



# LAN property for stochastic differential equations with additive fractional noise and continuous time observation

Yanghui Liu<sup>a</sup>, Eulalia Nualart<sup>b,\*</sup>, Samy Tindel<sup>a</sup>

<sup>a</sup>Department of Mathematics, Purdue University, 150 N. University Street, W. Lafayette, IN 47907, USA

<sup>b</sup>Department of Economics and Business, Universitat Pompeu Fabra and Barcelona Graduate School of Economics, Ramón Trias Fargas 25-27, 08005 Barcelona, Spain

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## Abstract

We consider a stochastic differential equation with additive fractional noise with Hurst parameter  $H > 1/2$ , and a non-linear drift depending on an unknown parameter. We show the Local Asymptotic Normality property (LAN) of this parametric model with rate  $\sqrt{\tau}$  as  $\tau \rightarrow \infty$ , when the solution is observed continuously on the time interval  $[0, \tau]$ . The proof uses ergodic properties of the equation and a Girsanov-type transform. We analyze the particular case of the fractional Ornstein–Uhlenbeck process and show that the Maximum Likelihood Estimator is asymptotically efficient in the sense of the Minimax Theorem.

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\* Corresponding author.

E-mail addresses: [liu2048@purdue.edu](mailto:liu2048@purdue.edu) (Y. Liu), [eulalia.nualart@upf.edu](mailto:eulalia.nualart@upf.edu) (E. Nualart), [stindel@purdue.edu](mailto:stindel@purdue.edu) (S. Tindel).

URLs: <https://www.math.purdue.edu/~liu2048/> (Y. Liu), <https://www.upf.edu/web/eulalia-nualart> (E. Nualart), <https://www.math.purdue.edu/~stindel/> (S. Tindel).

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### 1. Introduction

Let  $B$  be a  $d$ -dimensional fractional Brownian motion (fBm) with Hurst parameter  $H > 1/2$ . Let us recall that  $B$  is a centered Gaussian process defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . The law of  $B$  is thus characterized by its covariance function, which is defined by

$$R_{s;t} \equiv \mathbf{E}[B_t^i B_s^j] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}) \mathbf{1}_{\{0\}}(i - j), \quad s, t \in \mathbb{R}. \tag{1}$$

The variance of the increments of  $B$  is then given by

$$\mathbf{E}[|B_t^i - B_s^i|^2] = |t - s|^{2H}, \quad s, t \in \mathbb{R}, \quad i = 1, \dots, d, \tag{2}$$

and this implies that almost surely the fBm paths are  $\gamma$ -Hölder continuous for any  $\gamma < H$ .

In this article, we consider a time horizon  $\tau \in (0, \infty)$  and the following  $\mathbb{R}^d$ -valued stochastic differential equation driven by  $B$ :

$$Y_t = y_0 + \int_0^t b(Y_s; \theta) ds + \sum_{j=1}^d \sigma_j B_t^j, \quad t \in [0, \tau]. \tag{3}$$

Here  $y_0 \in \mathbb{R}^d$  is a given initial condition,  $B = (B^1, \dots, B^d)$  is the aforementioned fractional Brownian motion, the unknown parameter  $\theta$  lies in a certain set  $\Theta$  which will be specified later on,  $\{b(\cdot; \theta), \theta \in \Theta\}$  is a known family of drift coefficients with  $b(\cdot; \theta) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , and  $\sigma_1, \dots, \sigma_d \in \mathbb{R}^d$  are assumed to be known diffusion coefficients.

There has been a wide interest in drift estimation for stochastic equations driven by fractional Brownian motion in the recent past, partly motivated by inverse problems in a biomedical context [20]. However, notice that the existing literature on the topic mainly focuses on the case where the dependence  $\theta \mapsto b(x; \theta)$  is linear and all coefficients are real-valued. In this situation least squares and maximum likelihood estimators (MLE) for the unknown parameter  $\theta$  can be computed explicitly, and numerous results are available: the case of continuous observations of  $Y$  and where the drift coefficient is linear in both  $\theta$  and  $x$  (that is, fractional Ornstein–Uhlenbeck type process) has been studied e.g. in [1,16,18,23], either in the ergodic ( $\theta < 0$ ) and non-ergodic ( $\theta > 0$ ) case. See also the monograph [30] and the references therein. Results on parameter estimation based on discrete observations of  $Y$  in the linear case can be found e.g. in [3,32]. The case of a dependence of the form  $(\theta, x) \mapsto \theta b(x)$  is handled e.g. in [21,31].

However, the case of a general multi-dimensional coefficient  $b(\theta, x)$  in Eq. (3) is also a very natural situation to consider, though we are only aware of the articles [6,27] giving some positive answers for such a dependence. The current contribution has thus to be thought of as a step in this direction. Indeed, our main aim is to show that the model given by Eq. (3) satisfies the Local Asymptotic Normality property (LAN) with rate  $\sqrt{\tau}$  as  $\tau \rightarrow \infty$ , when the process  $Y$  is observed continuously on the time interval  $[0, \tau]$ .

Before we proceed to a specific statement of our results, let us describe the assumptions we shall work with, which are similar to the ones used in [27]. We start with a standard hypothesis on the parameter set  $\Theta$ :

**Hypothesis 1.1.** The set  $\Theta$  is compactly embedded in  $\mathbb{R}^q$  for a given  $q \geq 1$ .

In order to describe the assumptions on our coefficients  $b$ , we will use the following notation for partial derivatives:

**Notation 1.2.** Let  $f : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}$  be a  $C^{p_1, p_2}$  function for  $p_1, p_2 \geq 0$ . Then for any tuple  $(i_1, \dots, i_p) \in \{1, \dots, d\}^p$ , we set  $\partial_x^{i_1 \dots i_p} f$  for  $\frac{\partial^p f}{\partial x_{i_1} \dots \partial x_{i_p}}$ . Analogously, we use the notation  $\partial_\theta^{i_1 \dots i_p} f$  for  $\frac{\partial^p f}{\partial \theta_{i_1} \dots \partial \theta_{i_p}}$ , where  $(i_1, \dots, i_p)$  is a tuple in  $\{1, \dots, q\}^p$ . Moreover, we will write  $\partial_x f$  resp.  $\partial_\theta f$  for the Jacobi-matrices  $(\partial_{x_1} f, \dots, \partial_{x_d} f)$  and  $(\partial_{\theta_1} f, \dots, \partial_{\theta_q} f)$ . Finally, for the sake of simplicity, we denote by  $\langle \cdot, \cdot \rangle$  the Euclidean scalar product in  $\mathbb{R}^q$  or  $\mathbb{R}^d$ , and by  $|\cdot|$  the corresponding Euclidean norm.

Let us now state the linear growth plus inward assumptions on our drift coefficient ensuring ergodic properties for the process  $Y$ :

**Hypothesis 1.3.** (i) There exists  $\alpha > 0$  such that for every  $x, y \in \mathbb{R}^d$  and  $\theta \in \Theta$  we have

$$\langle b(x; \theta) - b(y; \theta), x - y \rangle \leq -\alpha |x - y|^2.$$

(ii) We have  $b \in C^{2,1}(\mathbb{R}^d \times \Theta; \mathbb{R}^d)$ , with  $\partial_x b$  and  $\partial_{xx}^2 b$  uniformly bounded in  $(x, \theta)$ .

(iii) We have  $\hat{b} := \partial_\theta b \in C^{2,0}(\mathbb{R}^d \times \Theta; \mathbb{R}^d \times \mathbb{R}^q)$ , with  $\partial_x \hat{b}$  and  $\partial_{xx}^2 \hat{b}$  uniformly bounded in  $(x, \theta)$ .

(iv) There exists  $c > 0$  such that for every  $x, y \in \mathbb{R}^d$  and  $\theta, \theta_0 \in \Theta$ , the following Lipschitz and growth conditions hold:

$$\begin{aligned} |b(x; \theta)| &\leq c(1 + |x|), & |\hat{b}(x; \theta)| &\leq c(1 + |x|), \\ |\hat{b}(x; \theta) - \hat{b}(x; \theta_0)| &\leq c|\theta - \theta_0|(1 + |x|). \end{aligned}$$

Furthermore, we suppose that the coefficient  $\sigma$  fulfills an invertibility condition of the following form:

**Hypothesis 1.4.** The dimension  $d$  of driving fBm equals the dimension of the state space  $\mathbb{R}^d$  for  $Y$ . In addition, denoting by  $\sigma$  the  $d \times d$  matrix with columns  $\sigma_1, \dots, \sigma_d$ , we assume that  $\sigma$  is invertible.

Notice that **Hypotheses 1.3** and **1.4** entail the following result (see next section for more details): for a given  $\theta \in \Theta$  the solution to Eq. (3) satisfies a.s.  $\lim_{t \rightarrow \infty} |Y_t - \bar{Y}_t| = 0$ , where  $\bar{Y} = \{\bar{Y}_t, t \geq 0\}$  is a stationary and ergodic stochastic process.

Let us now introduce the concept of the LAN property in our context. Towards this aim, for any  $\theta \in \Theta$  and  $\lambda < H$ , we denote by  $\mathbf{P}_\theta$  (resp.  $\mathbf{P}_\theta^\tau$ ) the probability laws of the solution to Eq. (3) in the spaces  $C^\lambda(\mathbb{R}_+; \mathbb{R}^d)$  (resp.  $C^\lambda([0, \tau]; \mathbb{R}^d)$ ). Moreover, we assume that the process  $Y$  is observed continuously in  $[0, \tau]$ . In this context, the definition of the LAN property takes the following form:

**Definition 1.5.** We say that the parametric statistical model  $\{\mathbf{P}_\theta, \theta \in \Theta\}$  satisfies the LAN property at  $\theta \in \Theta$  if there exist:

- (i) A  $q \times q$  invertible matrix  $\varphi_\tau(\theta)$ ,
- (ii) A  $q \times q$  positive definite matrix  $\Sigma(\theta)$ ,

such that for any  $u \in \mathbb{R}^q$ , the following limit holds true as  $\tau \rightarrow \infty$ :

$$\log \left( \frac{d\mathbf{P}_{\theta + \varphi_\tau(\theta)u}^\tau}{d\mathbf{P}_\theta^\tau} \right) \xrightarrow{\mathcal{L}(\mathbf{P}_\theta)} u^T \mathcal{N}(0, \Sigma(\theta)) - \frac{1}{2} u^T \Sigma(\theta) u,$$

where  $\mathcal{N}(0, \Sigma(\theta))$  is a centered  $q$ -dimensional Gaussian random variable with covariance matrix  $\Sigma(\theta)$ .

The LAN property is a fundamental concept in asymptotic theory of statistics, which was developed by Le Cam [24]. The essence of LAN is that the local log-likelihood ratio is asymptotically normally distributed, with a locally constant covariance matrix and a mean equal to minus one half the variance. The main application of the LAN property is the following Minimax Theorem. It asserts (if LAN holds true) an asymptotic (minimax) lower bound for the risk with respect to a loss function, for any sequence  $\hat{\theta}_\tau$  of estimators of  $\theta$ .

A loss function is defined as a function  $l : \mathbb{R}^q \rightarrow [0, +\infty)$  satisfying the following properties:

- For any  $u \in \mathbb{R}^q$ ,  $l(u) = l(-u)$ ,  $l$  is continuous at 0,  $l(0) = 0$  but is not identically 0.
- For all  $c > 0$  the sets  $\{u : l(u) < c\}$  are convex.
- $l(u)$  has growth as  $|u| \rightarrow \infty$  less than  $e^{\epsilon|u|^2}$ , for any  $\epsilon > 0$ .

**Theorem 1.6** (Minimax Theorem [17, Theorem II.12.1], [22, Theorem 2.4]). *Suppose that the family of probability measures  $(\mathbf{P}_\theta)_{\theta \in \Theta}$  satisfies the LAN property at a point  $\theta$ . Let  $(\hat{\theta}_\tau)_{\tau \geq 0}$  be a family of estimators of the parameter  $\theta$ . Then for any loss function  $l$  we have:*

$$\lim_{\delta \rightarrow 0} \liminf_{\tau \rightarrow \infty} \sup_{\theta' \in \Theta: |\theta' - \theta| < \delta} \mathbf{E}_{\theta'} \left[ l \left( \varphi_\tau^{-1}(\theta) \left( \hat{\theta}_\tau - \theta' \right) \right) \right] \geq \mathbf{E}_\theta [l(Z)], \tag{4}$$

where  $\mathcal{L}(Z) = \mathcal{N}(0, \Sigma(\theta)^{-1})$ .

Observe that when  $l(u) = |u|^2$ , the lower bound in (4) is simply the trace of  $\Sigma(\theta)^{-1}$ . A sequence of estimators that attains this asymptotic bound for some loss function  $l$  is called asymptotically efficient for this loss function. Therefore, Theorem 1.6 opens the way to a theory which mimics the concept of efficient estimators from the Cramér–Rao lower bound.

As one can see, the LAN property is an important tool in order to quantify the identifiability of a system. The aim of this paper is thus to show the following result for our Eq. (3).

**Theorem 1.7.** *Assume Hypothesis 1.3, and recall that  $\mathbf{P}_\theta^\tau$  stands for the probability law of the solution to Eq. (3) in the space  $\mathcal{C}^\lambda([0, \tau]; \mathbb{R}^d)$ . Then, for any  $\theta \in \Theta$  and  $u \in \mathbb{R}^q$  fixed, as  $\tau \rightarrow \infty$ , we have:*

$$\log \frac{d\mathbf{P}_{\theta + \frac{u}{\sqrt{\tau}}}^\tau}{d\mathbf{P}_\theta^\tau} \xrightarrow{\mathcal{L}(\mathbf{P}_\theta)} u^T \mathcal{N}(0, \Sigma(\theta)) - \frac{1}{2} u^T \Sigma(\theta) u, \tag{5}$$

where the matrix  $\Sigma(\theta)$  is defined by:

$$\Sigma(\theta) := C_H \int_{\mathbb{R}_+^2} \frac{\mathbf{E}_\theta [(\hat{b}(\bar{Y}_0; \theta) - \hat{b}(\bar{Y}_{r_1}; \theta))^T (\sigma^{-1})^T \sigma^{-1} (\hat{b}(\bar{Y}_0; \theta) - \hat{b}(\bar{Y}_{r_2}; \theta))]}{r_1^{1/2+H} r_2^{1/2+H}} dr_1 dr_2, \tag{6}$$

where the constant  $C_H$  is defined by:

$$C_H = \frac{\sin^2(\pi(H - 1/2)) \Gamma^2(H + 1/2)}{2H\pi^2 \sin(\pi H) \Gamma(2H)} = [\sin(\pi H) \Gamma(1/2 - H)^2 \Gamma(2H + 1)]^{-1}. \tag{7}$$

The LAN property for stochastic processes has been widely explored in the literature. To mention a few references close to our contribution, the case of ergodic diffusion processes observed continuously (with an unknown parameter on the drift coefficient) is dealt with in e.g. [22, Proposition 2.2]. The proof follows from Girsanov’s theorem, ergodic properties and the central limit theorem for Brownian martingales. The rate of convergence achieved in [22, Proposition 2.2] is  $\sqrt{\tau}$  like in our Theorem 1.7, with a function  $\Gamma(\theta)$  given by:

$$\Sigma(\theta) = \mathbf{E}_\theta [\hat{b}(\bar{Y}; \theta)^T \sigma^{-1}(\bar{Y})^T \sigma^{-1}(\bar{Y}) \hat{b}(\bar{Y}; \theta)],$$

which should be compared to our expression (6). The case where the Brownian diffusion process is observed discretely was solved in [14] using an integration by parts formula taken from the Malliavin calculus. Let us also mention that the LAN property is obtained in [7] for some discretely observed fractional noises including fractional Brownian motion. This last result is achieved by a direct expansion of log-likelihood functions, plus a thorough analysis based on properties of Toeplitz matrices.

Let us say a few words about the strategy we have followed in order to handle the case of Eq. (3). In order to compare likelihood functions for different values of the parameter  $\theta$ , we take the obvious choice of applying Girsanov's theorem for fractional Brownian motion (following the steps of [26]). We are then left with two technical difficulties in order to get asymptotic results: (i) Handle the singularities popping out of the fractional derivatives in the Girsanov exponent. (ii) Get ergodic results in Hölder type norms for our process  $Y$ . Let us also mention that CLTs for martingales, which were an essential tool in the diffusion case, are unavailable in our fBm situation.

Interestingly enough, our general Theorem 1.7 applies to the fractional Ornstein–Uhlenbeck case (that is,  $d = 1$  and  $b(x; \theta) = -\theta x$  for  $\theta > 0$ .) In this context we find (i) that  $\Sigma(\theta)$  does not depend on  $H$ . (ii) that the MLE reaches the lower bound in (4), and is thus asymptotically minimax efficient.

Finally, the generalization of Theorem 1.7 to a fBm  $B$  with Hurst parameter  $H < 1/2$  is obviously a natural problem to consider. To this aim and following the strategy we have just summarized above, one can try to apply Girsanov's transform and ergodic theorems for the solution process  $Y$  to (3). Then the main difficulty to implement this strategy stems from the fact that one is always left with evaluations of integrals similar to the right-hand side of relation (6). Indeed, when  $H < 1/2$  it is easily seen that this kind of integral exhibits some divergences as  $r_1, r_2 \rightarrow \infty$ . After this paper was completed, a study concerning the case  $H \in (1/4, 1/2)$  was completed (see [5]) for a 1-dimensional fractional Brownian motion (namely  $d = 1$  in our notation). The global strategy followed in [5] is the same as ours, and an extra care is taken in order to bound the numerator in the right hand side of (6) thanks to Malliavin calculus techniques. The fact that  $d = 1$  is important in [5], since an explicit expression for the Malliavin derivative of  $\bar{Y}$  is available in this case. Therefore, to the best of our knowledge, the LAN property for a multidimensional equation like (3) and a Hurst parameter  $H < 1/2$  is still open.

Our paper is structured as follows: Section 2 is devoted to necessary preliminary results. Then we prove Theorem 1.7 in Section 3, following the strategy described above. Eventually, in Section 4, we analyze the particular case of the fractional Ornstein–Uhlenbeck process.

Let us close this introduction by giving a set of notations which will prevail until the end of the paper.

**Notation 1.8.** We use the following conventions: for 2 quantities  $a$  and  $b$ , we write  $a \lesssim b$  if there exists a universal constant  $c$  such that  $a \leq cb$ . In the same way, we write  $a \asymp b$  whenever  $a \lesssim b$  and  $b \lesssim a$ . If  $f$  is a vector-valued function defined on an interval  $[0, \tau]$  and  $s, t \in [0, \tau]$ ,  $\delta f_{st}$  designates the increment  $f_t - f_s$ .

## 2. Auxiliary results

In this section we first recall some basic facts about stochastic analysis for a fBm  $B$ , and also about Young integrals. Then we shall derive some pathwise and probabilistic estimates for Eq. (3) under the coercive Hypothesis 1.3.

2.1. Stochastic analysis related to  $B$

The ergodic properties of Eq. (3) are accurately encoded by the fBm representation given in [15], which goes back to the original paper [25]. We present this construction here for a one-dimensional fBm, for sake of simplicity. Obvious generalizations to the  $d$ -dimensional case are left to the reader.

Let  $W$  be a two-sided Brownian motion, and  $H$  a Hurst parameter in  $(0, 1)$ . We consider the process  $B$  defined for  $t \in \mathbb{R}$  by:

$$\begin{aligned}
 B_t &= c_1(H) \int_{\mathbb{R}_-} (-r)^{H-1/2} [dW_{t+r} - dW_r] \\
 &= c_1(H) \left\{ \int_{-\infty}^0 [(t-r)^{H-1/2} - (-r)^{H-1/2}] dW_r - \int_0^t (t-r)^{H-1/2} dW_r \right\},
 \end{aligned}
 \tag{8}$$

where the constant  $c_1(H)$  is defined by:

$$c_1(H) = \frac{[2H \sin(\pi H) \Gamma(2H)]^{1/2}}{\Gamma(H + 1/2)}.
 \tag{9}$$

Then  $B$  is well-defined as a fBm, that is a centered Gaussian process with covariance given by relation (1). Eq. (8) is often referred to as Mandelbrot’s representation of fractional Brownian motion.

The process  $B$  defined as (8) is closely related to the following fractional derivatives: for  $\alpha \in (0, 1)$  and  $\varphi \in C_c^\infty(\mathbb{R})$  we set

$$[D_+^\alpha \varphi]_t = \frac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} \frac{\varphi_t - \varphi_{t-r}}{r^{1+\alpha}} dr, \quad \text{and} \quad [I_+^\alpha \varphi]_t = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}_+} \varphi_{t-r} r^{\alpha-1} dr.
 \tag{10}$$

With this notation in mind, the following proposition identifies a convenient operator which transforms  $W$  into  $B$  as in (8). We refer to [15, Lemma 3.6] for further details.

**Proposition 2.1.** For  $w \in C_c^\infty(\mathbb{R})$  such that  $w_0 = 0$  and  $H \in (0, 1)$ , set

$$[K_H w]_t = c_1(H) \int_{\mathbb{R}_-} (-r)^{H-1/2} [\dot{w}_{t+r} - \dot{w}_r] dr.$$

Then the following holds true:

(i) The operator  $K_H$  admits the following expression:

$$[K_H w]_t = \begin{cases} c_2(H) \left( [I_+^{H-1/2} w]_t - [I_+^{H-1/2} w]_0 \right), & \text{for } H > \frac{1}{2} \\ -c_2(H) \left( [D_+^{1/2-H} w]_t - [D_+^{1/2-H} w]_0 \right), & \text{for } H < \frac{1}{2}, \end{cases}$$

where the constant  $c_2(H)$  is given by  $c_2(H) = c_1(H)\Gamma(H + \frac{1}{2})$ , with  $c_1(H)$  defined by (9).

(ii) Define for every  $w \in C_c^\infty(\mathbb{R})$  the norm

$$\|w\|_H = \sup_{s,t \in \mathbb{R}} \frac{|w(t) - w(s)|}{|t - s|^{(1-H)/2} (1 + |t| + |s|)^{1/2}}.$$

Define the Banach space  $\mathcal{H}_H$  to be the closure of  $C_c^\infty(\mathbb{R})$  under the norm  $\|\cdot\|_H$ . Then  $K_H$  can be extended continuously to  $\mathcal{H}_H$ .

(iii) For all  $H \in (0, 1)$  we have  $K_H^{-1} = -(c_2(H)c_2(1-H))^{-1} K_{1-H}$ .

Also notice that in the sequel, our underlying Wiener space will be defined as  $(\mathcal{H}_H, \mathbf{P})$ , where  $\mathcal{H}_H$  is introduced in Proposition 2.1(ii), and  $\mathbf{P}$  is the unique Wiener measure on  $\mathcal{H}_H$  (see [15] for further details). The law  $\mathbf{P}_\theta^\tau$  of the solution process  $Y|_{[0,\tau]}$  is defined as the image of  $\mathbf{P}$  by the map  $\omega \mapsto Y(\omega)|_{[0,\tau]}$ .

2.1.1. Girsanov transform

Let us set the ground for our Girsanov transform by relating a general drift with respect to a fractional Brownian motion with its counterpart with respect to  $W$  (we continue in the one-dimensional setting for the sake of simplicity). This proposition is a slight extension of [15, Lemma 4.2].

**Proposition 2.2.** *Let  $H \in (1/2, 1)$ ,  $b^1, b^2$  be two paths in  $I_+^{H-1/2}(L^2(\mathbb{R}))$ , and suppose that  $b^j = K_H w^j$  for  $j = 1, 2$ , where  $K_H$  is defined in Proposition 2.1. We also assume that for  $t \geq 0$ ,  $b^1$  and  $b^2$  are linked by the relation:*

$$b_0^2 = b_0^1, \quad \text{and} \quad b_t^2 = b_t^1 + \int_0^t g_{b,s} ds, \quad t \geq 0 \tag{11}$$

where  $g_b$  is a function in  $I_+^{H-1/2}(L^2(\mathbb{R}))$  and  $g_b = 0$  for  $t < 0$ . Then we also have:

$$w_t^2 = w_t^1 + \int_0^t g_{w,s} ds,$$

with

$$g_w = c_2(H)^{-1} D_+^{H-1/2} g_b,$$

where we recall that the constant  $c_2(H) = c_1(H)\Gamma(H + \frac{1}{2})$  has been introduced in Proposition 2.1.

**Proof.** Let us write  $G_{b,t} = \int_0^t g_{b,s} ds$  for  $t \geq 0$  and  $G_{b,t} = 0$  for  $t < 0$ , so that (11) can be read as:

$$b_t^2 = b_t^1 + G_{b,t}. \tag{12}$$

Moreover, invoking the fact that  $b^j = K_H w^j$  for  $j = 1, 2$ , one can recast (12) into:

$$K_H w^2 = K_H (w^1 + G_w), \quad \text{where} \quad G_w = K_H^{-1} G_b.$$

Therefore, thanks to the relation  $K_H^{-1} = -c_2(H)^{-1}c_2(1 - H)^{-1}K_{1-H}$ , we end up with:

$$g_w = -c_2(H)^{-1}c_2(1 - H)^{-1}D_+^1 K_{1-H} G_b. \tag{13}$$

Since  $G_b \in I^{H-1/2}(L_2)$ , the operator  $K_{1-H}$  can be interpreted as a fractional differentiation. Specifically, applying the definition of  $K_{1-H}$  contained in Proposition 2.1(i), we have

$$K_{1-H} G_b = -c_2(1 - H)((D^{H-1/2} G_b)_t - (D^{H-1/2} G_b)_0). \tag{14}$$

Substituting (14) into (13) we obtain

$$g_w = c_2(H)^{-1} D_+^1 D_+^{H-1/2} G_b = c_2(H)^{-1} D_+^{H-1/2} g_b,$$

which is our claim.  $\square$

The previous proposition yields a version of Girsanov’s theorem in a fractional Brownian context, inspired by [26] (see also [8,28]):

**Proposition 2.3.** *Let  $B$  be a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$  and let  $g$  be a process in  $I_+^{H-1/2}(L^2(\mathbb{R}))$  adapted to  $\sigma(W_s, s \leq t)$ . We set  $V_t = \int_0^t g_s ds$  and  $Q_t = B_t + V_t$  for  $t \geq 0$ . Then  $Q$  is a fractional Brownian motion on  $[0, \tau]$ , under the probability  $\tilde{\mathbf{P}}$  defined by  $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}|_{[0, \tau]} = e^{-L}$ , where*

$$L = \frac{1}{c_2(H)} \int_0^\tau [D_+^{H-1/2}g]_u dW_u + \frac{1}{2(c_2(H))^2} \int_0^\tau [D_+^{H-1/2}g]_u^2 du, \tag{15}$$

provided  $D_+^{H-1/2}g$  satisfies Novikov’s conditions (see [10, Theorem 7.1.1]) on  $[0, \tau]$ , that is, that there exists  $\lambda > 0$  such that

$$\sup_{t \in [0, \tau]} \mathbf{E}_\theta \left[ \exp \left( \lambda [D_+^{H-1/2}g]_t^2 \right) \right] < \infty. \tag{16}$$

### 2.2. Generalized Riemann–Stieltjes integrals

We are focusing here on the case of a Hurst parameter  $H > 1/2$ , and the stochastic integrals with respect to  $B$  should be understood in the Young (or Riemann–Stieltjes) sense. In order to recall the definition of this integral, we set

$$|f|_{\infty; [a, b]} = \sup_{t \in [a, b]} |f(t)|, \quad |f|_{\lambda; [a, b]} = \sup_{s, t \in [a, b]} \frac{|f(t) - f(s)|}{|t - s|^\lambda}, \tag{17}$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $\lambda \in (0, 1)$ . We also recall the classical definition of Hölder-continuous functions:

$$C^\lambda([a, b]; \mathbb{R}^n) = \{ f : [a, b] \rightarrow \mathbb{R}^n; |f|_{\infty; [a, b]} + |f|_{\lambda; [a, b]} < \infty \}.$$

Now, let  $f \in C^\lambda([a, b]; \mathbb{R})$  and  $g \in C^\mu([a, b]; \mathbb{R})$  with  $\lambda + \mu > 1$ . Then it is well known that the Riemann–Stieltjes integral  $\int_a^b f_s dg_s$  exists, and can be expressed as a limit of Riemann sums.

In the sequel, the only property on Young’s integral we shall use is the classical chain rule for changes of variables. Indeed, let  $f \in C^\lambda([a, b]; \mathbb{R})$  with  $\lambda > 1/2$  and  $F \in C^1(\mathbb{R}; \mathbb{R})$ . Then we have:

$$F(f_y) - F(f_a) = \int_a^y F'(f_s) df_s, \quad y \in [a, b]. \tag{18}$$

### 2.3. Basic properties of solutions to SDEs

In order to deal with stationary solutions, let us first extend our equation as a process indexed by  $(-\infty, \tau]$ , by considering a process  $Y$  solution to:

$$dY_t = b(Y_t; \theta) dt + \sum_{j=1}^d \sigma_j dB_t^j, \quad t \in [0, \tau], \quad \text{and} \quad Y_t = y_0, \quad t \in (-\infty, 0]. \tag{19}$$

We first state the existence and uniqueness result for Eq. (19) borrowed from [15, Lemma 3.9].

**Proposition 2.4.** *Under Hypothesis 1.3, there exists a unique continuous pathwise solution to Eq. (19) on any arbitrary interval  $[0, \tau]$ . Moreover the map  $Y : (y_0, B) \in \mathbb{R}^d \times \mathcal{C}([0, \tau]; \mathbb{R}^d) \rightarrow \mathcal{C}([0, \tau]; \mathbb{R}^d)$  is locally Lipschitz continuous.*



In addition, exploiting the integrability properties of the stationary fractional Ornstein–Uhlenbeck process, the following uniform estimates hold true. Their proof is omitted, since it is only a slight variation on the proof of [15, Proposition 3.12].

**Proposition 2.5.** Assume Hypothesis 1.3 holds true and let  $Y$  be the solution to Eq. (19). Then for any  $\theta \in \Theta$  and  $p \geq 1$  there exist constants  $c_p, k_p > 0$  such that

$$\mathbf{E}[|Y_t|^p] \leq c_p, \quad \text{and} \quad \mathbf{E}[|\delta Y_{st}|^p] \leq k_p |t - s|^{pH},$$

for all  $s, t \geq 0$ , where we recall that we have set  $\delta Y_{st} = Y_t - Y_s$  as mentioned in Notation 1.8.

Let us now state a path-wise estimates on the increments of  $Y$ , where we translate the moment estimates of Proposition 2.5 into path-wise bounds along the same lines as in [19].

**Proposition 2.6.** Assume Hypothesis 1.3 holds true and let  $Y$  be the solution to Eq. (19). Then for all  $\varepsilon \in (0, H)$  there exists a random variable  $Z_\varepsilon \in \cap_{p \geq 1} L^p(\Omega)$  such that almost surely we have:

$$|Y_t| \leq Z_\varepsilon (1 + t)^{2\varepsilon}, \quad \text{and} \quad |\delta Y_{st}| \leq Z_\varepsilon (1 + t)^{2\varepsilon} |t - s|^{H-\varepsilon}, \tag{20}$$

uniformly for  $0 \leq s \leq t$ .

**Proof.** We focus on the bound for  $\delta Y_{st}$ , the estimate on  $|Y_t|$  being shown in the same manner. Set  $\gamma = H - \varepsilon$  and consider  $n \geq 1$ . Thanks to Garsia’s lemma (see e.g. [11]) and recalling our Notation 1.8, for all  $s, t \in [0, n]$  we have  $|\delta Y_{st}| \lesssim \mathcal{N}_{\gamma,p,n} |t - s|^\gamma$ , where  $p \geq 1$  and  $\mathcal{N}_{\gamma,p,n}$  is defined by:

$$\mathcal{N}_{\gamma,p,n} \equiv \left( \int_0^n \int_0^n \frac{|\delta Y_{uv}|^p}{|v - u|^{\gamma p + 2}} \, dudv \right)^{\frac{1}{p}}.$$

Furthermore, owing to Proposition 2.5 we immediately get:

$$\mathbf{E}^{1/p} [\mathcal{N}_{\gamma,p,n}^p] \lesssim n^\varepsilon,$$

under the constraint  $p \geq \varepsilon^{-1}$ . Set now  $Z_\varepsilon = \sup_{n \in \mathbb{N}} \frac{|\mathcal{N}_{\gamma,p,n}|}{n^{2\varepsilon}}$ . Then for all  $q > \varepsilon^{-1}$  we have:

$$\mathbf{E}[Z_\varepsilon^q] = \mathbf{E} \left[ \sup_{n \in \mathbb{N}} \frac{|\mathcal{N}_{\gamma,p,n}|^q}{n^{2\varepsilon q}} \right] \leq \sum_{n=1}^\infty \frac{\mathbf{E}[|\mathcal{N}_{\gamma,p,n}|^q]}{n^{2\varepsilon q}} \lesssim \sum_{n=1}^\infty \frac{1}{n^{\varepsilon q}} < \infty, \tag{21}$$

namely we have found  $Z_\varepsilon \in \cap_{p > \varepsilon^{-1}} L^p(\Omega)$ . Obviously this also yields  $Z_\varepsilon \in \cap_{p \geq 1} L^p(\Omega)$ . Finally, replace  $n \geq 1$  by  $[t] + 2$  to obtain our pathwise bound (20).  $\square$

We shall need the following bound for the solution. This will be invoked while checking Novikov’s condition.

**Lemma 2.7.** Assume Hypothesis 1.3 and let  $Y$  be the solution to (19). Then, for all  $t \geq 0$ ,  $Y_t$  satisfies the following bound:

$$|Y_t| \leq \kappa_1 e^{\kappa_2 t} \sup_{s \in [0,t]} |B_s|, \tag{22}$$

where  $\kappa_1$  and  $\kappa_2$  are two positive constants depending on  $b$  and  $\sigma$ . Furthermore, the increments of  $Y$  are such that for all  $r < s$ :

$$|\delta Y_{rs}| \leq \kappa_3 (s - r) (1 + e^{\kappa_4 s} \sup_{u \in [0,s]} |B_u|) + \kappa_5 |\delta B_{rs}|. \tag{23}$$

**Proof.** Using (19) and the linear growth of  $b$ , we have

$$|Y_t| \leq |y_0| + ct + \|\sigma\| |B_t| + c \int_0^t |Y_s| ds.$$

Gronwall’s lemma concludes the proof of (22). In order to prove (23), we just write

$$\delta Y_{rs} = \int_r^s b(Y_u; \theta) du + \sum_{j=1}^d \sigma_j \delta B_{rs}. \tag{24}$$

According to Hypothesis 1.3(iv) and (22) we have

$$|b(Y_u; \theta)| \leq c_b(1 + |Y_u|) \leq c_b(1 + \kappa_1 e^{\kappa_2 u} \sup_{v \leq u} |B_v|).$$

Plugging this bound into (24) we easily get (23).  $\square$

### 2.4. Ergodic properties of the SDE

We recall here some basic facts about the limiting behavior of Eq. (19), mainly taken from [12]. We still work on the Wiener space  $(\mathcal{H}_H, \mathbf{P})$  introduced in Section 2.1, and seen as the canonical probability space. Together with the shift operators  $\theta_t : \Omega \rightarrow \Omega$  defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in \Omega,$$

our probability space defines an ergodic metric dynamical system, see e.g. [13]. In particular, the measure  $\mathbf{P}$  is invariant to the shift operators  $\theta_t$ , i.e. the shifted process  $(B_s(\theta_t \cdot))_{s \in \mathbb{R}}$  is still an  $m$ -dimensional fractional Brownian motion and for any integrable random variable  $F : \Omega \rightarrow \mathbb{R}$  we have

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau F(\theta_t(\omega)) dt = \mathbf{E}[F], \tag{25}$$

for  $\mathbf{P}$ -almost all  $\omega \in \Omega$ .

Under our coercive hypothesis on the drift coefficient of Eq. (19), the following limit theorem is borrowed from [12, Section 4]. Notice that its proof is based on contraction properties for the stochastic equation.

**Theorem 2.8.** *Let Hypothesis 1.3 hold, and consider the unique solution  $Y$  to Eq. (19) as given in Proposition 2.4. Then there exists a random variable  $\bar{Y} : \Omega \rightarrow \mathbb{R}^d$  such that*

$$\lim_{t \rightarrow \infty} |Y_t(\omega) - \bar{Y}(\theta_t \omega)| = 0$$

for  $\mathbf{P}$ -almost all  $\omega \in \Omega$ . Moreover, we have  $\mathbf{E}[|\bar{Y}|^p] < \infty$  for all  $p \geq 1$ .

With Theorem 2.8 in hand, let us label a notation for further use.

**Notation 2.9.** *In the sequel we will denote the stationary process  $\bar{Y}(\theta_t \omega)$  by  $\bar{Y}_t$ .*

It is worth observing that the convergence of  $Y$  towards  $\bar{Y}$  can also be quantified in Hölder norm:

**Proposition 2.10.** Assume Hypothesis 1.3 holds true. Let  $\beta \in (0, 1)$ . There exists a random variable  $Z$  admitting moments of any order and 3 constants  $c_1, c_2, c_3 > 0$  such that for all  $0 \leq s \leq t$  we have:

$$|Y_t - \bar{Y}_t| \leq Z e^{-c_1 t} \quad \text{and} \quad |\delta[Y - \bar{Y}]_{st}| \leq c_2 Z e^{-c_3 s} (t - s)^\beta.$$

**Proof.** The difference  $Y_t - \bar{Y}_t$  satisfies

$$Y_t - \bar{Y}_t = y_0 - \bar{Y}_0 + \int_0^t [b(Y_s; \theta) - b(\bar{Y}_s; \theta)] ds$$

Applying the change of variable formula in the Young setting (18) and using Hypothesis 1.3(i), we get:

$$\begin{aligned} |Y_t - \bar{Y}_t|^2 &= |y_0 - \bar{Y}_0|^2 + 2 \int_0^t \langle b(Y_s; \theta) - b(\bar{Y}_s; \theta), Y_s - \bar{Y}_s \rangle ds \\ &\leq |y_0 - \bar{Y}_0|^2 - 2\alpha \int_0^t |Y_s - \bar{Y}_s|^2 ds. \end{aligned}$$

The first claim follows by a direct application of Gronwall’s lemma.

Similarly, for our additional parameter  $\beta \in (0, H)$ , we have

$$|\delta[Y - \bar{Y}]_{st}| = \left| \int_s^t [b(Y_u; \theta) - b(\bar{Y}_u; \theta)] du \right| \leq c_b (I_{st})^{1-\beta} I_{st}^\beta,$$

where  $c_b$  is the uniform bound on the Lipschitz constant of  $b$ , and where

$$I_{st} = \int_s^t |Y_u - \bar{Y}_u| du.$$

We now bound  $I_{st}$  in two different ways:

(i) Since we have just proved that  $|Y_u - \bar{Y}_u| \leq Z e^{-c_1 u}$ , we get

$$I_{st} \leq Z \int_s^t e^{-c_1 u} du \leq Z \int_0^\infty e^{-c_1 u} du = \frac{Z}{c_1}. \tag{26}$$

(ii) Still using the relation  $|Y_u - \bar{Y}_u| \leq Z e^{-c_1 u}$ , it is also readily checked that:

$$I_{st} \leq Z \int_s^t e^{-c_1 u} du \leq Z e^{-c_1 s} (t - s).$$

Plugging those two bounds into (26), we get

$$I_{st} \leq \frac{Z e^{-c_1 s}}{c_1^{1-\beta}} (t - s)^\beta,$$

from which our second claim is easily deduced.  $\square$

We now recall, similarly to [27], that the integrability of  $\bar{Y}$  implies the ergodicity of Eq. (19):

**Proposition 2.11.** Assume Hypothesis 1.3 holds true. Consider a finite horizon  $\rho > 0$ , a generic parameter  $\theta \in \Theta$ , and the set  $C^\beta([-\rho, 0])$  of Hölder continuous functions on the interval  $[-\rho, 0]$  for an exponent  $\beta < H$ . Let  $F$  be a functional on  $C^\beta([-\rho, 0])$  such that  $F$  is Fréchet differentiable and

$$|F(g)| + \|DF(g)\| \leq c \left( 1 + |g|_{\beta; [-\rho, 0]}^N \right), \quad g \in C^\beta([-\rho, 0]),$$

for some  $c > 0$ ,  $N \in \mathbb{N}$ . Then the following limit holds true:

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau F(Y_{|[t-\rho, t]}) dt = \mathbf{E}[f(\bar{Y})_{|[-\rho, 0]}], \quad \mathbf{P}\text{-a.s.} \tag{27}$$

**Proof.** This is a direct consequence of Proposition 2.10 and the fact that relation (25) holds true for the ergodic system  $(w, \bar{Y})$ .  $\square$

### 3. Proof of Theorem 1.7

We can now gather the information we have obtained on the system (19) in order to complete the proof of Theorem 1.7. This will be divided in different steps.

*Step 1.* The first step of the proof consists in applying Girsanov’s theorem.

We first notice that  $D_+^{H-1/2}b(Y; \theta)$  is well defined on  $[0, \tau]$ , which follows since the function  $u \mapsto b(Y_u; \theta)$  is  $(H - \varepsilon)$ -Hölder, and thus  $(H - 1/2)$ -Hölder.

Fix  $\theta \in \Theta$ , and set  $\theta_\tau = \theta + \tau^{-1/2}u$ . Suppose that Novikov’s condition holds. Then our Girsanov-type Proposition 2.3 implies that

$$\begin{aligned} \log \left( \frac{d\mathbf{P}_{\theta_\tau}^\tau}{d\mathbf{P}_\theta^\tau} \right) &= -\frac{1}{c_2(H)} \int_0^\tau \langle \sigma^{-1}([D_+^{H-1/2}b(Y; \theta)]_t - [D_+^{H-1/2}b(Y; \theta_\tau)]_t), dW_t \rangle \\ &\quad - \frac{1}{2(c_2(H))^2} \int_0^\tau |\sigma^{-1}([D_+^{H-1/2}b(Y; \theta)]_t - [D_+^{H-1/2}b(Y; \theta_\tau)]_t)|^2 dt. \end{aligned} \tag{28}$$

In the following, we show that Novikov’s condition (16) holds for  $\bar{b}_t := b(Y_t; \theta) - b(Y_t; \theta_\tau)$ , that is, that there exists  $\lambda > 0$  such that

$$\sup_{t \in [0, \tau]} \mathbf{E}_\theta \left[ \exp \left( \lambda |\sigma^{-1}[D_+^{H-1/2}\bar{b}]_t|^2 \right) \right] < \infty. \tag{29}$$

Towards this aim, we extend the definition of  $Y$  to  $\mathbb{R}_-$  by setting  $Y_t = y_0$  for all  $t \leq 0$ . Then,

$$|\sigma^{-1}[D_+^{H-1/2}\bar{b}]_t| = c_H \left| \int_{\mathbb{R}_+} \frac{\sigma^{-1}(\bar{b}_t - \bar{b}_{t-r})}{r^{H+1/2}} dr \right| \leq c_H(A_{1,t} + A_{2,t}), \tag{30}$$

where

$$A_{1,t} = \left| \int_0^t \frac{\sigma^{-1}(\bar{b}_t - \bar{b}_{t-r})}{r^{H+1/2}} dr \right| \quad \text{and} \quad A_{2,t} = \left| \int_t^\infty \frac{\sigma^{-1}(\bar{b}_t - \bar{b}_0)}{r^{H+1/2}} dr \right|.$$

Using the mean value theorem and the uniform boundedness of  $\partial_x b$ , the second term is easily bounded as:

$$A_{2,t} \leq c_H \frac{|Y_t - y_0|}{t^{H-1/2}}.$$

In addition, Eq. (18), the linear growth of  $b$  and Lemma 2.7 yield

$$|Y_t - y_0| \leq ct(1 + e^{ct} \sup_{r \in [0, t]} |B_r|) + c|B_t|.$$

Therefore, we obtain that

$$|A_{2,t}|^2 \leq c \frac{|Y_t - y_0|^2}{t^{2H-1}} \leq ce^{ct} (1 + |B|_{\infty; [0, t]}^2 + |B|_{H-\varepsilon; [0, t]}^2). \tag{31}$$

Similarly, we get that for all  $0 < \varepsilon < \frac{1}{2}$ ,

$$A_{1,s} \leq c \sup_{0 < r \leq s} \frac{|\delta Y_{s-r,s}|}{r^{H-\varepsilon}} \int_0^s \frac{1}{r^{\frac{1}{2}+\varepsilon}} dr = c_H s^{\frac{1}{2}-\varepsilon} \sup_{0 < r \leq s} \frac{|\delta Y_{s-r,s}|}{r^{H-\varepsilon}}.$$

Moreover, inequalities (22) and (23) yield:

$$|\delta Y_{s-r,s}| \leq cr(1 + e^{cs} \sup_{u \in [0,t]} |B_u|) + c|\delta B_{s-r,s}|,$$

and thus  $A_{1,s}$  satisfies the following bound:

$$A_{1,t}^2 \leq ce^{ct} (1 + |B|_{\infty;[0,t]}^2 + |B|_{H-\varepsilon;[0,t]}^2). \tag{32}$$

Plugging our estimates (31) and (32) into (30), we have thus obtained

$$|\sigma^{-1}[D_+^{H-1/2} \hat{b}]_t|^2 \leq ce^{ct} (1 + |B|_{\infty;[0,t]}^2 + |B|_{H-\varepsilon;[0,t]}^2).$$

Finally, Fernique’s theorem [9] implies (29).

*Step 2.* We next linearize relation (28). To this aim, recall that  $\hat{b}$  is defined in Hypothesis 1.3 as  $\hat{b} = \partial_\theta b$ . We add and subtract the  $d$ -dimensional vector

$$[D_+^{H-1/2} \hat{b}(Y; \theta)]_t (\theta_\tau - \theta) = \frac{1}{\sqrt{\tau}} [D_+^{H-1/2} \hat{b}(Y; \theta)]_t u,$$

where we abbreviate  $\langle [D_+^{H-1/2} \hat{b}(Y; \theta)]_t, u \rangle_{\mathbb{R}^d}$  into  $[D_+^{H-1/2} \hat{b}(Y; \theta)]_t u$  for notational sake. This easily yields:

$$\log \left( \frac{d\mathbf{P}_{\theta_\tau}^\tau}{d\mathbf{P}_\theta^\tau} \right) = I_1 + I_2 - \frac{1}{2} I_3 - I_4, \tag{33}$$

where

$$\begin{aligned} I_1 &= \frac{1}{\tau^{1/2} c_2(H)} \int_0^\tau \langle \sigma^{-1}[D_+^{H-1/2} \hat{b}(Y; \theta)]_t u, dW_t \rangle \\ &\quad - \frac{1}{2(c_2(H))^2 \tau} \int_0^\tau |\sigma^{-1}[D_+^{H-1/2} \hat{b}(Y; \theta)]_t u|^2 dt \\ I_2 &= \frac{1}{c_2(H)} \int_0^\tau \langle \sigma^{-1}([D_+^{H-1/2} b(Y; \theta_\tau)]_t - [D_+^{H-1/2} b(Y; \theta)]_t \\ &\quad - [D_+^{H-1/2} \hat{b}(Y; \theta)]_t (\theta_\tau - \theta)), dW_t \rangle \\ I_3 &= \frac{1}{(c_2(H))^2} \int_0^\tau |\sigma^{-1}([D_+^{H-1/2} b(Y; \theta_\tau)]_t - [D_+^{H-1/2} b(Y; \theta)]_t \\ &\quad - [D_+^{H-1/2} \hat{b}(Y; \theta)]_t (\theta_\tau - \theta))|^2 dt \end{aligned}$$

and

$$\begin{aligned} I_4 &= \frac{1}{(c_2(H))^2} \int_0^\tau \langle \sigma^{-1}([D_+^{H-1/2} b(Y; \theta_\tau)]_t - [D_+^{H-1/2} b(Y; \theta)]_t \\ &\quad - [D_+^{H-1/2} \hat{b}(Y; \theta)]_t (\theta_\tau - \theta)), \sigma^{-1}[D_+^{H-1/2} \hat{b}(Y; \theta)]_t (\theta_\tau - \theta) \rangle dt. \end{aligned}$$

*Step 3.* In this step we set the ground for the identification of the main contribution to our log-likelihood. That is, we wish to show that as  $\tau \rightarrow \infty$  we have:

$$\frac{1}{\tau(c_2(H))^2} \int_0^\tau |\sigma^{-1}[D_+^{H-1/2} \hat{b}(Y; \theta)]_t u|^2 dt \xrightarrow{\mathbf{P}_\theta} u^T \Sigma(\theta) u, \tag{34}$$

where  $\Sigma(\theta)$  is given by (6). Observe that (34) together with the multivariate central limit theorem for Brownian martingales (cf. [22, Proposition 1.21]), imply that  $I_1$  is the term that contributes to the limit in (5). In Steps 4 and 5 below we will show that  $I_2, I_3$  and  $I_4$  are negligible terms with respect to this main contribution.

In order to prove (34), we first write

$$\frac{1}{\tau(c_2(H))^2} \int_0^\tau |\sigma^{-1}[D_+^{H-1/2}\hat{b}(Y; \theta)]_t u|^2 dt = u^T I(\tau) u,$$

where we have set

$$I(\tau) = \frac{1}{\tau(c_2(H))^2} \int_0^\tau [D_+^{H-1/2}\hat{b}(Y; \theta)^T]_t (\sigma^{-1})^T \sigma^{-1} [D_+^{H-1/2}\hat{b}(Y; \theta)]_t dt. \tag{35}$$

To prove the convergence (34) we are then reduced to show the limit

$$\mathbf{P}_\theta\text{-}\lim_{\tau \rightarrow \infty} I(\tau) = \Sigma(\theta). \tag{36}$$

Next recall that for  $\alpha \in (0, 1)$  and  $\varphi \in C_c^\infty(\mathbb{R})$  the fractional derivative operator  $D_+^\alpha$  is defined by (10). In particular we can write:

$$[D_+^{H-1/2}\hat{b}(Y; \theta)]_t = \frac{H-1/2}{\Gamma(3/2-H)} \int_{\mathbb{R}_+} \frac{\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta)}{r^{H+1/2}} dr. \tag{37}$$

Now substituting (37) into (35) we obtain:

$$I(\tau) = \frac{c_4(H)}{\tau} \int_0^\tau \int_{\mathbb{R}_+^2} \frac{\mathcal{E}_t(r, \tilde{r})}{\tilde{r}^{H+1/2} r^{H+1/2}} dr d\tilde{r} dt, \tag{38}$$

where we have denoted the constant  $c_4(H) = (\frac{H-1/2}{\Gamma(3/2-H)c_2(H)})^2$ , and where  $\mathcal{E}$  stands for the function:

$$\mathcal{E}_t(r, \tilde{r}) = \left( \hat{b}(Y_t; \theta) - \hat{b}(Y_{t-\tilde{r}}; \theta) \right)^T (\sigma^{-1})^T \sigma^{-1} \left( \hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta) \right). \tag{39}$$

We now specify the constant  $c_4(H)$  appearing in (38). Indeed, owing to the definition of  $c_2(H)$  given in Proposition 2.1 and resorting to the elementary identity:

$$\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi, \quad z > -1, \tag{40}$$

we get the following alternative expression for  $c_4(H)$ :

$$c_4(H) = \left( \frac{\sin(\pi(H - \frac{1}{2}))}{\pi c_1(H)} \right)^2. \tag{41}$$

In addition, taking into account the expression for the constant  $c_1(H)$  given in relation (9), it is readily checked that  $c_4(H) = C_H$ , where  $C_H$  is the constant displayed in our main Theorem 1.7.

We now further analyze  $I(\tau)$  as given by (38) by exchanging the order of the integrals. Specifically, we introduce the following notation:

$$\mu(dr, d\tilde{r}) = \frac{1}{\tilde{r}^{H+1/2}} \frac{1}{r^{H+1/2}} dr d\tilde{r} \quad \text{and} \quad J_\tau(r, \tilde{r}) = \frac{1}{\tau} \int_0^\tau \mathcal{E}_t(r, \tilde{r}) dt. \tag{42}$$

Then a standard application of Fubini’s theorem to relation (38), plus the fact that  $c_4(H) = C_H$ , yield:

$$I(\tau) = C_H \int_{\mathbb{R}_+^2} J_\tau(r, \tilde{r}) \mu(dr, d\tilde{r}). \tag{43}$$

Moreover, sending  $\tau$  to  $\infty$  and invoking Proposition 2.11 for the term  $\mathcal{E}_\tau(r, \tilde{r})$  defined by (39), we get the following almost sure limit:

$$\lim_{\tau \rightarrow \infty} J_\tau(r, \tilde{r}) = J(r, \tilde{r}), \tag{44}$$

where we denote:

$$J(r, \tilde{r}) = \mathbf{E}_\theta \left[ \left( \hat{b}(\bar{Y}_0; \theta) - \hat{b}(\bar{Y}_{-\tilde{r}}; \theta) \right)^T (\sigma^{-1})^T \sigma^{-1} \left( \hat{b}(\bar{Y}_0; \theta) - \hat{b}(\bar{Y}_{-r}; \theta) \right) \right].$$

Provided one can pass the limit through the integral in (43), we thus get the almost sure limit:

$$\lim_{\tau \rightarrow \infty} I(\tau) = I := C_H \int_{\mathbb{R}_+^2} J(r, \tilde{r}) \mu(dr, d\tilde{r}). \tag{45}$$

Notice that, thanks to relation (20) for small  $r, \tilde{r}$  and Cauchy–Schwarz’s inequality for large  $r$  or  $\tilde{r}$ , it can be easily checked that  $I$  is a convergent integral.

Let us identify the right-hand side of (45) with the announced expression (6) for  $\Sigma(\theta)$ . To this aim, we consider the expansion

$$\begin{aligned} J(r, \tilde{r}) = & \mathbf{E}_\theta [\hat{b}(\bar{Y}_0; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_0; \theta)] - \mathbf{E}_\theta [\hat{b}(\bar{Y}_{-\tilde{r}}; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_0; \theta)] \\ & - \mathbf{E}_\theta [\hat{b}(\bar{Y}_0; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_{-r}; \theta)] + \mathbf{E}_\theta [\hat{b}(\bar{Y}_{-\tilde{r}}; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_{-r}; \theta)]. \end{aligned}$$

Invoking the stationarity of  $\bar{Y}$  in each of the four terms on the right-hand side above yields the following identity:

$$\begin{aligned} J(r, \tilde{r}) = & \mathbf{E}_\theta [\hat{b}(\bar{Y}_0; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_0; \theta)] - \mathbf{E}_\theta [\hat{b}(\bar{Y}_0; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_{\tilde{r}}; \theta)] \\ & - \mathbf{E}_\theta [\hat{b}(\bar{Y}_r; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_0; \theta)] + \mathbf{E}_\theta [\hat{b}(\bar{Y}_r; \theta)^T (\sigma^{-1})^T \sigma^{-1} \hat{b}(\bar{Y}_{\tilde{r}}; \theta)], \end{aligned}$$

or otherwise stated:

$$J(r, \tilde{r}) = \mathbf{E}_\theta [(\hat{b}(\bar{Y}_0; \theta) - \hat{b}(\bar{Y}_r; \theta))^T (\sigma^{-1})^T \sigma^{-1} (\hat{b}(\bar{Y}_0; \theta) - \hat{b}(\bar{Y}_{\tilde{r}}; \theta))].$$

The expression (6) of  $\Sigma(\theta)$  is obtained by substituting this expression of  $J$  into (45) and exchanging the order of the integrals in  $\tilde{r}$  and  $r$ .

*Step 4.* Summarizing our considerations, in order to prove (34) we are now reduced to take limits in  $\tau$  in relation (43). To this aim, take an arbitrary constant  $\varepsilon > 0$ . In the following, we show that there exists  $\tau(\omega) < \infty, \mathbf{P}_\theta$  a.s. such that  $|I(\tau) - I| < \varepsilon$  for all  $\tau > \tau(\omega)$ , where  $I$  is defined by (45).

Since  $I$  is a convergent integral, we can take a constant  $A = A(\varepsilon) > 0$  such that

$$\left| I - C_H \int_{[0, A]^2} J(r, \tilde{r}) \mu(dr, d\tilde{r}) \right| < \varepsilon \quad \text{and} \quad \int_A^\infty r^{-H-1/2} dr < \varepsilon. \tag{46}$$

For  $V, V' \subset \mathbb{R}_+$  we set

$$I(\tau, V \times V') = C_H \int_{V \times V'} J_\tau(r, \tilde{r}) \mu(dr, d\tilde{r}). \tag{47}$$

In the following, we show that one can take limits in (43) if the integral domain  $\mathbb{R}_+^2$  is replaced by  $[0, A]^2$ . Namely, we will prove that

$$\lim_{\tau \rightarrow \infty} I(\tau, [0, A]^2) = C_H \int_{[0, A]^2} J(r, \tilde{r}) \mu(dr, d\tilde{r}). \tag{48}$$

We first derive some estimates for  $\mathcal{E}_t(r, \tilde{r})$  and  $J_\tau(r, \tilde{r})$ , where we recall that  $\mathcal{E}_t(r, \tilde{r})$  is defined in (39). Take  $0 < \delta < 1/2$ . Proposition 2.6 on the  $(H - \delta)$ -Hölder continuity of the solution process  $Y$  allows to write the relation

$$|\mathcal{E}_t(r, \tilde{r})| \leq K |Y|_{H-\delta, [t-A, t]}^2 r^{H-\delta} \tilde{r}^{H-\delta}, \tag{49}$$

which holds true for all  $0 \leq r, \tilde{r} \leq A$ . Applying relation (49) to  $J_\tau(r, \tilde{r})$  in (42), we then get the following upper bound for  $J_\tau(r, \tilde{r})$ :

$$|J_\tau(r, \tilde{r})| \leq \frac{1}{\tau} \int_0^\tau K |Y|_{H-\delta, [t-A, t]}^2 r^{H-\delta} \tilde{r}^{H-\delta} dt. \tag{50}$$

Notice that Proposition 2.11 implies the almost sure convergence

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau K |Y|_{H-\delta, [t-A, t]}^2 dt = \mathbf{K} \mathbf{E} \left[ |\bar{Y}|_{H-\delta, [-A, 0]}^2 \right].$$

In particular, we can find  $\tau_1(\omega) > 0$  such that for  $\tau > \tau_1(\omega)$

$$\frac{1}{\tau} \int_0^\tau K |Y|_{H-\delta, [t-A, t]}^2 r^{H-\delta} \tilde{r}^{H-\delta} dt \leq g(r, \tilde{r}) := \left( 1 + \mathbf{K} \mathbf{E} \left[ |\bar{Y}|_{H-\delta, [-A, 0]}^2 \right] \right) r^{H-\delta} \tilde{r}^{H-\delta}. \tag{51}$$

So the estimate (50) implies that for  $\tau > \tau_1(\omega)$

$$|J_\tau(r, \tilde{r})| \leq g(r, \tilde{r}). \tag{52}$$

On the other hand, it is clear that  $\int_{[0, A]^2} g(r, \tilde{r}) \mu(dr, d\tilde{r}) < \infty$ . Thus taking into account relation (44) and (52), one can apply the dominated convergence theorem in (47) with  $V = V' = [0, A]$ . This yields (48). In the sequel, we will thus consider  $\tau_2(\omega) < \infty$  such that for  $\tau > \tau_2(\omega)$  we have

$$\left| I(\tau, [0, A]^2) - C_H \int_{[0, A]^2} J(r, \tilde{r}) \mu(dr, d\tilde{r}) \right| < \varepsilon. \tag{53}$$

In the following, we show that  $|I(\tau) - I(\tau, [0, A]^2)| < \varepsilon$  for  $\tau$  sufficiently large. We first show that the quantities  $\frac{1}{\tau} \int_0^\tau |\hat{b}(Y_t; \theta)|^2 dt$  and  $\frac{1}{\tau} \int_0^\tau |\hat{b}(Y_{t-r}; \theta)|^2 dt$  can be bounded by some constants. In fact, by Proposition 2.11 again we have the almost sure convergence

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau |\hat{b}(Y_t; \theta)|^2 dt = \mathbf{E} |\hat{b}(\bar{Y}_0; \theta)|^2.$$

In particular, there exists  $\tau_3(\omega)$  satisfying  $\tau_2(\omega) \leq \tau_3(\omega) < \infty$ , such that the following upper bound estimate holds true for all  $\tau > \tau_3(\omega)$ :

$$\frac{1}{\tau} \int_0^\tau |\hat{b}(Y_t; \theta)|^2 dt \leq K_1 := 1 + \mathbf{E} |\hat{b}(\bar{Y}_0; \theta)|^2. \tag{54}$$

On the other hand, due to the fact that  $Y_u = Y_0$  for  $u \leq 0$ , the following holds true for  $r \leq \tau$ :

$$\begin{aligned} \frac{1}{\tau} \int_0^\tau |\hat{b}(Y_{t-r}; \theta)|^2 dt &= \frac{1}{\tau} \int_r^\tau |\hat{b}(Y_{t-r}; \theta)|^2 dt + \frac{1}{\tau} \int_0^r |\hat{b}(Y_0; \theta)|^2 dt \\ &\leq K_1 + |\hat{b}(Y_0; \theta)|^2 := K_2, \end{aligned} \tag{55}$$

while (55) is also trivially true for  $r > \tau$ . In conclusion, the estimate (55) is valid for all  $\tau \geq \tau_3(\omega)$  and all  $r \geq 0$ .

We will now bound the quantity  $|I(\tau) - I(\tau, [0, A]^2)|$  by sums of elements of the form  $I(\tau, V \times V')$ . Recall that  $I(\tau, V \times V')$  is defined in (47). Those elements will then be estimated



thanks to (54) and (55). To this aim, we first recall that  $J_\tau(r, \tilde{r})$  is defined by (42). Therefore, an elementary application of Hölder’s inequality yields

$$|J_\tau(r, \tilde{r})| \leq K \left( \frac{1}{\tau} \int_0^\tau |\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta)|^2 dt \right)^{1/2} \times \left( \frac{1}{\tau} \int_0^\tau |\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-\tilde{r}}; \theta)|^2 dt \right)^{1/2}. \tag{56}$$

Then applying (56) to the definition (47) of  $I(\tau, V \times V')$  we obtain

$$|I(\tau, V \times V')| \leq K F_V F_{V'}, \tag{57}$$

where the quantity  $F_V$  is defined by

$$F_V = \int_V \left( \frac{1}{\tau} \int_0^\tau |\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta)|^2 dt \right)^{1/2} r^{-H-1/2} dr. \tag{58}$$

As a last preliminary step, let us bound trivially  $\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta)$  as follows:

$$|\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta)|^2 \leq 2|\hat{b}(Y_t; \theta)|^2 + 2|\hat{b}(Y_{t-r}; \theta)|^2.$$

So invoking relations (54) and (55), we obtain the estimate

$$F_V \leq 2 \int_V \left( \frac{1}{\tau} \int_0^\tau (|\hat{b}(Y_t; \theta)|^2 + |\hat{b}(Y_{t-r}; \theta)|^2) dt \right)^{1/2} r^{-H-1/2} dr \leq 2(K_1 + K_2)^{1/2} \int_V r^{-H-1/2} dr. \tag{59}$$

We now analyze  $F_V$  for three different choices of set  $V$ .

(i) Take  $V_1 = (A, \infty)$ . Then applying (59) and taking into account the second relation in (46) we obtain

$$F_{V_1} \leq 2(K_1 + K_2)^{1/2} \varepsilon := K_3 \varepsilon. \tag{60}$$

(ii) Similarly, consider  $V_2 = (1, A)$ , then the estimate (59) implies

$$F_{V_2} \leq K(K_1 + K_2)^{1/2} := K_4. \tag{61}$$

(iii) Take now  $V_3 = (0, 1)$ . Similarly to (49), for  $r \in V_3$  we have:

$$|\hat{b}(Y_t; \theta) - \hat{b}(Y_{t-r}; \theta)|^2 \leq K |Y|_{H-\delta, [t-1, t]}^2 r^{2(H-\delta)}.$$

Hence invoking expression (58) of  $F_V$  and along the same lines as for (51) we obtain

$$F_{V_3} \leq K \left( \frac{1}{\tau} \int_0^\tau |Y|_{t-1, t}^2 dt \right)^{1/2} \int_{V_3} r^{-1/2-\delta} dr \leq K(1 + \mathbf{E}|\bar{Y}|_{[-1, 0]}^2)^{1/2} := K_5, \tag{62}$$

for all  $\tau > \tau_3(\omega)$ .

With these preparations, we now consider the difference  $I(\tau) - I(\tau, [0, A]^2)$  for  $\tau > \tau_3(\omega)$ . Owing to relation (57) and recalling that  $V_1 = (A, \infty)$ ,  $V_2 = (1, A)$  and  $V_3 = (0, 1)$  we have

$$|I(\tau) - I(\tau, [0, A]^2)| = |I(\tau, V_1 \times (0, A)) + I(\tau, (0, A) \times V_1) + I(\tau, V_1 \times V_1)| \leq 2F_{V_1}(F_{V_2} + F_{V_3}) + F_{V_1}^2.$$

We then apply (60), (61), (62), which yields

$$|I(\tau) - I(\tau, [0, A]^2)| \leq 2K_3(K_4 + K_5)\varepsilon + K_3^2\varepsilon^2 \leq K\varepsilon. \tag{63}$$

We can now conclude the proof of (34). Indeed, combining (46), (53) and (63), we easily get the estimate

$$|I(\tau) - I| < K\varepsilon \quad \text{for } \tau > \tau_3(\omega),$$

from which (34) is trivially deduced. Notice that we have obtained in fact a stronger statement than (34), since our limit holds in the almost sure sense.

*Step 5.* We next show that the term  $I_3$  in (33) converges to zero in  $\mathbf{P}_\theta$ -probability as  $\tau \rightarrow \infty$ . For this, set  $g_t(\theta) = \sigma^{-1}[D_+^{H-1/2}b(Y; \theta)]_t$ . One can recast  $I_3$  into:

$$I_3 = \int_0^\tau |g_t(\theta_\tau) - g_t(\theta) - \partial_\theta g_t(\theta)(\theta_\tau - \theta)|^2 dt,$$

and we also recall that  $\theta_\tau - \theta = \tau^{-1/2}u$ . In addition, a simple application of Taylor’s expansion for multivariate functions yields the existence of a  $\lambda \in (0, 1)$  such that:

$$g_t(\theta_\tau) - g_t(\theta) = \partial_\theta g_t(\xi_\lambda)(\theta_\tau - \theta), \quad \text{where } \xi_\lambda = \theta + \lambda(\theta_\tau - \theta).$$

Notice that under our standing Hypothesis 1.3 we have  $\partial_\theta g_t(\xi_\lambda) = \sigma^{-1}[D_+^{H-1/2}\hat{b}(Y; \xi_\lambda)]_t$ . We thus get:

$$I_3 = \frac{1}{\tau} \int_0^\tau |\sigma^{-1}M_t(Y)u|^2 dt, \quad \text{where } M_t(Y) = \int_{\mathbb{R}_+} \frac{\delta[\hat{b}(Y; \xi_\lambda) - \hat{b}(Y; \theta)]_{t-r,t}}{r^{1/2+H}} dr.$$

Next we decompose  $M_t(Y)$  into  $M_{1,t}(Y) + M_{2,t}(Y)$ , where

$$M_{1,t}(Y) = \int_0^t \frac{\delta[\hat{b}(Y; \xi_\lambda) - \hat{b}(Y; \theta)]_{t-r,t}}{r^{1/2+H}} dr,$$

$$M_{2,t}(Y) = \int_t^\infty \frac{\delta[\hat{b}(Y; \xi_\lambda) - \hat{b}(Y; \theta)]_{t-r,t}}{r^{1/2+H}} dr.$$

Recall that we have extended the definition of  $Y$  to  $\mathbb{R}_-$  by setting  $Y_t = y_0$  for all  $t \leq 0$ . Therefore a simplified expression for  $M_{2,t}(Y)$  is as follows:

$$M_{2,t}(Y) = [\hat{b}(Y_t; \xi_\lambda) - \hat{b}(Y_t; \theta)] \int_t^\infty \frac{dr}{r^{H+1/2}},$$

and using Hypothesis 1.3 we obtain that:

$$\tau^{-1} \int_0^\tau |\sigma^{-1}M_{2,t}(Y)u|^2 dt \lesssim \tau^{-2} \int_0^\tau \frac{(1 + |Y_t|^2)}{t^{2H-1}} dt.$$

Thus, by Proposition 2.5, the  $L^1(\Omega)$ -norm of this term is bounded by  $c_H\tau^{-2H}$ . Hence, this term converges in  $\mathbf{P}_\theta$ -probability to zero as  $\tau \rightarrow \infty$ .

On the other hand, fix  $\alpha \in (\frac{1}{1+2H}, \frac{1}{2H})$ , and write

$$M_{1,t}(Y) \leq \int_0^t \frac{|\delta[\hat{b}(Y; \xi_\lambda) - \hat{b}(Y; \theta)]_{t-r,t}|^\alpha \cdot |\delta[\hat{b}(Y; \xi_\lambda) - \hat{b}(Y; \theta)]_{t-r,t}|^{1-\alpha}}{r^{1/2+H}} dr.$$

Then, appealing to Hypothesis 1.3, we get that

$$|M_{1,t}(Y)| \lesssim \frac{1}{\tau^{\alpha/2}} \int_0^t \frac{(1 + |Y_t|^\alpha + |Y_{t-r}|^\alpha) |Y_t - Y_{t-r}|^{1-\alpha}}{r^{H(1-\alpha)}r^{1/2+\alpha H}} dr.$$

Therefore,

$$\begin{aligned}
 & |M_{1,t}(Y)|^2 \\
 & \lesssim \frac{1}{\tau^\alpha} \int_0^t \int_0^t \frac{(1 + |Y_t|^\alpha + |Y_{t-r_1}|^\alpha) |Y_t - Y_{t-r_1}|^{1-\alpha}}{r_1^{H(1-\alpha)} r_1^{1/2+\alpha H}} \\
 & \quad \times \frac{(1 + |Y_t|^\alpha + |Y_{t-r_2}|^\alpha) |Y_t - Y_{t-r_2}|^{1-\alpha}}{r_2^{H(1-\alpha)} r_2^{1/2+\alpha H}} dr_1 dr_2.
 \end{aligned}$$

Thus, again by Proposition 2.5, the  $L^1(\Omega)$ -norm of the term  $\frac{1}{\tau} \int_0^\tau |\sigma^{-1} M_{1,t}(Y)u|^2 dt$  is bounded by

$$\frac{c_H}{\tau^{1+\alpha}} \int_0^\tau \int_0^t \int_0^t \frac{1}{r_1^{1/2+\alpha H}} \frac{1}{r_2^{1/2+\alpha H}} dr_1 dr_2 dt = \frac{c_H}{\tau^{1+\alpha}} \int_0^\tau t^{1-2H\alpha} dt = \frac{c_H}{\tau^{\alpha(2H+1)-1}},$$

which converges to zero as  $\tau \rightarrow \infty$  since  $\alpha > \frac{1}{2H+1}$ .

Step 6. We finally show that  $I_2$  and  $I_4$  are also negligible terms. Since  $I_3$  is the quadratic variation of the martingale  $I_2$ , this implies that  $I_2$  converges to zero in  $\mathbf{P}_\theta$ -probability as  $\tau \rightarrow \infty$ . On the other hand, applying the Cauchy–Schwarz inequality to  $I_4$ , we get that

$$|I_4| \leq \left( \frac{1}{\tau} \int_0^\tau |\sigma^{-1} [D_+^{H-1/2} \hat{b}(Y; \theta)]_t u|^2 dt \right)^{1/2} \times I_3^{1/2},$$

Thus, by the results in Steps 3 and 4, we obtain that  $I_4$  converges to zero in  $\mathbf{P}_\theta$ -probability as  $\tau \rightarrow \infty$ , which completes the proof of Theorem 1.7.

#### 4. Ornstein–Uhlenbeck case

Though our result encompasses a wide range of coefficients  $b$ , it is worth illustrating it on a simple linear case, that is for the real-valued fractional Ornstein–Uhlenbeck process corresponding to  $b(x; \theta) = -\theta x$  and  $\sigma \equiv \mathbf{1}$ . In addition, we will assume that our parameter  $\theta$  is an element of the set  $\Theta$ , which is a compact interval in  $(0, \infty)$ . Specifically, the equation followed by  $Y$  is the following:

$$Y_t = -\theta \int_0^t Y_s ds + B_t, \quad t \in [0, \tau], \quad \theta > 0, \tag{64}$$

and the stationary solution is given by:

$$\bar{Y}_t = \int_{-\infty}^t e^{-\theta(t-s)} dB_s.$$

##### 4.1. Computation of the LAN variance

One of the advantages of the Ornstein–Uhlenbeck case is that it allows explicit computations of the LAN variance  $\Sigma(\theta)$ . We summarize this possibility in the following proposition:

**Proposition 4.1.** *Let  $Y$  be the fractional Ornstein–Uhlenbeck process solution to (64). Then the LAN property (5) is satisfied with*

$$\Sigma(\theta) = \frac{1}{2\theta}.$$

**Remark 4.2.** As mentioned in the introduction, we observe that in this case  $\Sigma(\theta)$  does not depend on  $H$ , and is equal to the asymptotic variance of the LAN property of the Ornstein–Uhlenbeck process driven by a standard Brownian motion.

**Proof of Proposition 4.1.** By (6), we have the following expression for  $\Sigma(\theta)$

$$\Sigma(\theta) = C_H \int_{\mathbb{R}_+^2} \frac{\mathbf{E}_\theta[\bar{Y}_0^2] - \mathbf{E}_\theta[\bar{Y}_0\bar{Y}_{r_1}] - \mathbf{E}_\theta[\bar{Y}_0\bar{Y}_{r_2}] + \mathbf{E}_\theta[\bar{Y}_0\bar{Y}_{r_2-r_1}]}{r_1^{1/2+H} r_2^{1/2+H}} dr_1 dr_2,$$

where  $C_H$  is given by (7).

We start by computing the covariance of  $\bar{Y}$  between times 0 and  $t$ . Namely, thanks to [29, (3.4)], we obtain

$$\begin{aligned} \mathbf{E}_\theta[\bar{Y}_0\bar{Y}_t] &= \frac{1}{c_5(H)} \int_{\mathbb{R}} |\xi|^{1-2H} \mathcal{F}(\mathbf{1}_{(-\infty,t)} e^{-\theta(t-\cdot)})(\xi) \overline{\mathcal{F}(\mathbf{1}_{(-\infty,0)} e^\theta)(\xi)} d\xi \\ &= \frac{1}{c_5(H)} \int_{\mathbb{R}} |\xi|^{1-2H} \frac{e^{it\xi}}{(i\xi + \theta)} \frac{1}{(-i\xi + \theta)} d\xi \\ &= \frac{1}{c_5(H)} \int_{\mathbb{R}} \frac{|\xi|^{1-2H}}{\theta^2 + \xi^2} e^{it\xi} d\xi, \end{aligned}$$

where  $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(t) e^{it\xi} dt$  and

$$c_5(H) = \frac{2\pi}{\Gamma(2H + 1) \sin(\pi H)}. \tag{65}$$

Therefore, using Fubini’s theorem, we obtain

$$\begin{aligned} \Sigma(\theta) &= \frac{C_H}{c_5(H)} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} \frac{|\xi|^{1-2H}}{\theta^2 + \xi^2} \frac{(1 - e^{ir_1\xi} - e^{ir_2\xi} + e^{i(r_2-r_1)\xi})}{r_1^{1/2+H} r_2^{1/2+H}} d\xi dr_1 dr_2 \\ &= \frac{C_H}{c_5(H)} \int_{\mathbb{R}} \frac{|\xi|^{1-2H}}{\theta^2 + \xi^2} \int_{\mathbb{R}_+^2} \frac{(1 - e^{ir_1\xi})(1 - e^{-ir_2\xi})}{r_1^{1/2+H} r_2^{1/2+H}} dr_1 dr_2 d\xi \\ &= \frac{C_H}{c_5(H)} \int_{\mathbb{R}} \frac{|\xi|^{1-2H}}{\theta^2 + \xi^2} \left| \int_0^\infty \frac{1 - e^{ir\xi}}{r^{1/2+H}} dr \right|^2 d\xi \\ &= \frac{2C_H}{c_5(H)} \int_0^\infty \frac{\xi^{1-2H}}{\theta^2 + \xi^2} \left| \int_0^\infty \frac{1 - e^{ir\xi}}{r^{1/2+H}} dr \right|^2 d\xi. \end{aligned}$$

Thus, setting  $y = r\xi$  in the integral above, we get

$$\begin{aligned} \Sigma(\theta) &= \frac{2C_H}{c_5(H)} \int_0^\infty \frac{1}{\theta^2 + \xi^2} d\xi \left| \int_0^\infty \frac{1 - e^{iy}}{y^{1/2+H}} dy \right|^2 \\ &= \frac{\pi C_H}{\theta c_5(H)} \left| \int_0^\infty \frac{1 - e^{iy}}{y^{1/2+H}} dy \right|^2. \end{aligned} \tag{66}$$

We next write

$$\left| \int_0^\infty \frac{1 - e^{iy}}{y^{1/2+H}} dy \right|^2 = \left( \int_0^\infty \frac{1 - \cos(y)}{y^{1/2+H}} dy \right)^2 + \left( \int_0^\infty \frac{\sin(y)}{y^{1/2+H}} dy \right)^2.$$

For the first integral on the right hand side we use the integration by parts  $u = 1 - \cos(y)$ ,  $dv = y^{-\frac{1}{2}-H}$ ,  $du = \sin(y)$ ,  $v = \frac{y^{\frac{1}{2}-H}}{(\frac{1}{2}-H)}$ . For the second integral, we use  $u = \sin(y)$ ,  $du = \cos(y)$ ,

to get that

$$\begin{aligned} \left| \int_0^\infty \frac{1 - e^{iy}}{y^{1/2+H}} dy \right|^2 &= \left( \frac{1}{(H - \frac{1}{2})} \int_0^\infty \frac{\sin(y)}{y^{H-1/2}} dy \right)^2 + \left( \frac{1}{(H - \frac{1}{2})} \int_0^\infty \frac{\cos(y)}{y^{H-1/2}} dy \right)^2 \\ &= \left( \frac{\pi}{2(H - \frac{1}{2})\Gamma(H - \frac{1}{2}) \sin\left(\frac{\pi(H-\frac{1}{2})}{2}\right)} \right)^2 + \left( \frac{\pi}{2(H - \frac{1}{2})\Gamma(H - \frac{1}{2}) \cos\left(\frac{\pi(H-\frac{1}{2})}{2}\right)} \right)^2 \\ &= \frac{\pi^2}{4\Gamma^2(H + \frac{1}{2})\sin^2\left(\frac{\pi(H-\frac{1}{2})}{2}\right) \cos^2\left(\frac{\pi(H-\frac{1}{2})}{2}\right)} \\ &= \frac{\pi^2}{\Gamma^2(H + \frac{1}{2})\sin^2\left(\pi(H - \frac{1}{2})\right)}, \end{aligned}$$

where in the second equation we have used some standard formulas for improper integrals (see e.g. [33, Page 331–332]). Plugging this identity into the expression (66) we have obtained for  $\Sigma(\theta)$ , and using the definition of  $C_H$  in (7) and of  $c_5(H)$  in (65), we obtain that

$$\Sigma(\theta) = \frac{\pi C_H}{\theta c_5(H)} \frac{\pi^2}{\Gamma^2(H + \frac{1}{2})\sin^2\left(\pi(H - \frac{1}{2})\right)} = \frac{\Gamma(2H + 1)}{4\theta H \Gamma(2H)} = \frac{1}{2\theta},$$

which concludes the result.  $\square$

#### 4.2. Efficiency of the mle

Consider now the MLE  $\hat{\theta}_\tau$  of  $\theta$  from the observation of a fractional Ornstein–Uhlenbeck process  $Y$  in  $[0, \tau]$ , as defined in [18]. It is well-known (see [2,4,18]) that  $\hat{\theta}_\tau$  is uniformly consistent on  $\Theta$  (recall that  $\Theta$  is assumed to be a compact subinterval of  $(0, \infty)$  in the Ornstein–Uhlenbeck sense). That is, for any  $\lambda > 0$ , we have

$$\lim_{\tau \rightarrow \infty} \sup_{\theta \in \Theta} \mathbf{P}_\theta^\tau \left( \left| \hat{\theta}_\tau - \theta \right| > \lambda \right) = 0.$$

The estimator  $\hat{\theta}_\tau$  is also uniformly asymptotically normal:

$$\mathcal{L}(\mathbf{P}_\theta) - \lim_{\tau \rightarrow \infty} \sqrt{\tau}(\hat{\theta}_\tau - \theta) = \mathcal{N}(0, 2\theta),$$

where the limit is uniform in  $\theta \in \Theta$ . Moreover, we have a uniform convergence of moments. Namely, for any  $p > 0$ , one gets:

$$\lim_{\tau \rightarrow \infty} \sup_{\theta \in \Theta} \left| \mathbf{E}_\theta \left[ \left| \sqrt{\tau}(\hat{\theta}_\tau - \theta) \right|^p \right] - \mathbf{E}_\theta \left[ \left| \sqrt{2\theta} Z \right|^p \right] \right| = 0, \tag{67}$$

where  $\mathcal{L}(Z) = \mathcal{N}(0, 1)$ . In particular, the last two results already suggest that the rate of convergence for the LAN property in this case is  $\tau^{-1/2}$ , as mentioned in [30, p. 162]. Our Theorem 1.7 confirms this intuition, and one can see that the MLE reaches this optimal rate of order  $\tau^{-1/2}$ .

Moreover, as a consequence of Theorem 1.7 and Proposition 4.1, we obtain the asymptotic efficiency for polynomial loss functions of the MLE for the fractional Ornstein–Uhlenbeck process, in the sense of Theorem 1.6.

**Proposition 4.3.** *Let  $Y$  be the fractional Ornstein–Uhlenbeck process solution to (64). Then the MLE is asymptotically minimax efficient for any loss function  $\ell(u) = |u|^p$ ,  $p > 0$ .*

**Proof.** Fix  $\theta \in \Theta$ . Then, for all  $\delta > 0$  and  $p > 0$ ,

$$\begin{aligned} & \sup_{\theta' \in \Theta: |\theta' - \theta| < \delta} \left| \mathbf{E}_{\theta'} \left[ \left| \sqrt{\tau}(\hat{\theta}_\tau - \theta') \right|^p \right] - \mathbf{E}_\theta \left[ \left| \sqrt{2\theta} Z \right|^p \right] \right| \\ & \leq \sup_{\theta' \in \Theta} \left| \mathbf{E}_{\theta'} \left[ \left| \sqrt{\tau}(\hat{\theta}_\tau - \theta') \right|^p \right] - \mathbf{E}_{\theta'} \left[ \left| \sqrt{2\theta'} Z \right|^p \right] \right| \\ & \quad + \sup_{\theta' \in \Theta: |\theta' - \theta| < \delta} \left| \mathbf{E}_{\theta'} \left[ \left| \sqrt{2\theta'} Z \right|^p \right] - \mathbf{E}_\theta \left[ \left| \sqrt{2\theta} Z \right|^p \right] \right|, \end{aligned}$$

where  $\mathcal{L}(Z) = \mathcal{N}(0, 1)$ . Then, the uniform convergence of the moments (67) implies that

$$\begin{aligned} & \lim_{\tau \rightarrow \infty} \sup_{\theta' \in \Theta: |\theta' - \theta| < \delta} \left| \mathbf{E}_{\theta'} \left[ \left| \sqrt{\tau}(\hat{\theta}_\tau - \theta') \right|^p \right] - \mathbf{E}_\theta \left[ \left| \sqrt{2\theta} Z \right|^p \right] \right| \\ & \leq \sup_{\theta' \in \Theta: |\theta' - \theta| < \delta} \left| \mathbf{E}_{\theta'} \left[ \left| \sqrt{2\theta'} Z \right|^p \right] - \mathbf{E}_\theta \left[ \left| \sqrt{2\theta} Z \right|^p \right] \right|. \end{aligned}$$

Therefore,

$$\lim_{\delta \rightarrow 0} \lim_{\tau \rightarrow \infty} \sup_{\theta' \in \Theta: |\theta' - \theta| < \delta} \left| \mathbf{E}_{\theta'} \left[ \left| \sqrt{\tau}(\hat{\theta}_\tau - \theta') \right|^p \right] - \mathbf{E}_\theta \left[ \left| \sqrt{2\theta} Z \right|^p \right] \right| = 0.$$

Thus, by Proposition 4.1, the lower bound in (4) is achieved by the MLE when  $\ell(u) = |u|^p$ , which completes the proof.  $\square$

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