by (57) satisfies $E(i_1,\dots,i_4)\le C|i_3-i_4|^{4H-4}$. Finally, since $\sum\sum_{|i_3-i_4|\ge 2}|i_3-i_4|^{4H-4}=O(n)$, we obtain $\sum_{i_1,\dots,i_4\in J_{3,1}}E(i_1,\dots,i_4)=O(n)$.

Let us summarize now the results we have obtained so far: we have shown, respectively at Sections 4.2 and 4.3, that the terms $\mathcal{T}_{j,\text{reg}}$ and $\mathcal{T}_{j,\text{sing}}$ defined by Eq. (50) are $o(n^2)$. Going back to the definition of $\mathcal{T}$ (see Eq. (48)), this also shows that this quantity is of order $o(n^2)$. Recall now that $\mathbf{E}[\tilde{Z}_n^4(\eta)]_{(c)}$ can be decomposed into 6 terms, corresponding to our connected diagrams, each of the same kind as the particular example $\mathcal{T}$ we have chosen. We have thus proved that $\mathbf{E}[\tilde{Z}_n^4(\eta)]_{(c)}=o(n^2)$ uniformly in η, which yields relation (47). This finishes the proof of Theorem 3 for $H<3/4$.

5. Asymptotic error distribution of the Euler scheme: $H \geq 3/4$

In this case, we derive the limit distribution in a different way, i.e. by analyzing the difference between the trapezoidal and the Euler scheme. Recall that for $T = 1$ this difference is given by

$$\frac{1}{2}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}), \tag{62}$$

and we will see that, thanks to a simple geometric trick (borrowed from [22]), the latter quantity has the same law as

$$\frac{1}{4}\sum_{i=0}^{n-1}\left(|B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n}|^2 - |B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}|^2\right).$$

This allows us to apply easily the limit theorems for the quadratic variation of fBm, see e.g. [3,23] and the references therein, yielding the Lemma below, in which the following distribution appears:

Definition 16 (*Rosenblatt Random Variable*). A standard Rosenblatt random variable with parameter $H_0 = 2H - 1$ is given by

$$\frac{(4H-3)^{1/2}}{4H\,(2H-1)^{1/2}}\int_0^1\int_0^1\left(\int_{\max\{r,s\}}^1 \frac{\partial K^H}{\partial u}(u,s)\,\frac{\partial K^H}{\partial u}(u,r)\,\mathrm{d}u\right)\mathrm{d}W_r\mathrm{d}W_s$$

where W is a standard Brownian motion,

$$K^H(t,s) = c_H s^{1/2-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}\,\mathrm{d}u\,1_{[0,t)}(s)$$

and

$$c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)}\right)^{1/2}.$$

Lemma 17. *The following limits in law hold true:*

(i) *Let $H = 3/4$. Then we have*

$$\frac{\sqrt{2}n}{\sqrt{c_1(H)\log(n)}}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}) \xrightarrow{\mathcal{L}} Z,$$

where $c_1(H) = 9/16$ and Z is a standard normal random variable.

(ii) *Let $H \in (3/4, 1)$. Then*

$$\frac{\sqrt{2}n}{\sqrt{c_2(H)}}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}) \xrightarrow{\mathcal{L}} \frac{1}{\sqrt{2}}(R_1 - R_2),$$

where $c_2(H) = 2H^2\,(2H-1)\,/\,(4H-3)$ and R_1 and R_2 are two independent standard Rosenblatt variables of index $2H - 1$.

Proof. (i) Let β be a fractional Brownian motion with Hurst index H and define

$$V_n = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{|\beta_{(i+1)/n}-\beta_{i/n}|^2}{n^{-2H}}-1\right) = -1+n^{2H-1}\sum_{i=0}^{n-1}|\beta_{(i+1)/n}-\beta_{i/n}|^2.$$

If $H = 3/4$ it follows from [3,23] that

$$\sqrt{\frac{n}{c_1(H)\log(n)}}V_n \xrightarrow{\mathcal{L}} Z, \tag{63}$$

where Z is a standard normal random variable. Moreover, for $H \in (3/4, 1)$ we know from [3,23] that

$$\sqrt{\frac{n^{4-4H}}{c_2(H)}}V_n \xrightarrow{\mathcal{L}} R, \tag{64}$$

where R is a standard Rosenblatt random variable with index $2H-1$.

Now let $\tilde{\beta}$ be another fractional Brownian motion with the same Hurst index as β, but independent of β, and define

$$V_n' = n^{2H-1}\sum_{i=0}^{n-1}\left(|\beta_{(i+1)/n}-\beta_{i/n}|^2 - |\tilde{\beta}_{(i+1)/n}-\tilde{\beta}_{i/n}|^2\right).$$

The continuous mapping theorem and (63) imply that

$$\sqrt{\frac{n}{c_1(H)\log(n)}}V_n' \xrightarrow{\mathcal{L}} Z_1 - Z_2 \tag{65}$$

for $H = 3/4$, where Z_1 and Z_2 are two independent standard normal random variables. From (64) we obtain that

$$\sqrt{\frac{n^{4-4H}}{c_2(H)}}V_n' \xrightarrow{\mathcal{L}} (R_1 - R_2), \tag{66}$$

where R_1 and R_2 are two independent standard Rosenblatt random variables with index $2H-1$.
(ii) Now, set $B^{(1)} = (\beta+\widetilde{\beta})/\sqrt{2}$ and $B^{(2)} = (\beta-\widetilde{\beta})/\sqrt{2}$. Then $B^{(1)}$ and $B^{(2)}$ are two independent fractional Brownian motions with the same Hurst parameter. Moreover, we have

$$n^{2H-1}\sum_{k=0}^{n-1}(B^{(1)}_{(k+1)/n}-B^{(1)}_{k/n})(B^{(2)}_{(k+1)/n}-B^{(2)}_{k/n}) \stackrel{\mathcal{L}}{=} \frac{1}{2}V_n'.$$

Thus, we have for $H = 3/4$ that

$$\frac{2n}{\sqrt{c_1(H)\log n}}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n}-B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n}-B^{(2)}_{i/n}) \stackrel{\mathcal{L}}{=} \sqrt{\frac{n}{c_1(H)\log(n)}}V_n',$$

and the first claim follows from (65) and the fact that $Z_1 - Z_2$ has the same distribution as $\sqrt{2}Z_1$.

Moreover, since

$$\frac{2n}{\sqrt{c_2(H)}}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n}-B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n}-B^{(2)}_{i/n}) \stackrel{\mathcal{L}}{=} \frac{n^{2-2H}}{\sqrt{c_2(H)}}V_n'$$

the second claim follows from (66). □

Since the trapezoidal scheme has a better convergence rate than the Euler scheme for $H \geq 3/4$, which we already used in Section 3.4, the error of the latter scheme is dominated by (62). Thus, the asymptotic error distribution of the Euler scheme can be determined by the above Lemma, which will be carried out in the following two subsections.

5.1. Error distribution of the Euler scheme for $H = 3/4$

By scaling we can assume without loss of generality that $T = 1$. Recall that here we have

$$\mathbf{E}|X_1 - X_1^n|^2 = \frac{9}{128} \cdot \log(n)n^{-2} + o(\log(n)n^{-2})$$

for the error of the Euler scheme. Using the trapezoidal approximation $\widehat{X}_1^n$ we can write

$$\begin{aligned} X_1 - X_1^n &= X_1 - \widehat{X}_1^n + \widehat{X}_1^n - X_1^n \\ &= \frac{1}{2}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}) + \rho_n, \end{aligned}$$

where $\rho_n = X_1 - \widehat{X}_1^n$. Hence, setting $\kappa_n := n[\frac{9}{128}\log(n)]^{-1/2}$, we obtain

$$\kappa_n(X_1 - X_1^n) = \frac{\kappa_n}{2}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}) + \kappa_n\rho_n.$$

Now note that $\kappa_n\rho_n \to 0$ in $L^2(\Omega)$ by Theorem 2 and

$$\frac{\sqrt{2}n}{\sqrt{c_1(H)\log n}}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n}) \xrightarrow{\mathcal{L}} Z,$$

where $c_1(H) = 9/16$ by Lemma 17. Since

$$\frac{\kappa_n}{2} = \frac{n}{\sqrt{\log(n)}}\frac{\sqrt{32}}{\sqrt{9}} = \frac{n}{\sqrt{\log(n)}}\frac{\sqrt{2}}{\sqrt{c_1(H)}},$$

it finally follows that

$$n(\log(n))^{-1/2}(X_1 - X_1^n) \xrightarrow{\mathcal{L}} \sqrt{\frac{9}{128}} \cdot Z,$$

where Z is a standard normal random variable.

5.2. Error distribution of the Euler scheme for $H > 3/4$

Here we have

$$\mathbf{E}|X_1 - X_1^n|^2 = \alpha_2(H) \cdot n^{-2} + o(n^{-2})$$

with

$$\alpha_2(H) = \frac{1}{4}\frac{H^2(2H-1)}{4H-3}.$$

Proceeding as above, the limit distribution of the error of the Euler scheme is determined by the limit distribution of

$$\frac{n}{2\sqrt{\alpha_2(H)}}\sum_{i=0}^{n-1}(B^{(1)}_{(i+1)/n}-B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n}-B^{(2)}_{i/n}).$$

Since

$$\frac{1}{2\sqrt{\alpha_2(H)}}=\sqrt{\frac{4H-3}{H^2(2H-1)}}=\frac{\sqrt{2}}{\sqrt{c_2(H)}},$$

it follows by Lemma 17 that

$$\frac{n}{2\sqrt{\alpha_2(H)}}(X_1-X_1^n)\xrightarrow{\mathcal{L}}\frac{1}{\sqrt{2}}(R_1-R_2).$$

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