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Malliavin Calculus for Fractional Delay Equations

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Abstract In this paper we study the existence of a unique solution to a general class of Young delay differential equations driven by a Hölder continuous function with parameter greater that $1/2$ via the Young integration setting. Then some estimates of the solution are obtained, which allow to show that the solution of a delay differential equation driven by a fractional Brownian motion (fBm) with Hurst parameter $H > 1/2$ has a C^∞-density. To this purpose, we use Malliavin calculus based on the Fréchet differentiability in the directions of the reproducing kernel Hilbert space associated with fBm.

Keywords Delay equation · Young integration · Fractional Brownian motion · Malliavin calculus

Mathematics Subject Classification (2000) 60H10 · 60H05 · 60H07

1 Introduction

We shall consider in this article an equation of the form:

$$dy_t = f\big(\mathcal{Z}_t^y\big)\, dB_t + b\big(\mathcal{Z}_t^y\big)\, dt, \quad t \in [0, T], \tag{1}$$

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where B is a d-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$, $f : \mathcal{C}_1^\gamma([-h,0];\mathbb{R}^n) \to \mathbb{R}^{n\times d}$ and $b : \mathcal{C}_1^\gamma([-h,0];\mathbb{R}^n) \to \mathbb{R}^n$ satisfy some suitable regularity conditions, $\mathcal{C}_1^\gamma$ designates the space of γ-Hölder continuous functions of one variable (see Sect. 2.1 below) and $\mathcal{Z}_t^y : [-h,0] \to \mathbb{R}^n$ is defined by $\mathcal{Z}_t^y(s) = y_{t+s}$. In the previous equation, we also assume that an initial condition $\xi \in \mathcal{C}_1^\gamma$ is given on the interval $[-h,0]$. Notice that equation (1) is a general expression for fractional delay equations, including for instance the case of functions f of the following form:

$$f : \mathcal{C}_1^\gamma\big([-h,0];\mathbb{R}^n\big) \to \mathbb{R}^{n\times d}, \quad \text{with } f\big(\mathcal{Z}_t^y\big) = \sigma\left(\int_{-h}^0 y_{t+\theta}\,\nu(d\theta)\right), \tag{2}$$

for a regular enough function σ, and a finite signed measure ν on $[-h,0]$. This special case of interest will be treated in detail in the sequel. Our considerations also include a function f defined by $f(\mathcal{Z}_t^y) = \sigma(\mathcal{Z}_t^y(-u_1),\dots,\mathcal{Z}_t^y(-u_k)) = \sigma(y_{t-u_1},\dots,y_{t-u_k})$ for a given $k \geq 1$, $0 \leq u_1 < \cdots < u_k \leq h$ and a smooth enough function $\sigma : \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times d}$.

The kind of delay stochastic differential system described by (1) is widely studied when driven by a standard Brownian motion (see [26] for a nice survey), but the results in the fractional Brownian case are scarce: we are only aware of [12] for the case $H > 1/2$ and $f(\mathcal{Z}^y) = \sigma(\mathcal{Z}^y(-r))$, $0 \leq r \leq h$, the further investigation [13] which establishes a continuity result in terms of the delay r, and a reflected version investigated in [4]. As far as the rough case is concerned (see [22] and [23]), an existence and uniqueness result is given in [27] for a Hurst parameter $H > 1/3$, and [36] extends this result to $H > 1/4$. In spite of this lack of theoretical results, the need for suitable oscillating dynamical models with delays is obvious in the applied literature, and involves problems in signal or disease transmission [6, 32, 35], biochemical reactions [2] or gene regulation [5, 25]. The demand for noisy versions of these systems is therefore natural, as nicely stressed in [19].

Our paper can also be seen as part of a global project aiming at an understanding of physically relevant systems driven by fractional noises. Just to mention a few examples concerning fractional Brownian motion with Hurst parameter $H > 1/2$, let us quote ordinary differential equations [18, 30, 31, 38], some interesting cases of PDEs [3, 7, 10, 11, 15, 16, 24, 34], as well as Volterra type systems [8, 9].

The current article can thus be seen as a step in the study of processes defined as the solution to fractional delay differential systems, and we shall investigate the behavior of the density of the $\mathbb{R}^n$-valued random variable y_t for a fixed $t \in (0,T]$, where y is the solution to (1). More specifically, we shall prove the following theorem, which can be seen as the main result of the article:

Theorem 1.1 *Consider an equation of the form* (1) *for an initial condition ξ lying in the space $\mathcal{C}_1^\gamma([-h,0];\mathbb{R}^n)$. Assume $b \equiv 0$, and that f is of the form* (2) *for a given finite signed measure ν on $[-h,0]$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ an infinitely differentiable function, bounded together with all its derivatives and satisfying the non-degeneracy condition*

$$\sigma(\eta)\sigma(\eta)^* \geq \varepsilon Id_{\mathbb{R}^n}, \quad \textit{for all } \eta \in \mathbb{R}^n.$$

Suppose moreover that $H > H_0$, *where* $H_0 = (7+\sqrt{17})/16 \approx 0.6951$. *Let* $t \in (0,T]$ *be an arbitrary time, and* y *be the unique solution to* (1) *in* $\mathcal{C}_1^{\kappa}([0,T];\mathbb{R}^n)$, *for a given* $1/2 < \kappa < H$. *Then the law of* y_t *is absolutely continuous with respect to Lebesgue measure in* $\mathbb{R}^n$, *and its density is a* $\mathcal{C}^\infty$-*function.*

Notice that this kind of result, which has its own interest as a natural step in the study of processes defined by delay systems, is also a useful result when one wants to study other natural properties of the equation, such as convergence to equilibrium (see e.g. [17]). Let us also observe that the case $b \equiv 0$ has been considered here for sake of simplicity, but the extension of our result to a non-trivial drift (namely a coefficient $b : \mathcal{C}_1^{\gamma}([-h,0];\mathbb{R}^n) \to \mathbb{R}^n$ of the form $b(Z) = \mu(\int_{-h}^0 Z_\theta\, \nu(d\theta))$ for an infinitely differentiable function μ bounded together with all its derivatives) is just a matter of easy additional computations. Finally, the reader may wonder about our restriction $H > H_0$ above. To this respect, let us make the following observations:

(i) We believe that this restriction is due to the method we have used in order to bound delayed linear equations, which is the best one we had in mind but might not be completely optimal. In any case, we don't see any obvious reason for which our smoothness result for the density shouldn't hold true for $1/2 < H \le H_0$.
(ii) As mentioned in Remark 3.15, the assumption $H > H_0$ also stems the fact that we consider a delay which depends *continuously* on the past. For a discrete type delay of the form $\sigma(y_t, y_{t-r_1}, \dots, y_{t-r_q})$, with $q \ge 1$ and $r_1 < \cdots < r_q \le h$, we shall see in Remark 4.7 that one can show the smoothness of the density up to $H > 1/2$, as for ordinary differential equations.
(iii) Interestingly enough, a behavior dichotomy between the discrete and continuous situation has already been observed in [26] for the flow properties of the equation in case of a Brownian noise. However, in the latter case one improves the continuity properties of the equation by considering continuous delays (contrarily to our situation).

Let us say a few words about the strategy we shall follow in order to get our Theorem 1.1. First of all, some of the steps we are following are rather standard in the pathwise stochastic calculus context:

- As mentioned before, there are not too many results about delay systems governed by a fractional Brownian motion. In particular, equation (1) has never been considered (to the best of our knowledge) with such a general delay dependence. We shall thus first show how to define and solve this differential system, by means of a slight variation of the Young integration theory (called algebraic integration), introduced in [14] and also explained in [16]. This setting allows to solve equations like (1) in Hölder spaces thanks to contraction arguments, as will be explained in Sect. 3.1. In fact, observe that our resolution will be entirely pathwise, and we shall deal with a general equation of the form

$$dy_t = f\big(\mathcal{Z}_t^y\big)\, dx_t + b\big(\mathcal{Z}_t^y\big)\, dt, \qquad t \in [0,T], \tag{3}$$

for a given path $x \in \mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^d)$ with $\gamma > 1/2$, where the integral with respect to x has to be understood in the Young sense [37]. Furthermore, in equations like

(3), the drift term $b(\mathcal{Z}^y)$ is usually harmless, but induces some cumbersome notations. Thus, for sake of simplicity, we shall rather deal in the sequel with a reduced delay equation of the type:

$$y_t = a + \int_0^t f\left(\mathcal{Z}_s^y\right) dx_s, \quad t \in [0, T].$$

- Once this last equation is properly defined and solved, the differentiability of the solution y_t in the Malliavin calculus sense will also be obtained in a pathwise manner, similarly to the case treated in [31].

The essential part of our technical efforts for the current project are thus concentrated on the smoothness property for the density of y_t. Indeed, as for other stochastic systems defined in a pathwise manner, the main difficulty in order to get smooth densities is to provide moment estimates for the Malliavin derivative of the solution y_t. In our situation, owing to the fact that we have chosen a delay depending continuously on the past, this essential step is nontrivial, and is carefully detailed in Propositions 3.4 and 3.14. Notice also that the way to obtain the smoothness Theorem 1.1 from those moments estimates, which follows roughly the methodology of [21], requires some additional work in the context of a fractional Brownian motion. This step will be carried out in Sect. 4.3.

Here is how our article is structured: Sect. 2 is devoted to recall some basic facts about Young integration. We solve, estimate and differentiate a general class of delay equations driven by a Hölder noise in Sect. 3. Then in Sect. 4 we apply those general results to fBm and prove our main Theorem 1.1.

2 Algebraic Young Integration

The Young integration can be introduced in several ways (convergence of Riemann sums, fractional calculus setting [38]). We have chosen here to follow the algebraic approach introduced in [14] and developed e.g. in [16], since this formalism will help us later in our analysis.

2.1 Increments

Let us begin with the basic algebraic structures which will allow us to define a pathwise integral with respect to irregular functions: first of all, for an arbitrary real number $T > 0$, a topological vector space V and an integer $k \geq 1$ we denote by $\mathcal{C}_k(V)$ (or by $\mathcal{C}_k([0,T];V)$) the set of continuous functions $g : [0,T]^k \to V$ such that $g_{t_1\cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*. Note that $\mathcal{C}_1(V)$ is the family of all continuous functions from $[0,T]$ into V, and we will set $\mathcal{C}_*(V) = \bigcup_{k\geq 1} \mathcal{C}_k(V)$. An important elementary operator is δ, which is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1\cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_1\cdots \hat{t}_i \cdots t_{k+1}}, \tag{4}$$

where $\hat{t}_i$ means that this particular argument is omitted. A fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Ker}\delta$, for any $k \geq 1$, and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Im}\delta$ for $k \geq 2$, that is

$$\mathcal{ZC}_k(V) = \{g \in \mathcal{C}_k(V);\ \delta g = 0\},$$
$$\mathcal{BC}_k(V) = \{g \in \mathcal{C}_k(V); g = \delta f \text{ for } f \in \mathcal{C}_{k-1}(V)\}.$$

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then, for any $s, u, t \in [0, T]$, we have

$$(\delta g)_{st} = g_t - g_s, \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{5}$$

Furthermore, it is easily checked that $\mathcal{ZC}_k(V) = \mathcal{BC}_k(V)$ for any $k \geq 2$. In particular, the following basic property holds:

Lemma 2.1 *Let $k \geq 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (non-unique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Observe that Lemma 2.1 implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non-unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an exact increment of a function, i.e., a finite difference.

Remark 2.2 Here is a first elementary but important link between these algebraic structures and integration theory: let f and g be two smooth real valued function on $[0, T]$. Define then $I \in \mathcal{C}_2(V)$ by

$$I_{st} = \int_s^t df_v \int_s^v dg_w, \quad \text{for } s, t \in [0, T].$$

Then, some trivial computations show that

$$(\delta I)_{sut} = [g_u - g_s][f_t - f_u] = (\delta f)_{ut}(\delta g)_{su}.$$

This is a helpful property of the operator δ: it transforms iterated integrals into products of increments, and we will be able to take advantage of both regularities of f and g in these products of the form $\delta f\, \delta g$.

For sake of simplicity, let us specialize now our setting to the case $V = \mathbb{R}^m$ for an arbitrary $m \geq 1$. Notice that our future discussions will mainly rely on k-increments with $k \leq 2$, for which we will use some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $0 \leq a_1 < a_2 \leq T$ and $f \in \mathcal{C}_2([a_1, a_2]; V)$, let

$$\|f\|_{\mu,[a_1,a_2]} = \sup_{r,t \in [a_1,a_2]} \frac{|f_{rt}|}{|t-r|^\mu}, \quad \text{and}$$

$$\mathcal{C}_2^{\mu}\big([a_1,a_2];V\big)=\big\{f\in\mathcal{C}_2(V);\ \|f\|_{\mu,[a_1,a_2]}<\infty\big\}.$$

Obviously, the usual Hölder spaces $\mathcal{C}_1^{\mu}([a_1,a_2];V)$ will be determined in the following way: for a continuous function $g\in\mathcal{C}_1([a_1,a_2];V)$, we simply set

$$\|g\|_{\mu,[a_1,a_2]}=\|\delta g\|_{\mu,[a_1,a_2]}, \tag{6}$$

and we will say that $g\in\mathcal{C}_1^{\mu}([a_1,a_2];V)$ iff $\|g\|_{\mu,[a_1,a_2]}$ is finite. Notice that $\|\cdot\|_{\mu,[a_1,a_2]}$ is only a semi-norm on $\mathcal{C}_1^{\mu}([a_1,a_2];V)$, but we will generally work on spaces of the type

$$\mathcal{C}_{v,a_1,a_2}^{\mu}(V)=\big\{g:[a_1,a_2]\to V;\ g_{a_1}=v,\ \|g\|_{\mu,[a_1,a_2]}<\infty\big\}, \tag{7}$$

for a given $v\in V$, or

$$\mathcal{C}_{\varrho,a_1,a_2}^{\mu}\big(\mathbb{R}^d\big):=\big\{\zeta\in\mathcal{C}_1^{\mu}\big([a_1-h,a_2];\mathbb{R}^d\big);\ \zeta=\varrho\text{ on }[a_1-h,a_1]\big\}, \tag{8}$$

where $0\le a_1<a_2$ and $\varrho\in\mathcal{C}_1^{\mu}([a_1-h,a_1];\mathbb{R}^d)$. These last two spaces are complete metric spaces with the distance d_{μ,a_1,a_2}. Here, $d_{\mu,a_1,a_2}(f,g)=\|f-g\|_{\mu,[a_1,a_2]}$ on $\mathcal{C}_{v,a_1,a_2}^{\mu}(V)$; and $d_{\mu,a_1,a_2}(f,g)=\|f-g\|_{\mu,[a_1-h,a_2]}$ on the space $\mathcal{C}_{\varrho,a_1,a_2}^{\mu}(\mathbb{R}^d)$.

In some cases we will only write $\mathcal{C}_k^{\mu}(V)$ instead of $\mathcal{C}_k^{\mu}([a_1,a_2];V)$ when this does not lead to an ambiguity in the domain of definition of the functions under consideration. For $h\in\mathcal{C}_3([a_1,a_2];V)$ set in the same way

$$\begin{aligned}\|h\|_{\gamma,\rho,[a_1,a_2]}&=\sup_{s,u,t\in[a_1,a_2]}\frac{|h_{sut}|}{|u-s|^{\gamma}|t-u|^{\rho}},\\ \|h\|_{\mu,[a_1,a_2]}&=\inf\Big\{\sum_i\|h_i\|_{\rho_i,\mu-\rho_i};\ h=\sum_i h_i,\ 0<\rho_i<\mu\Big\},\end{aligned} \tag{9}$$

where the last infimum is taken over all sequences $\{h_i\in\mathcal{C}_3(V)\}$ such that $h=\sum_i h_i$ and for all choices of the numbers $\rho_i\in(0,\mu)$. Then $\|\cdot\|_{\mu,[a_1,a_2]}$ is easily seen to be a norm on $\mathcal{C}_3([a_1,a_2];V)$, and we set

$$\mathcal{C}_3^{\mu}\big([a_1,a_2];V\big):=\big\{h\in\mathcal{C}_3\big([a_1,a_2];V\big);\ \|h\|_{\mu,[a_1,a_2]}<\infty\big\}.$$

Now, let $\mathcal{C}_3^{1+}([a_1,a_2];V)=\bigcup_{\mu>1}\mathcal{C}_3^{\mu}([a_1,a_2];V)$ and $\mathcal{Z}\mathcal{C}_3^{1+}([a_1,a_2];V)=\mathcal{C}_3^{1+}([a_1,a_2];V)\cap\ker\delta$.

With these notations in mind, the crucial point in our approach to pathwise integration of irregular processes is that, under mild smoothness conditions, the operator δ can be inverted. This inverse is called Λ, and is defined in the following proposition, whose proof can be found in [14].

Proposition 2.3 *Let $0\le a_1<a_2\le T$. Then there exists a unique linear map $\Lambda:\mathcal{Z}\mathcal{C}_3^{1+}([a_1,a_2];V)\to\mathcal{C}_2^{1+}([a_1,a_2];V)$ such that*

$$\delta\Lambda=Id_{\mathcal{Z}\mathcal{C}_3^{1+}([a_1,a_2];V)}.$$

In other words, for any $h \in \mathcal{C}_3^{1+}([a_1,a_2];V)$ such that $\delta h = 0$ there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^{1+}([a_1,a_2];V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathcal{Z}\mathcal{C}_3^{\mu}([a_1,a_2];V)$ to $\mathcal{C}_2^{\mu}([a_1,a_2];V)$ and we have

$$\|\Lambda h\|_{\mu,[a_1,a_2]} \le \frac{1}{2^\mu - 2}\|h\|_{\mu,[a_1,a_2]}, \quad h \in \mathcal{Z}\mathcal{C}_3^{\mu}\big([a_1,a_2];V\big). \tag{10}$$

Moreover, the operator Λ can be related to the limit of some Riemann sums, which gives a second link (after Remark 2.2) between the previous algebraic developments and some kind of generalized integration.

Corollary 2.4 *For any 1-increment $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^{1+}$, set $h = (Id - \Lambda\delta)g$. Note that $\delta h = 0$ due to the fact that $\delta\Lambda = Id_{\mathcal{Z}\mathcal{C}_3^{1+}(V)}$. Thus there exists $f \in \mathcal{C}_1$ such that $h = \delta f$. Moreover, we have*

$$(\delta f)_{st} = \lim_{|\Pi_{st}|\to 0}\sum_{i=0}^{n-1} g_{t_i\,t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero. Thus, the 1-increment δf is the indefinite integral of the 1-increment g.

2.2 Young Integration

In this section, we will define a generalized integral $\int_s^t f_u\,dg_u$ for a $C_1^\kappa([0,T];\mathbb{R}^{n\times d})$-function f, and a $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$-function g, with $\kappa + \gamma > 1$, by means of the algebraic tools introduced in Sect. 2.1. To this purpose, we will first assume that f and g are smooth functions, in which case the integral of f with respect to g can be defined in the Lebesgue–Stieltjes sense, and then we will express this integral in terms of the operator Λ. This will lead to a natural extension of the notion of integral, which coincides with the usual Young integral. In the sequel, in order to avoid some cumbersome notations, we will sometimes write $\mathcal{J}_{st}(f\,dg)$ instead of $\int_s^t f_u\,dg_u$.

Let us consider then for the moment two smooth functions f and g defined on $[0,T]$. One can write, thanks to some elementary algebraic manipulations,

$$\mathcal{J}_{st}(f\,dg) \equiv \int_s^t f_u\,dg_u = f_s(\delta g)_{st} + \int_s^t (\delta f)_{su}\,dg_u = f_s(\delta g)_{st} + \mathcal{J}_{st}(\delta f\,dg). \tag{11}$$

Let us analyze now the term $\mathcal{J}(\delta f\,dg)$, which is an element of $\mathcal{C}_2(\mathbb{R}^n)$. Invoking Remark 2.2, it is easily seen that, for $s,u,t \in [0,T]$,

$$h_{sut} \equiv \big[\delta\big(\mathcal{J}(\delta f\,dg)\big)\big]_{sut} = (\delta f)_{su}(\delta g)_{ut}.$$

The increment h is thus an element of $\mathcal{C}_3(\mathbb{R}^n)$ satisfying $\delta h = 0$ (recall that $\delta\delta = 0$). Let us estimate now the regularity of h: if $f \in C_1^\kappa([0,T];\mathbb{R}^{n\times d})$ and $g \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$, from the definition (9), it is readily checked that $h \in \mathcal{C}_3^{\gamma+\kappa}(\mathbb{R}^n)$.

Hence $h \in \mathcal{Z}\mathcal{C}_3^{\gamma+\kappa}(\mathbb{R}^n)$, and if $\kappa + \gamma > 1$ (which is the case if f and g are regular), Proposition 2.3 yields that $\mathcal{J}(\delta f\, dg)$ can also be expressed as

$$\mathcal{J}(\delta f\, dg) = \Lambda(h) = \Lambda(\delta f\, \delta g),$$

and thus, plugging this identity into (11), we get

$$\mathcal{J}_{st}(f\, dg) = f_s(\delta g)_{st} + \Lambda_{st}(\delta f\, \delta g). \tag{12}$$

Now we can see that the right hand side of the last equality is rigorously defined whenever $f \in \mathcal{C}_1^\kappa([0,T];\mathbb{R}^{n\times d})$, $g \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$, and this is the definition we will use in order to extend the notion of integral:

Theorem 2.5 *Let $f \in \mathcal{C}_1^\kappa([0,T];\mathbb{R}^{n\times d})$ and $g \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$, with $\kappa+\gamma>1$. Set*

$$\mathcal{J}_{st}(f\, dg) = f_s(\delta g)_{st} + \Lambda_{st}\big(\delta f\, \delta g\big). \tag{13}$$

Then

(1) *Whenever f and g are smooth functions, $\mathcal{J}_{st}(f\, dg)$ coincides with the usual Riemann integral.*
(2) *The generalized integral $\mathcal{J}(f\, dg)$ satisfies*

$$\big|\mathcal{J}_{st}(f\, dg)\big| \le \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa}\|f\|_\kappa \|g\|_\gamma |t-s|^{\gamma+\kappa},$$

for a constant $c_{\gamma,\kappa}$ whose exact value is $(2^{\gamma+\kappa}-2)^{-1}$.
(3) *We have*

$$\mathcal{J}_{st}(f\, dg) = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n-1} f_{t_i}\, \delta g_{t_i\, t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero. In particular, $\mathcal{J}_{st}(f\, dg)$ coincides with the Young integral as defined in [37].

Proof The first claim is just what we proved at equation (12). The second assertion follows directly from the definition (13) and the inequality (10) concerning the operator Λ. Finally, our third property is a direct consequence of Corollary 2.4 and the fact that $\delta(f\, \delta g) = -\delta f\, \delta g$, which means that

$$\mathcal{J}(f\, dg) = [Id - \Lambda\delta](f\, \delta g).$$ □

A Fubini type theorem for Young's integral will be needed in the last section of this paper. Its proof below is a good example of the importance of Proposition 2.3 and Theorem 2.5.

Proposition 2.6 *Assume that $\gamma > \lambda > 1/2$. Let f and g be two functions in $\mathcal{C}_1^\gamma([0,T]:\mathbb{R})$ and $h: \{(t,s)\in[0,T]^2; 0\le s\le t\le T\} \to \mathbb{R}$ a function such that*

$h(\cdot,t)$ (resp. $h(t,\cdot)$) belongs to $\mathcal{C}_1^\lambda([t,T];\mathbb{R})$ (resp. $\mathcal{C}_1^\lambda([0,t];\mathbb{R})$) uniformly in $t\in[0,T]$, and

$$\big\|h(r_1,\cdot)-h(r_2,\cdot)\big\|_{\lambda,[0,r_1\wedge r_2]}\le C|r_1-r_2|^\lambda. \tag{14}$$

Then

$$\int_s^t\left(\int_s^r h(r,u)\,dg_u\right)df_r=\int_s^t\left(\int_u^t h(r,u)\,df_r\right)dg_u,\quad 0\le s\le t\le T. \tag{15}$$

Proof Fix $s,t\in[0,T]$, with $s<t$, and divide the proof in several steps.

Step 1. Here we see that $\int_s^t\int_s^r h(r,u)\,dg_u\,df_r$ is well-defined. Note that we only need to show that $\int_s^\cdot h(\cdot,u)\,dg_u$ belongs to $\mathcal{C}_1^\lambda([s,T];\mathbb{R})$ due to Theorem 2.5.

Let $r_1,r_2\in[s,t]$, $r_1<r_2$, then Theorem 2.5.(2) gives

$$\begin{aligned}
&\left|\int_s^{r_2}h(r_2,u)\,dg_u-\int_s^{r_1}h(r_1,u)\,dg_u\right|\\
&\quad\le\left|\int_s^{r_1}\big(h(r_2,u)-h(r_1,u)\big)\,dg_u\right|+\left|\int_{r_1}^{r_2}h(r_2,u)\,dg_u\right|\\
&\quad\le\|g\|_\gamma\big(\big\|h(r_2,\cdot)-h(r_1,\cdot)\big\|_{\infty,[0,r_1]}(r_1-s)^\gamma\\
&\qquad+c_{\gamma,\lambda}\big\|h(r_2,\cdot)-h(r_1,\cdot)\big\|_{\lambda,[0,r_1]}(r_1-s)^{\gamma+\lambda}\big)\\
&\qquad+\|g\|_\gamma\big(\big\|h(r_2,\cdot)\big\|_{\infty,[0,r_2]}(r_2-r_1)^\gamma+c_{\gamma,\lambda}\big\|h(r_2,\cdot)\big\|_{\lambda,[0,r_2]}(r_2-r_1)^{\gamma+\lambda}\big).
\end{aligned}$$

Hence (14) implies our claim. The definition of $\int_s^t\int_u^t h(r,u)\,df_r\,dg_u$ follows along the same lines.

Step 2. Let $\Pi_{st}=\{t_0=s,\dots,t_n=t\}$ be a partition of the interval $[s,t]$. Then, according to Theorem 2.5, for any $v\in[0,t)$ we have

$$\int_s^v h(t,u)\,dg_u=\lim_{|\Pi_{st}|\to0}\sum_{i=0}^{n-1}h(t,t_i)\,(\delta g)_{t_i\wedge v,t_{i+1}\wedge v}. \tag{16}$$

Our assumption (14) allows us now to take limits in the equation above, so that we obtain, for any $0\le s<t\le T$,

$$q^1_{st}:=\int_s^t h(t,u)\,dg_u=\lim_{|\Pi_{st}|\to0}\sum_{i=0}^{n-1}h(t,t_i)\,\delta g_{t_i,t_{i+1}}:=q^2_{st}. \tag{17}$$

In order to see that the relation above holds in $\mathcal{C}_2^\lambda([0,T];\mathbb{R})$, it is now enough to check that both q^1 and q^2 in (17) are elements of $\mathcal{C}_2^\lambda([0,T];\mathbb{R})$.

However, the fact that $q^1\in\mathcal{C}_2^\lambda([0,T];\mathbb{R})$ can be proved along the same lines as in Step 1. The assertion $q^2\in\mathcal{C}_2^\lambda([0,T];\mathbb{R})$ can be proved by observing that the limit defining q^2_{st} do not depend on the sequence of partitions under consideration. In particular, consider the sequence $(\pi^n)_n$ of dyadic partitions of $[0,T]$, that is

$$\pi^n=\big\{0=t_0^n\le t_1^n\le\cdots\le t_{2^n}^n=T\big\},\quad\text{with } t_i^n=\frac{iT}{2^n},$$

and set, for all $s,t\in[0,T]$, $\pi^n_{st}=\pi^n\cap(s,t)$. Then $q^2_{st}=\lim_{n\to\infty}\sum_{t_i\in\pi^n_{st}} h(t,t^n_i)\times\delta g_{t^n_i,t^n_{i+1}}$ for all $0\le s<t\le T$, and the same kind of arguments as in [9, Theorem 2.2] yield our claim $q^2\in\mathcal{C}^\lambda_2([0,T];\mathbb{R})$. We have thus proved that (17) holds in $\mathcal{C}^\lambda_2([0,T];\mathbb{R})$.

Step 3. From Proposition 2.3, Step 2 and (13) we have

$$\begin{aligned}\int_s^t\int_s^r h(r,u)\,dg_u\,df_r &= \lim_{|\Pi_{st}|\to 0}\int_s^t\left(\sum_{i=0}^{n-1} h(r,t_i)(g_{t_{i+1}\wedge r}-g_{t_i\wedge r})\right)df_r\\ &= \lim_{|\Pi_{st}|\to 0}\sum_{i=0}^{n-1}\int_{t_i}^t h(r,t_i)(g_{t_{i+1}\wedge r}-g_{t_i})\,df_r\\ &= \lim_{|\Pi_{st}|\to 0}\sum_{i=0}^{n-1}\left[\left(\int_{t_{i+1}}^t h(r,t_i)\,df_r\right)(g_{t_{i+1}}-g_{t_i})\right.\\ &\quad\left.+\int_{t_i}^{t_{i+1}} h(r,t_i)(g_r-g_{t_i})\,df_r\right].\end{aligned}$$

Moreover, thanks to the Hölder properties of f and g, we have

$$\sum_{i=0}^{n-1}\left|\int_{t_i}^{t_{i+1}} h(r,t_i)(g_r-g_{t_i})\,df_r\right|\le C\sum_{i=0}^{n-1}(t_{i+1}-t_i)^{\gamma+\lambda}\to 0$$

as $|\Pi_{st}|\to 0$, and thus

$$\int_s^t\int_s^r h(r,u)\,dg_u\,df_r=\lim_{|\Pi_{st}|\to 0}\sum_{i=0}^{n-1}\left(\int_{t_i}^t h(r,t_i)\,df_r\right)(g_{t_{i+1}}-g_{t_i}).$$

Consequently, Step 2 and Theorem 2.5 imply that (15) is satisfied and therefore the proof is complete. □

Remark 2.7 Our Fubini type theorem could also have been obtained with the following strategy: show formula (15) for smooth functions f, g and h. Then use a density argument in order to cover all suitable Hölder cases. However, the density of smooth functions in spaces of the form $\mathcal{C}^\gamma_2$ has not been investigated yet (to our knowledge). Therefore the inclusion of this density argument would lead to a longer proof than the one we have chosen for Proposition 2.6.

The following integration by parts and Itô's formulas will be also needed in the last part of this paper.

Proposition 2.8 *Let f and g be two functions in $\mathcal{C}^\gamma_1([0,T];\mathbb{R})$, with $\gamma>1/2$. Then*

$$f_tg_t=f_0g_0+\int_0^t f_u\,dg_u+\int_0^t g_u\,df_u,\quad t\in[0,T].$$

Proof Set $q_t := f_t g_t - \int_0^t f_u\, dg_u - \int_0^t g_u\, df_u$, $t \in [0,T]$. It is easy to see that this function belongs to $\mathcal{C}_1^{2\gamma}([0,T];\mathbb{R})$ because of the equalities

$$f_t g_t - f_s g_s = f_s(\delta g)_{st} + g_s(\delta f)_{st} + (\delta g)_{st}(\delta f)_{st}$$

and

$$\int_s^t f_u\, dg_u + \int_s^t g_u\, df_u = f_s(\delta g)_{st} + g_s(\delta f)_{st} + \Lambda_{st}(\delta f\, \delta g) + \Lambda_{st}(\delta g\, \delta f),$$

which follows from (13). Now, since $q \in \mathcal{C}_1^{2\gamma}([0,T];\mathbb{R})$, with $2\gamma > 1$, q is a constant function. Otherwise stated, $q_t = q_0 = f_0 g_0$. Therefore the announced result is true. □

Proposition 2.9 *Let g and h be in $\mathcal{C}_1^{\gamma}([0,T],\mathbb{R})$ and $f \in \mathcal{C}_b^2(\mathbb{R})$. Also let $x_t = x_0 + \int_0^t g_s\, dh_s$, $t \in [0,T]$. Then*

$$f(x_t) = f(x_0) + \int_0^t f'(x_u) g_u\, dh_u, \quad t \in [0,T].$$

Proof Proceeding as in the proof of Proposition 2.8 and using the mean value theorem, we can show that

$$q_t = f(x_t) - \int_0^t f'(x_s) g_s\, dh_s, \quad t \in [0,T],$$

is a 2γ-Hölder-continuous function. Therefore the result holds. □

Remark 2.10 Proposition 2.9 has been proven in [38] using Riemann sums.

3 Young Delay Equation

Recall first that we wish to consider a differential equation of the form:

$$\begin{aligned} y_t &= \xi_0 + \int_0^t f\left(\mathcal{Z}_u^y\right) dx_u, \quad t \in [0,T], \\ \mathcal{Z}_0^y &= \xi. \end{aligned} \tag{18}$$

In the previous equation, the integral has to be interpreted in the Young sense of (13), the initial condition ξ is an element of $\mathcal{C}_1^{\gamma}([-h,0];\mathbb{R}^n)$, the driving noise x is in $\mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^d)$, with $\gamma > 1/2$. We seek a solution y in the space $\mathcal{C}_{\xi,0,T}^{\lambda}(\mathbb{R}^n)$ for $1/2 < \lambda < \gamma$, and f is a given function $f : \mathcal{C}_1^{\lambda}([-h,0];\mathbb{R}^n) \to \mathbb{R}^{n\times d}$. In this section, we shall solve equation (18) thanks to a contraction argument, and then study its differentiability with respect to the driving noise x. Of course, the main application we have in mind is the case where x is a d-dimensional fractional Brownian motion, and this particular case will be considered in Sect. 4.

3.1 Existence and Uniqueness of the Solution

In order to solve equation (18), some smoothness and boundedness assumptions have to be made on our coefficient f. In fact, we shall rely on the following hypothesis:

Hypothesis 1 *There exist a positive constant M and $\lambda \in (1/2, \gamma)$ such that*

$$\left|f(\zeta)\right| \leq M, \quad \text{and} \quad \left|f(\zeta_2) - f(\zeta_1)\right| \leq M \sup_{\theta \in [-h,0]} \left|\zeta_2(\theta) - \zeta_1(\theta)\right|$$

uniformly in $\zeta, \zeta_1, \zeta_2 \in \mathcal{C}_1^\lambda([-h, 0]; \mathbb{R}^n)$.

Actually we will assume that f satisfies a stronger Lipschitz type hypothesis on the space $\mathcal{C}_1^\lambda(\mathbb{R}^n)$. Let us state first a preliminary result before we come to this second assumption:

Lemma 3.1 *Let $a = (a_1, a_2)$, with $0 \leq a_1 < a_2 \leq T$, let also $Z \in \mathcal{C}_1^\lambda([a_1 - h, a_2]; \mathbb{R}^n)$ and set*

$$\left[\mathcal{U}^{(a)} Z\right]_s = f\left(\mathcal{Z}_s^Z\right), \quad s \in [a_1, a_2].$$

Then Hypothesis 1 implies that $\mathcal{U}^{(a)}$ is a map from $\mathcal{C}_1^\lambda([a_1 - h, a_2]; \mathbb{R}^n)$ into $\mathcal{C}_1^\lambda([a_1, a_2]; \mathbb{R}^{n\times d})$, satisfying:

$$\left\|\mathcal{U}^{(a)} Z\right\|_{\lambda,[a_1,a_2]} \leq M \, \|Z\|_{\lambda,[a_1-h,a_2]}.$$

Proof The proof of this result is an immediate consequence of the definition (6) of Hölder's norms on $\mathcal{C}_1$ and Hypothesis 1. □

With this preliminary result in hand, we can now introduce our second hypothesis on the coefficient f.

Hypothesis 2 *Taking up the notations of Lemma 3.1, consider an initial condition $\rho \in \mathcal{C}_1^\lambda([a_1 - h, a_1])$. We assume that, for any $N \geq 1$, there is a positive constant c_N such that:*

$$\left\|\mathcal{U}^{(a)}(Z_1) - \mathcal{U}^{(a)}(Z_2)\right\|_{\lambda,[a_1,a_2]} \leq c_N \|Z_1 - Z_2\|_{\lambda,[a_1-h,a_2]},$$

for all $0 \leq a_1 \leq a_2 \leq T$ and $Z_1, Z_2 \in \mathcal{C}_{\rho,a_1,a_2}^\lambda(\mathbb{R}^n)$, satisfying

$$\max\left\{\|Z_1\|_{\lambda,[a_1-h,a_2]};\ \|Z_2\|_{\lambda,[a_1-h,a_2]}\right\} \leq N,$$

where λ is given in Hypothesis 1.

Observe that Hypothesis 2 holds in particular if, for $\lambda > 0$, the map $\mathcal{U}^{(a)}$ admits a derivative which is locally bounded, uniformly in $a \in [0, T]$.

Now that we have stated our main assumptions, the following theorem is the main result of this section.

Theorem 3.2 *Under Hypotheses 1 and 2, the delay equation (18) has a unique solution in $\mathcal{C}^{\lambda}_{\xi,0,T}(\mathbb{R}^n)$.*

Before giving the proof of this theorem, we establish an auxiliary result. This will be helpful in order to get the existence of an invariant ball under the contracting map which gives raise to the solution of our equation.

Lemma 3.3 *Let $x \in \mathcal{C}_1^{\gamma}([a_1,a_2];\mathbb{R}^d)$ with $\gamma > 1/2$ and $0 \le a_1 < a_2$, $\lambda \in (1/2,\gamma)$ and $v \in \mathbb{R}^n$. Set $a=(a_1,a_2)$, recall notation (7) and define $\mathcal{V}^{(a)} : \mathcal{C}_1^{\lambda}([a_1,a_2];\mathbb{R}^{n\times d}) \to \mathcal{C}^{\lambda}_{v,a_1,a_2}(\mathbb{R}^n)$ by:*

$$\left[\mathcal{V}^{(a)} Z\right]_s = v + \mathcal{J}_{a_1 s}(Z\,dx), \qquad s \in [a_1,a_2],$$

where $\mathcal{J}_{a_1 s}(Z\,dx)$ stands for the Young integral defined by (13). Then

$$\left\|\mathcal{V}^{(a)} Z\right\|_{\lambda,[a_1,a_2]} \le \|x\|_{\gamma}\left(\|Z\|_{\infty,[a_1,a_2]}(a_2-a_1)^{\gamma-\lambda} + c_{\lambda+\gamma}\|Z\|_{\lambda,[a_1,a_2]}(a_2-a_1)^{\gamma}\right),$$

with $c_{\lambda+\gamma} = (2^{\lambda+\gamma}-2)^{-1}$.

Proof Let $a_1 \le s \le t \le T$. Then Theorem 2.5 point (3) implies that

$$\left[\mathcal{V}^{(a)} Z\right]_t - \left[\mathcal{V}^{(a)} Z\right]_s = \mathcal{J}_{st}(Z\,dx).$$

Our claim is then a direct consequence of Theorem 2.5 point (2) and of the definition (6). □

Proof of Theorem 3.2 This proof is divided in several steps.

Step 1: *Existence of invariant balls.* Let us first consider an interval of the form $[0,\varepsilon]$, which means that, when we include the delay of the equation, we shall consider processes defined on $[-h,\varepsilon]$. More specifically, let us recall that the spaces $\mathcal{C}^{\lambda}_{\xi,0,\varepsilon}(\mathbb{R}^n)$ have been defined by relation (8). Then we consider a map $\Gamma : \mathcal{C}^{\lambda}_{\xi,0,\varepsilon} \to \mathcal{C}^{\lambda}_{\xi,0,\varepsilon}$, where we have set $\mathcal{C}^{\lambda}_{\xi,0,\varepsilon} = \mathcal{C}^{\lambda}_{\xi,0,\varepsilon}(\mathbb{R}^n)$ for notational sake, defined in the following way: if $z \in \mathcal{C}^{\lambda}_{\xi,0,\varepsilon}$, then $\Gamma(z) = \hat{z}$, where $\hat{z}_t = \xi_t$ for $t \in [-h,0]$, and:

$$(\delta\hat{z})_{st} = \mathcal{J}_{st}(Z\,dx), \qquad \text{with } Z_u = f\left(\mathcal{Z}^z_u\right), \text{ for } s,t \in [0,\varepsilon]. \tag{19}$$

That is (recalling that $\mathcal{Z}^z_u(s) = z_{u+s}$ for $s \in [-h,0]$),

$$\hat{z}_t - \xi_0 = (\delta\hat{z})_{0t} = \int_0^t f\left(\mathcal{Z}^z_u\right) dx_u, \qquad t \in [0,\varepsilon].$$

We shall now look for an invariant ball in the space $\mathcal{C}^{\lambda}_{\xi,0,\varepsilon}$ for the map Γ.

So let us pick an element z, such that $\|z\|_{\lambda,[-h,\varepsilon]} \le N_1$ and set $\Gamma(z) = \hat{z}$. On $[-h,0]$, we have $\hat{z} = \xi$, and hence $\|\delta\hat{z}\|_{\lambda,[-h,0]} = \|\delta\xi\|_{\lambda,[-h,0]} \equiv N_\xi$. We shall thus choose $N_1 \ge 2N_\xi$.

On $[0,\varepsilon]$, we have now, invoking Lemma 3.3:

$$\left\|\delta\hat{z}\right\|_{\lambda,[0,\varepsilon]} \le \|Z\|_\infty \|x\|_\gamma \varepsilon^{\gamma-\lambda} + c_{\gamma,\lambda}\|Z\|_{\lambda,[0,\varepsilon]}\|x\|_\gamma \varepsilon^\gamma. \tag{20}$$

Furthermore, according to Hypothesis 1, we have $\|Z\|_\infty \le M$ and thanks to Lemma 3.1, we also have $\|Z\|_{\lambda,[0,\varepsilon]} \le M\,\|z\|_{\lambda,[-h,\varepsilon]} \le M\,N_1$, by assumption. Then we can recast the previous inequality into:

$$\left\|\delta\hat{z}\right\|_{\lambda,[0,\varepsilon]} \le M\|x\|_\gamma \varepsilon^{\gamma-\lambda}\left[1 + c_{\gamma,\lambda}N_1\varepsilon^\lambda\right]. \tag{21}$$

Let us choose now ε and N_1 in the following manner (notice that ε does **not** depend on the initial condition ξ):

$$\varepsilon = \left[4Mc_{\gamma,\lambda}\|x\|_\gamma\right]^{-1/\gamma} \wedge 1, \quad \text{and} \quad N_1 \ge 4M\|x\|_\gamma. \tag{22}$$

With this choice of ε, N_1, inequality (21) becomes $\|\delta\hat{z}\|_{\lambda,[0,\varepsilon]} \le N_1/2$. Summarizing the considerations above, we have thus found that

$$\begin{aligned} &\varepsilon = \left[4Mc_{\gamma,\lambda}\|x\|_\gamma\right]^{-1/\gamma} \wedge 1, \qquad N_1 \ge \sup\left\{2N_\xi;\ 4M\|x\|_\gamma\right\} \\ &\quad\Longrightarrow \quad \sup\left\{\left\|\delta\hat{z}\right\|_{\lambda,[-h,0]};\ \left\|\delta\hat{z}\right\|_{\lambda,[0,\varepsilon]}\right\} \le \frac{N_1}{2}. \end{aligned} \tag{23}$$

Consider now $s<t$, with $s\in[-h,0]$ and $t\in[0,\varepsilon]$. Then, owing to the previous relation, we have:

$$\left|(\delta\hat{z})_{st}\right| \le \left|(\delta\hat{z})_{s0}\right| + \left|(\delta\hat{z})_{0t}\right| \le \frac{N_1}{2}\left(s^\lambda + t^\lambda\right) \le N_1|t-s|^\lambda,$$

which, together with the last inequality, proves that $B(0,N_1)$ in $\mathcal{C}^\lambda_{\xi,0,\varepsilon}$ is left invariant by Γ, under the assumptions of (23).

Assume now that we have been able to produce a solution $y^{(1)}$ to equation (18) on the interval $[-h,\varepsilon]$. We try now to iterate the invariant ball argument on $[\varepsilon-h;2\varepsilon]$. The arguments above go through with very little changes: we are now working on delayed Hölder spaces of the form $\mathcal{C}^\lambda_{y^{(1)},\varepsilon,2\varepsilon}$, and the map Γ is defined by $\Gamma(z)=\hat{z}$, with $\hat{z}=y^{(1)}$ on $[\varepsilon-h;\varepsilon]$, and $\delta\hat{z}$ having the same expression as in (19) on $[\varepsilon,2\varepsilon]$. We wish to find a ball $B(0,N_2)$ in $\mathcal{C}^\lambda_{y^{(1)},\varepsilon,2\varepsilon}$, left invariant by the map Γ. With the same computations as for the interval $[-h,\varepsilon]$, the assumptions of inequality (23) become

$$\varepsilon = \left[4Mc_{\gamma,\lambda}\|x\|_\gamma\right]^{-1/\gamma} \wedge 1, \qquad N_2 \ge \sup\left\{2N_{y^{(1)}};\ 4M\|x\|_\gamma\right\}.$$

Notice again that we are able to choose here the same ε as before, by changing N_1 into N_2 according to the value of $\|y^{(1)}\|_{\lambda,[\varepsilon-h,\varepsilon]}$. It is now readily checked that $B(0,N_2)$ is invariant under Γ, and this calculation is also easily repeated on any interval $[k\varepsilon-h,(k+1)\varepsilon]$ for any $k\ge 0$, until the whole interval $[0,T]$ is covered.

Step 2: *Fixed point argument.* We shall suppose here that we have been able to construct the unique solution y to (18) on $[-h;l\varepsilon]$, and we shall build the fixed point argument on $[l\varepsilon-h;(l+1)\varepsilon]$. On the latter interval, the initial condition of the paths we shall consider is $\xi^{l,1} \equiv y$ on $[l\varepsilon - h; l\varepsilon]$. If Γ is the map defined on $\mathcal{C}^{\lambda}_{\xi^{l,1},l\varepsilon,(l+1)\varepsilon}$ by (19), then we know that $B(0, N_{l+1})$ is invariant by Γ.

In order to settle our fixed point argument, we shall first consider an interval of the form $[l\varepsilon - h; l\varepsilon + \eta]$, for a parameter $0 < \eta \le \varepsilon$ to be determined. On $\mathcal{C}^{\lambda}_{\xi^{l,1},l\varepsilon,l\varepsilon+\eta}$, we define a map, called again Γ, according to (19). Pick then two functions $z^1, z^2 \in \mathcal{C}^{\lambda}_{\xi^{l,1},l\varepsilon,l\varepsilon+\eta}$, set $\hat{z}^i = \Gamma(z^i)$ for $i = 1, 2$ and $\zeta = \hat{z}^2 - \hat{z}^1$. Then $\zeta \in \mathcal{C}^{\lambda}_{0,l\varepsilon,l\varepsilon+\eta}$, and if $l\varepsilon \le s < t \le l\varepsilon + \eta$, we have

$$(\delta\zeta)_{st} = \mathcal{J}_{st}\big(\big(Z^2 - Z^1\big)\,dx\big), \quad \text{where } Z^i = f\big(\mathcal{Z}^{z^i}\big).$$

Thus, just like in (20), we have

$$\begin{aligned}\|\delta\zeta\|_{\lambda,[l\varepsilon-h,l\varepsilon+\eta]} &\le \big\|Z^1 - Z^2\big\|_{\infty,[l\varepsilon,l\varepsilon+\eta]}\|x\|_{\gamma}\,\eta^{\gamma-\lambda} \\ &\quad + c_{\gamma,\lambda}\big\|Z^1 - Z^2\big\|_{\lambda,[l\varepsilon,l\varepsilon+\eta]}\|x\|_{\gamma}\,\eta^{\gamma}.\end{aligned}$$

Furthermore, $\|Z^1 - Z^2\|_{\infty,[l\varepsilon,l\varepsilon+\eta]} \le \|Z^1 - Z^2\|_{\lambda,[l\varepsilon,l\varepsilon+\eta]}\,\eta^{\lambda}$. Hence,

$$\|\delta\zeta\|_{\lambda,[l\varepsilon-h,l\varepsilon+\eta]} \le (1 + c_{\gamma,\lambda})\big\|Z^1 - Z^2\big\|_{\lambda,[l\varepsilon,l\varepsilon+\eta]}\,\|x\|_{\gamma}\,\eta^{\gamma}.$$

We also have $Z^1 - Z^2 = f(\mathcal{Z}^{z^1}) - f(\mathcal{Z}^{z^2})$, and thanks to Hypothesis 2, we obtain

$$\|\delta\zeta\|_{\lambda,[l\varepsilon-h,l\varepsilon+\eta]} \le (1 + c_{\gamma,\lambda})\|x\|_{\gamma}\,c_{N_{l+1}}\,\eta^{\gamma}\,\big\|z^1 - z^2\big\|_{\lambda,[l\varepsilon-h,l\varepsilon+\eta]}.$$

Therefore, we are able to apply the fixed point argument in the usual way as soon as

$$(1 + c_{\gamma,\lambda})\,c_{N_{l+1}}\,\|x\|_{\gamma}\,\eta^{\gamma} \le \frac{1}{2}, \quad \text{or} \quad \eta = \big[2(1 + c_{\gamma,\lambda})\,c_{N_{l+1}}\,\|x\|_{\gamma}\big]^{-1/\gamma} \wedge \varepsilon.$$

With this value of η, we are thus able to get a unique solution to (18) on $[l\varepsilon - h; l\varepsilon + \eta]$.

Let us proceed now to the case of $[l\varepsilon + \eta - h, l\varepsilon + 2\eta]$. The arguments are roughly the same as in the previous case, but one has to be careful about the change in the initial condition. In fact, the initial condition here should be $\xi^{l,2} \equiv y$ on $[l\varepsilon + \eta - h, l\varepsilon + \eta]$. However, we can also choose to extend this initial condition backward, and set it as $\xi^{l,2} \equiv y$ on $[l\varepsilon - h, l\varepsilon + \eta]$. We then define the usual map Γ as in (19), and we have to prove that $B(0, N_{l+1})$ is left invariant by Γ. To this purpose, take $z \in \mathcal{C}^{\lambda}_{\xi^{l,2},l\varepsilon+\eta,l\varepsilon+2\eta}$ in $B(0, N_{l+1})$, and set $\hat{z} = \Gamma(z)$. Observe then that, for any $t \in [l\varepsilon + \eta, l\varepsilon + 2\eta]$, we have

$$\begin{aligned}\hat{z}_t &= \xi^2_{l\varepsilon+\eta} + \int_{l\varepsilon+\eta}^{t} f\big(\mathcal{Z}^z_u\big)\,dx_u = \xi^1_{l\varepsilon} + \int_{l\varepsilon}^{l\varepsilon+\eta} f\big(\mathcal{Z}^y_u\big)\,dx_u + \int_{l\varepsilon+\eta}^{t} f\big(\mathcal{Z}^z_u\big)\,dx_u \\ &= \xi^1_{l\varepsilon} + \int_{l\varepsilon}^{t} f\big(\mathcal{Z}^z_u\big)\,dx_u,\end{aligned}$$

where we have used the fact that $\xi^{l,2} \equiv y$ on $[l\varepsilon - h, l\varepsilon + \eta]$ solves (18). It is now easily seen that $\hat{z}$ is in $B(0, N_{l+1})$, and this allows to settle our fixed point argument as in the previous case, with the same interval length η. This step can now be iterated until the whole interval $[l\varepsilon; (l+1)\varepsilon]$ is covered. □

3.2 Moments of the Solution

The moments of the solution to (18) can be bounded in the following way:

Proposition 3.4 *Under the same assumptions as in Theorem 3.2, let y be the solution of equation (18) on the interval $[0,T]$, with an initial condition $\xi \in \mathcal{C}_1^\lambda([-h,0];\mathbb{R}^n)$. Then there exists a strictly positive constant $c = c(\gamma,\lambda,M,T)$ such that*

$$\|y\|_{\lambda,[-h,T]} \le c \max\bigl[\|\xi\|_\lambda, \|x\|_\gamma^{\lambda/(\gamma+\lambda-1)}, \|x\|_\gamma\bigr].$$

Proof From the proof of Theorem 3.2, we know that $\|y\|_{\lambda,[-h,T]}$ is finite. Let us assume that this quantity is equal to K, and let us find an estimate on K. One can begin with a small interval, which will be called again $[0,\varepsilon]$, though it will not be the same interval as in the proof of Theorem 3.2. In any case, taking into account that y solves equation (18), we obtain similarly to (20),

$$\begin{aligned}\|\delta y\|_{\lambda,[0,\varepsilon]} &\le M\|x\|_\gamma \varepsilon^{\gamma-\lambda} + c_{\gamma,\lambda} M \|\delta y\|_{\lambda,[-h,\varepsilon]}\|x\|_\gamma \varepsilon^\gamma \\ &\le M\|x\|_\gamma \varepsilon^{\gamma-\lambda} + c_{\gamma,\lambda} M K\|x\|_\gamma \varepsilon^\gamma \equiv g(\varepsilon,K). \end{aligned} \tag{24}$$

Along the same line, for any $k \le [T/\varepsilon]$, we have

$$\|\delta y\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} \le g(\varepsilon,K).$$

Take now $s,t \in [0,T]$ such that $i\varepsilon \le s < (i+1)\varepsilon \le j\varepsilon \le t < (j+1)\varepsilon$. Set also $t_i = s$, $t_k = k\varepsilon$ for $i+1 \le k \le j$, and $t_{j+1} = t$. Then

$$\begin{aligned}\bigl|(\delta y)_{st}\bigr| = \left|\sum_{k=i}^{j} (\delta y)_{t_k t_{k+1}}\right| &\le g(\varepsilon,K)\sum_{k=i}^{j}(t_{k+1}-t_k)^\lambda \\ &\le g(\varepsilon,K)(j-i+1)^{1-\lambda}(t-s)^\lambda,\end{aligned}$$

where we have used the fact that $r \mapsto r^\lambda$ is a concave function. Note that the indices i,j above satisfy $(j-i+1) \le 2T/\varepsilon$. Plugging this into the last series of inequalities, we end up with

$$\|\delta y\|_{\lambda,[0,T]} \le \frac{g(\varepsilon,K)(2T)^{1-\lambda}}{\varepsilon^{1-\lambda}} = \left[\frac{M\,\|x\|_\gamma}{\varepsilon^{1-\gamma}} + c_{\gamma,\lambda}\, M\, K\, \|x\|_\gamma \varepsilon^{\gamma+\lambda-1}\right](2T)^{1-\lambda}.$$

Thus the parameters K and ε satisfy the relation

$$K \le \left[\frac{M\,\|x\|_\gamma}{\varepsilon^{1-\gamma}} + c_{\gamma,\lambda}\, M\, K\, \|x\|_\gamma \varepsilon^{\gamma+\lambda-1}\right](2T)^{1-\lambda} + \|\xi\|_\lambda, \tag{25}$$

In order to solve (25), choose ε such that

$$c_{\gamma,\lambda}\, M\, \|x\|_\gamma \varepsilon^{\gamma+\lambda-1}\,(2T)^{1-\lambda} = \frac{1}{2},$$

that is

$$\varepsilon = \left[2c_{\gamma,\lambda}\, M\, \|x\|_\gamma\,(2T)^{1-\lambda}\right]^{-1/(\gamma+\lambda-1)}.$$

Plugging this relation into (25), we obtain the result when $\varepsilon < T$.

Finally, $T < \varepsilon$ if and only if $T^\gamma < [2^{2-\lambda} c_{\gamma+\lambda} M \|x\|_\gamma]^{-1}$. Thus, by inequality (24), the proof is complete. □

3.3 Case of a Weighted Delay

In this subsection, we prove that our Hypotheses 1 and 2 are satisfied for the weighted delay alluded to in the introduction, that is for the function f given by equation (2).

Proposition 3.5 *Let ν be a finite signed measure on $[-h,0]$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ a four times differentiable bounded function with bounded derivatives. Then Hypotheses 1 and 2 are fulfilled for $f : \mathcal{C}_1^\lambda([-h,0];\mathbb{R}^n) \to \mathbb{R}^{n\times d}$ defined by:*

$$f(Z) = \sigma\left(\int_{-h}^0 Z(\theta)\nu(d\theta)\right),$$

with $Z \in \mathcal{C}_1^\lambda([-h,0];\mathbb{R}^n)$.

Proof We first show that Hypothesis 1 holds. More specifically, the condition $|f(\zeta)| \le M$ being obvious in our case, we focus on the second condition of Hypothesis 1. Let $Z_1, Z_2 \in \mathcal{C}_1^\lambda([-h,0];\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\begin{aligned}\left|f(Z_1) - f(Z_2)\right| &\le C\left|\int_{-h}^0 \big(Z_1(\theta) - Z_2(\theta)\big)\nu(d\theta)\right| \\ &\le C|\nu|\big([-h,0]\big)\Big(\sup_{\theta\in[-h,0]}\left|Z_1(\theta) - Z_2(\theta)\right|\Big),\end{aligned}$$

where $|\nu|$ is the total variation of ν. Therefore Hypothesis 1 is satisfied in this case.

Now we prove that $\mathcal{U}^{(a)}$ is Fréchet differentiable in order to analyze Hypothesis 2. Since the map $Z \mapsto \int_{-h}^0 Z(\cdot+\theta)\nu(d\theta)$ is easily shown to be a bounded linear operator from $\mathcal{C}_1^\lambda([a_1-h,a_2];\mathbb{R}^n)$ into $\mathcal{C}_1^\lambda([a_1,a_2];\mathbb{R}^n)$, we only need to show that

$$\sigma : \mathcal{C}^\lambda_{\rho,a_1,a_2}\big(\mathbb{R}^n\big) \to \mathcal{C}^\lambda_{\hat\rho,a_1,a_2}\big(\mathbb{R}^{n\times d}\big), \quad \text{with } \hat\rho \triangleq \sigma(\rho),$$

is Fréchet differentiable in the directions of $\mathcal{C}^\lambda_{0,a_1,a_2}(\mathbb{R}^n)$, with derivative $[D\sigma(Z)\ell](t) = \sigma'(Z(t))\ell(t)$. Toward this end, we have to show that, taking $Z \in \mathcal{C}^\lambda_{\rho,a_1,a_2}(\mathbb{R}^n)$ and $\ell \in \mathcal{C}^\lambda_{0,a_1,a_2}(\mathbb{R}^n)$, and setting

$$q_t = \sigma\big(Z(t)+\ell(t)\big) - \sigma\big(Z(t)\big) - \sigma'\big(Z(t)\big)\,\ell(t),$$

then

$$\lim_{\|\ell\|_{\lambda,[a_1-h,a_2]}\to 0} \frac{\|q\|_{\lambda,[a_1-h,a_2]}}{\|\ell\|_{\lambda,[a_1-h,a_2]}} = 0. \tag{26}$$

In order to prove relation (26), define a function $b : [0,1]^2 \to \mathbb{R}$ by:

$$b(\lambda,\mu) = Z(s) + \lambda\ell(s) + \mu\big[Z(t) - Z(s)\big] + \lambda\mu\big[\ell(t) - \ell(s)\big].$$

Observe then that $b(1,1) = Z(t) + \ell(t)$, $b(1,0) = Z(s) + \ell(s)$, $b(0,1) = Z(t)$ and $b(0,0) = Z(s)$. We will also set $H(\lambda,\mu) = \sigma(b(\lambda,\mu))$. Then

$$\begin{aligned}
&\sigma\big(Z(t)+\ell(t)\big) - \sigma\big(Z(t)\big) - \sigma'\big(Z(t)\big)\,\ell(t) \\
&\quad = \sigma\big(b(1,1)\big) - \sigma\big(b(0,1)\big) - \sigma'\big(b(0,1)\big)\big[b(1,1) - b(0,1)\big] \\
&\quad = \frac{1}{2}\int_0^1 \partial^2_{\lambda\lambda} H(\lambda,1)[1-\lambda]\,d\lambda,
\end{aligned}$$

and similarly, we have

$$\sigma\big(Z(s)+\ell(s)\big) - \sigma\big(Z(s)\big) - \sigma'\big(Z(s)\big)\ell(s) = \int_0^1 \partial^2_{\lambda\lambda} H(\lambda,0)[1-\lambda]\,d\lambda.$$

Hence, plugging these two relations in the definition of q, we end up with

$$\begin{aligned}
(\delta q)_{st} &= \int_0^1 \big(\partial^2_{\lambda\lambda} H(\lambda,1) - \partial^2_{\lambda\lambda} H(\lambda,0)\big)[1-\lambda]\,d\lambda \\
&= \int_0^1 \partial^3_{\lambda\lambda\mu} H(\lambda,0)[1-\lambda]\,d\lambda + \int_{[0,1]^2} \partial^4_{\lambda\lambda\mu\mu} H(\lambda,\mu)[1-\lambda][1-\mu]\,d\lambda\,d\mu.
\end{aligned}$$

The calculation of $\partial^3_{\lambda\lambda\mu} H(\lambda,0)$ and $\partial^4_{\lambda\lambda\mu\mu} H(\lambda,\mu)$ is a matter of long and tedious computations, which are left to the reader. Let us just mention that both expressions can be written as a sum of terms of which a typical example is

$$\sigma'''\big(b(\lambda,\mu)\big)\big[(\delta Z)_{st} + \mu(\delta Z)_{st}\big]\big[\ell(s) + \lambda(\delta\ell)_{st}\big]\,(\delta\ell)_{st}. \tag{27}$$

These terms are obviously quadratic in ℓ, and can be bounded uniformly in λ, μ, s, t under the hypothesis $\sigma \in C_b^4$. Notice that, in order to bound the term $|\ell(s)|$ in (27), we use the fact that ℓ has a null initial condition, which means in particular that $|\ell(s)| \le (a_2 - a_1 + h)^\lambda \|\ell\|_{\lambda,[a_1-h,a_2]}$. This finishes the proof of (26). The continuity of $D\sigma(Z)$ and the existence of the constant c_N introduced in Hypothesis 2 are now a question of trivial considerations, and this ends the proof of our proposition. □

Remark 3.6 The proof of Fréchet differentiability of f was not necessary for the existence-uniqueness result, which relied on some Lipschitz type condition. However, this stronger result turns out to be useful for the Malliavin calculus part, and this is why we prove it here. Nevertheless, notice that Theorem 3.2 holds true for a C_b^2 coefficient σ.

3.4 Differentiability of the Solution

In this section we study the differentiability of the solution of (18) as a function of the integrator x, following closely the methodology of [31]. In particular, our differentiability result will be achieved with the help of the map $F : \mathcal{C}^{\gamma}_{0,0,T}(\mathbb{R}^d) \times \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n) \to \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$ given by

$$\big[F(k,Z)\big]_t = Z_t - \mathcal{J}_{0t}\big(f\big(\mathcal{Z}^{Z+\tilde{\xi}}\big)d(x+k)\big), \quad t \in [0,T] \tag{28}$$

where $\tilde{\xi}_t = \xi_0$ for $t \in [0,T]$, and $\tilde{\xi}_t = \xi_t$ for $t \in [-h,0]$. Here we recall that ξ stands for an initial condition in $\mathcal{C}^{\lambda}_1([-h,0])$. In this section the coefficient f will satisfies the following:

Hypothesis 3 *Set $\mathbf{t} = (0,t)$, and recall that the map $\mathcal{U}^{(\mathbf{t})}$ has been defined in Lemma 3.1. We assume that $\mathcal{U}^{(\mathbf{t})} : \mathcal{C}^{\lambda}_{\xi,0,t}(\mathbb{R}^n) \to \mathcal{C}^{\lambda}([0,t];\mathbb{R}^{n\times d})$ is continuously Fréchet differentiable in the directions of $\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^n)$, for some $\lambda \in (1/2,\gamma)$. We call $\nabla\mathcal{U}^{(\mathbf{t})} : \mathcal{C}^{\lambda}_{\xi,0,t}(\mathbb{R}^n) \to \mathcal{L}(\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^n);\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^{n\times d}))$ its differential, where $\mathcal{L}(\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^n);\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^{n\times d}))$ denotes the linear operators from $\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^n)$ into $\mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^{n\times d})$. Moreover, we also assume that, for $s<t$ and $Z \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$,*

$$\big[\nabla\mathcal{U}^{(\mathbf{t})}(y)\big](Z) = \big[\nabla\mathcal{U}^{(\mathbf{s})}(y)\big](Z) \quad \text{on } [0,s],$$

where y is the solution of equation (18).

Remarks 3.7

(1) Notice that we have shown, during the proof of Proposition 3.5, that the weighted delay given by (2) also satisfies this last assumption.
(2) If $Z \in \mathcal{C}^{\lambda}_{0,0,t}(\mathbb{R}^n)$, then

$$\big\|\nabla\mathcal{U}^{(\mathbf{t})}(y)(Z)\big\|_{\lambda,[0,t]} \le \big|\nabla\mathcal{U}^{(\mathbf{T})}(y)\big|\|Z\|_{\lambda,[0,t]}.$$

Indeed, set $\tilde{Z}_s = Z_s$ for $s \in [0,t]$, and $\tilde{Z}_s = Z_t$ for $s > t$. Therefore Hypothesis 3 implies

$$\begin{aligned}\big\|\nabla\mathcal{U}^{(\mathbf{t})}(y)(Z)\big\|_{\lambda,[0,t]} &\le \big\|\nabla\mathcal{U}^{(\mathbf{T})}(y)\big(\tilde{Z}\big)\big\|_{\lambda,[0,T]} \le \big|\nabla\mathcal{U}^{(\mathbf{T})}(y)\big|\big\|\tilde{Z}\big\|_{\lambda,[0,T]} \\ &= \big|\nabla\mathcal{U}^{(\mathbf{T})}(y)\big|\|Z\|_{\lambda,[0,t]},\end{aligned}$$

and our claim is satisfied.

We are now ready to prove the differentiability properties for equation (18):

Lemma 3.8 *Under the Hypothesis 3, the map F given by (28) is continuously Fréchet differentiable.*

Proof Let us call respectively D_1 and D_2 the two directional derivatives. We first observe that, for $k, g \in \mathcal{C}^{\gamma}_{0,0,T}(\mathbb{R}^d)$ and $Z \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$, we have

$$F(k+g, Z) - F(k, Z) + \int_0^{\cdot} \big[\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big]_s \, dg_s = 0.$$

In other words, the partial derivative $D_1 F$ is defined by

$$D_1 F(k, Z)(g) = -\int_0^{\cdot} \big[\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big]_s \, dg_s = -\mathcal{J}_{0\cdot}\big(\big[\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big]\, dg\big).$$

We shall prove now that $D_1 F$ is continuous: consider $k, \tilde{k} \in \mathcal{C}^{\gamma}_{0,0,T}(\mathbb{R}^d)$ and $Z, \tilde{Z} \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$. For notational sake, set also $\|\cdot\|_\lambda$ for $\|\cdot\|_{\lambda,[0,T]}$. Then, according to Lemma 3.3, we obtain

$$\begin{aligned}
&\big\| D_1 F(k, Z)(\eta) - D_1 F\big(\tilde{k}, \tilde{Z}\big)(\eta)\big\|_\lambda \\
&\quad = \big\| \mathcal{J}\big(\big[\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big) - \mathcal{U}^{(\mathbf{T})}\big(\tilde{Z}+\tilde{\xi}\big)\big]\, d\eta_s\big)\big\|_\lambda \\
&\quad \le \|\eta\|_\gamma \big(\big\|\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big) - \mathcal{U}^{(\mathbf{T})}\big(\tilde{Z}+\tilde{\xi}\big)\big\|_\infty T^{\gamma-\lambda} \\
&\qquad + C_{\lambda+\gamma} T^\gamma \big\|\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big) - \mathcal{U}^{(\mathbf{T})}\big(\tilde{Z}+\tilde{\xi}\big)\big\|_\lambda\big),
\end{aligned}$$

which, owing to Hypothesis 3, implies that $D_1 F$ is continuous.

Concerning $D_2 F$ we have, for $k \in \mathcal{C}^{\gamma}_{0,0,T}(\mathbb{R}^d)$, $Z \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$ and $\tilde{Z} \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$, and thanks to Theorem 2.5:

$$\begin{aligned}
&\big\| F\big(k, Z+\tilde{Z}\big) - F(k, Z) - \tilde{Z} + \mathcal{J}\big(\big[\nabla\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big]\big(\tilde{Z}\big)\, d(x+k)\big)\big\|_\lambda \\
&\quad \le \|x+k\|_\gamma \big(\big\|\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{Z}+\tilde{\xi}\big) - \mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big) - \big[\nabla\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big]\big(\tilde{Z}\big)\big\|_\infty T^{\gamma-\lambda} \\
&\qquad + C_{\lambda+\gamma} T^\gamma \big\|\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{Z}+\tilde{\xi}\big) - \mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big) - \big[\nabla\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big]\big(\tilde{Z}\big)\big\|_\lambda\big).
\end{aligned}$$

Therefore, making use of Hypothesis 3, we have

$$D_2 F(k, Z)\big(\tilde{Z}\big) = \tilde{Z} - \int_0^{\cdot} \nabla\mathcal{U}^{(\mathbf{T})}\big(Z+\tilde{\xi}\big)\big(\tilde{Z}\big)_s \, d(x_s + k_s).$$

The continuity of $D_2 F$ can now be proven along the same lines as for $D_1 F$, and the computational details are left to the reader for sake of conciseness. The proof is now finished. □

The following will be used to show that $D_2 F(k, Z)$ is a linear homeomorphism.

Lemma 3.9 *Let $w \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$, y the solution of* (18) *and assume Hypotheses* 1, 2 *and* 3 *hold. Then the equation*

$$Z_t = w_t + \int_0^t \big(\big[\nabla\mathcal{U}^{(\mathbf{T})}(y)\big](Z)\big)_s \, dx_s, \quad 0 \le t \le T, \tag{29}$$

has a unique solution Z in $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$.

Proof Thanks to Lemma 3.3 and Remark 3.7.(2), one can proceed as in the proof of Theorem 3.2 to show that the result holds. □

Proposition 3.10 *Assume that Hypotheses 1 to 3 are satisfied. Let y be the solution of equation (18). Then the map $h \mapsto y(x+h)$ is Fréchet differentiable in the directions of $\mathcal{C}^{\gamma}_{0,0,T}(\mathbb{R}^d)$, as a $\mathcal{C}^{\lambda}_{\xi,0,T}(\mathbb{R}^n)$-valued function. Moreover, for $h, k \in \mathcal{C}^{\gamma}_{0,0,T}(\mathbb{R}^d)$, we have*

$$\begin{aligned}\big[Dy(x+h)(k)\big]_t &= \int_0^t \mathcal{U}^{(\mathbf{T})}\big(y(x+h)\big)_s \, dk_s \\ &\quad + \int_0^t \big[\nabla \mathcal{U}^{(\mathbf{T})}\big(y(x+h)\big)\big(Dy(x+h)(k)\big)\big]_s \, d(x_s + h_s). \end{aligned} \tag{30}$$

In particular, $[Dy(x+h)](k)$ is an element of $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$.

Remark 3.11 Let us recall that equation (30) has a unique solution, thanks to Lemma 3.9.

Proof of Proposition 3.10 Like in [31], the proof of this result is a consequence of the implicit function theorem, and we only need to show that $D_2F(0, y(x) - \tilde{\xi})$ is a linear homeomorphism from $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$ onto $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$. Indeed, in this case we deduce that $h \mapsto y(x+h)$ is Fréchet differentiable with

$$Dy(x+h)(k) = -\big(D_2F\big(h, y(x+h) - \tilde{\xi}\big)\big)^{-1} \circ D_1F\big(h, y(x+h) - \tilde{\xi}\big)(k), \tag{31}$$

which yields that (30) holds.

Finally, notice that $D_2F(0, y(x) - \tilde{\xi})$ is bijective and continuous according to Lemmas 3.8 and 3.9. Consequently the open mapping theorem implies that the application $D_2F(0, y(x) - \tilde{\xi})$ is also a homeomorphism. □

Interestingly enough, in the particular case of the weighted delay of Sect. 3.3, one can also derive a linear equation for the derivative $[Dy(x)]_t$, seen as a Hölder-continuous function.

Proposition 3.12 *Let σ and ν be as in Proposition 3.5. Let also f and y be defined by (2) and (18), respectively. Assume that ν is absolutely continuous with respect to the Lebesgue measure with Radon–Nykodim derivative in $L^p([-h,0])$ for $p > 1/(1-\gamma)$. Then, for $i \in \{1,\dots,n\}$ and $k \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$, we have*

$$Dy_t^i(x)(k) = \sum_{j=1}^d \int_0^t \Phi_t^{ij}(r) \, dk_r^j,$$

where, for $j \in \{i,\dots,d\}$ and $i \in \{1,\dots,n\}$, Φ^{ij} is defined by the equation

$$\begin{aligned}\Phi_t^{ij}(r) &= \big(\mathcal{U}^{(\mathbf{T})}(y)\big)_t^{ij} + \sum_{m=1}^n \sum_{l=1}^d \int_r^t \big(\big(\big[\nabla \mathcal{U}^{(\mathbf{T})}(y)\big]^m\big)^{il} \big(\Phi^{mj}(s)\big)\big)_s \, dx_s^l, \\ &\quad 0 \le r \le t \le T, \end{aligned} \tag{32}$$

and $\Phi_t(r) = 0$ for all $0 \le t < r \le T$.

Remark 3.13 Note that, for each $s \in [0,T]$, equation (32) has a unique solution in $\mathcal{C}^\lambda([s,T];\mathbb{R}^n)$ due to Lemma 3.9.

Proof of Proposition 3.12 In order to avoid cumbersome matrix notations, we shall prove this result for $n = d = 1$: notice that an easy consequence of the proof of Proposition 3.5 is that in our particular case,

$$\left[\nabla \mathcal{U}^{(\mathbf{T})}(Z)(k)\right]_t = \sigma'\left(\int_{-h}^0 Z_{t+\theta}\,\nu(d\theta)\right)\left(\int_{-h}^0 k_{t+\theta}\,\nu(d\theta)\right). \tag{33}$$

Set now $q_t = \sigma(\int_{-h}^0 y_{t+\theta}\,\nu(d\theta))$ and $q_t' = \sigma'(\int_{-h}^0 y_{t+\theta}\,\nu(d\theta))$, and write $y = y(x)$. Then equation (30) can be read as:

$$\left[Dy(k)\right]_t = \int_0^t q_s\,dk_s + U_t, \quad \text{with } U_t = \int_0^t q_s'\left(\int_{-h}^0 \left[Dy(k)\right]_{s+\theta}\,\nu(d\theta)\right)dx_s. \tag{34}$$

The Fubini type relation given in Lemma 2.6 allows then to show, as in [31, Proposition 4], that

$$\left[Dy(k)\right]_t = \int_0^t \Phi_t(r)\,dk_r, \tag{35}$$

for a certain function Φ, λ-Hölder continuous in all its variables. In order to identify the process Φ, plug relation (35) into equation (34) and apply Fubini's theorem, which yields

$$U_t = \int_{-h}^0 \nu(d\theta)\int_0^t q_s'\left(\int_0^{(s+\theta)_+} \Phi_{s+\theta}(r)\,dk_r\right)dx_s.$$

It should be noticed that this point is where we use the fact that $\nu(d\theta) = \mu(\theta)\,d\theta$ with $\mu \in L^p([-h,0])$. Indeed, in order to apply Lemma 2.6 to x, k and $\eta \mapsto F(\eta) = \int_{-h}^\eta \mu(\theta)\,d\theta$, we will assume (though this is not completely optimal) that F is γ-Hölder continuous. However, a simple application of Hölder's inequality yields

$$\left|F(\eta_2) - F(\eta_1)\right| \le |\eta_2 - \eta_1|^{(p-1)/p}\,\|\mu\|_{L^p([-h,0])}.$$

It is now easily seen that the condition $(p-1)/p > \gamma$ imposes $p > 1/(1-\gamma)$.

Owing now to (a slight extension of) Lemma 2.6, we can write

$$U_t = \int_{-h}^0 \nu(d\theta)\int_0^{(t+\theta)_+} m_t(r,\theta)\,dk_r, \quad \text{with } m_t(r,\theta) = \int_{r-\theta}^t q_s'\,\Phi_{s+\theta}(r)\,dx_s.$$

Apply Fubini's theorem again in order to integrate with respect to k in the last place: we obtain

$$\begin{aligned} U_t &= \int_0^t \left(\int_{-[(t-r)\wedge h]}^0 m_t(r,\theta)\,\nu(d\theta)\right)dk_r \\ &= \int_0^t \left(\int_{-[(t-r)\wedge h]}^0 \nu(d\theta)\int_{r-\theta}^t q_s'\,\Phi_{s+\theta}(r)\,dx_s\right)dk_r, \end{aligned}$$

and going back to (34), which is valid for any λ-Hölder-continuous function k, we see that Φ_t is defined on $[0,t]$ by the equation

$$\Phi_t(r) = q_t + \int_{-[(t-r)\wedge h]}^{0} \left(\int_{r-\theta}^{t} q_s' \Phi_{s+\theta}(r)\, dx_s \right) \nu(d\theta),$$

and $\Phi_t(r) = 0$ if $r > t$. A last application of Fubini's theorem allows us then to recast the above equation as

$$\Phi_t(r) = q_t + \int_r^t q_s' \left(\int_{-[h\wedge(s-r)]}^{0} \Phi_{s+\theta}(r)\, \nu(d\theta) \right) dx_s.$$

Notice now that, if $\theta \le -(s-r)$ in the above equation, then $s+\theta \le r$, which means that $\Phi_{s+\theta}(r) = 0$. Hence, we end up with an equation of the form

$$\Phi_t(r) = q_t + \int_r^t q_s' \left(\int_{-h}^{0} \Phi_{s+\theta}(r) \nu(d\theta) \right) dx_s,$$

which is easily seen to be of the form (32). □

3.5 Moments of Linear Equations

In order to obtain the regularity of the density for equation (18), we should bound the moments of the solution to equation (29). This is obtained in the following proposition:

Proposition 3.14 *Let $\tilde{f}$ be a mapping from $\mathcal{C}^{\lambda}_{\xi,0,T}(\mathbb{R}^n)$ into the linear operators from $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$ into $\mathcal{C}^{\lambda}([0,T];\mathbb{R}^{n\times d})$ such that, for $0 \le a < b \le T$, $\tilde{y} \in \mathcal{C}^{\lambda}_{\xi,0,T}(\mathbb{R}^n)$ and $\tilde{z} \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$,*

(1) $\|\tilde{f}(\tilde{y})\tilde{z}\|_{\infty,[a,b]} \le M\|\tilde{z}\|_{\infty,[a-h,b]}$.
(2) $\|\tilde{f}(\tilde{y})\tilde{z}\|_{\lambda,[a,b]} \le M\|\tilde{z}\|_{\lambda,[a-h,b]} + M\|\tilde{y}\|_{\lambda,[a-h,b]}\|\tilde{z}\|_{\infty,[a-h,b]}$.

Also let y be the solution of the equation (18), $w \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$ and $z \in \mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^n)$ the solution of the equation

$$z_t = w_t + \int_0^t \big(\tilde{f}(y)z\big)(s)\, dx_s, \qquad t \in [0,T].$$

Then

$$\|z\|_{\lambda,[0,T]} \le c_1 \|w\|_{\lambda,[0,T]} D^2_{\gamma,\lambda} e^{c_2 D_{\gamma,\lambda}},$$

for two strictly positive constants $c_i = c_i(T,\gamma,\lambda,M)$, $i = 1,2$ and

$$D_{\gamma,\lambda} = \big(\|\xi\|_{\lambda}\|x\|_{\gamma}\big)^{1/(\gamma+\lambda)} + \|x\|_{\gamma}^{1/\gamma} + \|x\|_{\gamma}^{(2\lambda+\gamma-1)/((\gamma+\lambda)(\gamma+\lambda-1))}.$$

Remarks 3.15

(1) Observe that if f is as in Proposition 3.5 and $\tilde{f} = \nabla\mathcal{U}^{(\mathbf{T})}$, then straightforward calculations show that Conditions (1) and (2) in the proposition are satisfied.
(2) The fact that $z_0 = 0$ implies that

$$\|z\|_{\infty,[0,T]} \le c_1 T^{\lambda} \|w\|_{\lambda,[0,T]} D_{\gamma,\lambda}^2 e^{c_2 D_{\gamma,\lambda}}.$$

(3) Let $\lambda = \gamma$. Then $(\gamma + 2\lambda - 1)/((\gamma+\lambda)(\gamma+\lambda-1))$ in Proposition 3.14 is smaller than 2 for $\gamma > H_0$, where $H_0 = (7+\sqrt{17})/16 \approx 0.6951$. This is the threshold above which our general delay equation will admit a smooth density.
(4) The unusual threshold H_0 above stems from the continuous dependence of the solution on its past, represented by the signed measure ν. In case of a discrete delay of the form $\sigma(y_t, y_{t-r_1}, \dots, y_{t-r_q})$, we shall see that all our considerations are valid for any $H > 1/2$.

Proof of Proposition 3.14 We first consider two generic positive numbers $k \in \mathbb{N}$ and ε, such that $(k+1)\varepsilon \le T$. Then Theorem 2.5, point (2), and Conditions (1) and (2) imply

$$\begin{aligned}
&\|z - w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}\\
&\quad\le \big\|\tilde{f}(y)z\big\|_{\infty,[k\varepsilon,(k+1)\varepsilon]} \|x\|_{\gamma} \varepsilon^{\gamma-\lambda} + c_{\gamma,\lambda} \big\|\tilde{f}(y)z\big\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} \|x\|_{\gamma} \varepsilon^{\gamma}\\
&\quad\le M\|z\|_{\infty,[0,(k+1)\varepsilon]} \|x\|_{\gamma} \varepsilon^{\gamma-\lambda}\\
&\qquad + c_{\gamma,\lambda} M \|x\|_{\gamma} \big(\|z\|_{\lambda,[0,(k+1)\varepsilon]} + \|z\|_{\infty,[0,(k+1)\varepsilon]} \|y\|_{\lambda,[0,T]}\big)\varepsilon^{\gamma}.
\end{aligned}$$

The following (arguably non-optimal) bound on $\|z\|_{\infty,[0,(k+1)\varepsilon]}$ can now be easily verified by induction:

$$\|z\|_{\infty,[0,(k+1)\varepsilon]} \le \sum_{i=1}^{k+1} 2^{k+1-i} \|z - z_{(i-1)\varepsilon}\|_{\infty,[(i-1)\varepsilon,i\varepsilon]} \le \sum_{i=1}^{k+1} 2^{k+1-i} \|z\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]}.$$

This yields

$$\begin{aligned}
&\|z - w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}\\
&\quad\le M\|x\|_{\gamma}\varepsilon^{\gamma} \left(\sum_{i=1}^{k+1} 2^{k+1-i} \|z - z_{(i-1)\varepsilon}\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]}\right)\\
&\qquad + c_{\gamma,\lambda} M \|x\|_{\gamma} \varepsilon^{\gamma} \big(\|z\|_{\lambda,[0,k\varepsilon]} + \|z\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}\big)\\
&\qquad + c_{\gamma,\lambda} M \|x\|_{\gamma} \|y\|_{\lambda,[0,T]} \varepsilon^{\gamma+\lambda} \left(\sum_{i=1}^{k+1} 2^{k+1-i} \|z - z_{(i-1)\varepsilon}\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]}\right).
\end{aligned} \tag{36}$$

Now the proof can be split in three steps.

Step 1. Bounds depending on ε. Let

$$\varepsilon = \Big(T + \big[6M\|x\|_{\gamma}(1 + c_{\gamma,\lambda})\big]^{1/\gamma} + \big[6M\|x\|_{\gamma} c_{\gamma,\lambda} \|y\|_{\lambda,[0,T]}\big]^{1/(\gamma+\lambda)}\Big)^{-1} \wedge T. \tag{37}$$

Note that in this case, inequality (36) yields

$$
\begin{aligned}
&\|z\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}\\
&\quad\le 2\|w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} + M\|x\|_{\gamma}\varepsilon^{\gamma}\left(\sum_{i=1}^{k} 2^{k+2-i}\|z\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]}\right)\\
&\qquad + c_{\gamma,\lambda}M\|x\|_{\gamma}\varepsilon^{\gamma}\left(2\|z\|_{\lambda,[0,k\varepsilon]} + \varepsilon^{\lambda}\|y\|_{\lambda,[0,T]}\sum_{i=1}^{k} 2^{k+2-i}\|z\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]}\right)\\
&\quad\le 2\|w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}\\
&\qquad + \sum_{i=1}^{k} 2^{k+2-i}\|z\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]}\big(M\|x\|_{\gamma}\varepsilon^{\gamma} + c_{\gamma,\lambda}M\|x\|_{\gamma}\varepsilon^{\gamma}\\
&\qquad + c_{\gamma,\lambda}M\|x\|_{\gamma}\varepsilon^{\gamma+\lambda}\|y\|_{\lambda,[0,T]}\big)\\
&\quad\le 2\|w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} + \sum_{i=1}^{k} 2^{k+1-i}\|z\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]},
\end{aligned}
\tag{38}
$$

where we have used (37) in the last step.

Step 2. Bounds for $\|z\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}$. Here we will use induction on k to show that

$$
\|z\|_{\lambda,[(i-1)\varepsilon,i\varepsilon]} \le \sum_{j=1}^{i} 2^{2i+1-2j}\|w\|_{\lambda,[(j-1)\varepsilon,j\varepsilon]}. \tag{39}
$$

By (38) we see that this inequality holds for $i=1$. Therefore we can assume that (39) holds for any positive integer $i\le k$ to show that it is also true for $i=k+1$.

The inequalities (38) and (39) lead us to write

$$
\begin{aligned}
&\|z\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]}\\
&\quad\le 2\|w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} + \sum_{i=1}^{k} 2^{k+1-i}\sum_{j=1}^{i} 2^{2i+1-2j}\|w\|_{\lambda,[(j-1)\varepsilon,j\varepsilon]}\\
&\quad\le 2\|w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} + \sum_{j=1}^{k}\|w\|_{\lambda,[(j-1)\varepsilon,j\varepsilon]}2^{k+2-2j}\sum_{i=1}^{k} 2^{i}\\
&\quad\le 2\|w\|_{\lambda,[k\varepsilon,(k+1)\varepsilon]} + \sum_{j=1}^{k}\|w\|_{\lambda,[(j-1)\varepsilon,j\varepsilon]}2^{2k+3-2j}.
\end{aligned}
$$

Now it is easy to see that (39) also holds for $i=k+1$.

Step 3. Final bound. Let k_0 such that $k_0\varepsilon < T < (k_0+1)\varepsilon$. Then, by Step 2 we have

$$\begin{aligned}
&\|z\|_{\lambda,[0,T]} \\
&\quad\le \|w\|_{\lambda,[0,T]} \sum_{k=1}^{k_0}\sum_{j=1}^{k} 2^{2k+1-2j} \\
&\quad\le \|w\|_{\lambda,[0,T]} (k_0)^2 2^{2k_0+1} \le \|w\|_{\lambda,[0,T]} (2T/\varepsilon)^2 2^{2T\varepsilon^{-1}+3}.
\end{aligned}$$

Thus the proof is finished by plugging relation (37) into the last expression, and invoking Proposition 3.4. □

The following result is a slight extension of Proposition 3.14, allowing to take into account the case of constant but non-vanishing functions.

Corollary 3.16 *Let $\tilde f$, $D_{\gamma,\lambda}$, w and y be as in Proposition 3.14. Furthermore, assume that $\tilde f$ is a mapping from $\mathcal{C}^{\lambda}_{\xi,0,T}(\mathbb{R}^n)$ into the linear operators from the constant functions on $[-h,T]$ into $\mathcal{C}^{\lambda}([0,T];\mathbb{R}^{n\times d})$ satisfying the Conditions (1) and (2) of Proposition 3.14 when $\tilde z$ is a constant function. Then the solution of the equation*

$$z_t = c + w_t + \int_0^t \big(\tilde f(y)z\big)(t)\,dx_t, \quad t\in[0,T],$$

satisfies the inequality

$$\|z\|_{\lambda,[0,T]} \le c_1 \left\| w + \int_0^{\cdot} \big(\tilde f(y)\tilde c\big)(t)\,dx_t \right\|_{\lambda,[0,T]} D^2_{\gamma,\lambda}\, e^{c_2 D_{\gamma,\lambda}},$$

where $\tilde c$ stands for the constant function $\tilde c_t \equiv c$.

Proof The proof is an immediate consequence of Proposition 3.14. Indeed, we only need to observe that

$$z_t - \tilde c_t = w_t + \int_0^t \big(\tilde f(y)\tilde c\big)(t)\,dx_t + \int_0^t \big(\tilde f(y)(z-\tilde c)\big)(t)\,dx_t, \quad t\in[0,T],$$

where $\tilde c(t) = c$, $t\in[0,T]$. □

4 Delay Equations Driven by a Fractional Brownian Motion

Here we consider the Young stochastic delay equation

$$\begin{aligned}
y_t &= \xi_0 + \int_0^t f\big(\mathcal{Z}_t^y\big)\,dB_t, \quad 0\le t\le T,\\
\mathcal{Z}_0^y &= \xi,
\end{aligned} \tag{40}$$

where $B=\{B_t; 0\le t\le T\}$ is a d-dimensional fractional Brownian motion (fBm) with parameter $H\in(1/2,1)$. The coefficient f satisfies Hypotheses 1–3 and ξ is a given deterministic function in $\mathcal{C}_1^{\gamma}([-h,0];\mathbb{R}^n)$, for some $\lambda<\gamma<H$. Remember that $\lambda\in(1/2,H)$ is introduced at the beginning of Sect. 3.

The fBm B is a centered Gaussian process with the covariance

$$R_H(t,s)\delta_{i,j}=E\big(B_s^iB_t^j\big)=\frac{1}{2}\delta_{i,j}\big(s^{2H}+t^{2H}-|t-s|^{2H}\big),$$

where $\delta_{i,j}$ represents the Dirac symbol. In particular, B has ν-Hölder-continuous paths for any exponent $\nu<H$. Consequently, from Theorem 3.2 and Hypotheses 1–3, equation (40) has a unique $\mathcal{C}^{\lambda}_{\xi,0,T}(\mathbb{R}^n)$-pathwise solution.

Here, our main goal is to analyze the existence of a smooth density of the solution of equation (40). This will be done via the Malliavin calculus or stochastic calculus of variations.

4.1 Preliminaries on Malliavin Calculus

In this subsection we introduce the framework and the results that we use in the remaining of this paper. Namely, we give some tools of the Malliavin calculus for fractional Brownian motion. Toward this end, we suppose that the reader is familiar with the basic facts of stochastic analysis for Gaussian processes as presented, for example, in Nualart [28].

Henceforth, we will consider the abstract Wiener space introduced in Nualart and Saussereau [31], in order to take advantage of the relation between the Fréchet derivatives of the solution to equation (40) (see Proposition 3.10) and its derivatives in the Malliavin calculus sense (see [28], Proposition 4.1.3). This abstract Wiener space is constructed as follows (for a more detailed exposition of it, the reader can consult [31]).

We assume that the underlying probability space $(\Omega,\mathcal{F},P)$ is such that Ω is the Banach space of all the continuous functions $C_0([0,T];\mathbb{R}^d)$, which are zero at time 0, endowed with the supremum norm. P is the only probability measure such that the canonical process $\{B_t; 0\le t\le T\}$ is a d-dimensional fBm with parameter $H\in(1/2,1)$ and the σ-algebra $\mathcal{F}$ is the completion of the Borel σ-algebra of Ω with respect to P.

Two important tools related to the fBm B are the completion $\mathcal{H}$ of the $\mathbb{R}^d$-valued step functions $\mathcal{E}$ with respect to the inner product $\langle(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]}),(\mathbf{1}_{[0,s_1]},\dots,\mathbf{1}_{[0,s_d]})\rangle=\sum_{i=1}^d R_H(s_i,t_i)$ (see [33]) and the isometry $K_H^*:\mathcal{H}\to L^2([0,T]^d)$, which satisfies

$$K_H^*(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]})=\big(\mathbf{1}_{[0,t_1]}(\cdot)K_H(t_1,\cdot),\dots,\mathbf{1}_{[0,t_d]}K_H(t_d,\cdot)\big),$$

where $K_H(t,s)=c_Hs^{1/2-H}\int_s^t(u-s)^{H-3/2}u^{H-1/2}\,du$ is a kernel verifying

$$R_H(t,s)=\int_0^{t\wedge s}K_H(t,r)K_H(s,r)\,dr.$$

It should be noticed at this point that K_H^* can be represented in the two following ways:

$$\big[K_H^*\varphi\big]_t = \int_t^T \varphi_r\,\partial_r K(r,t)\,dr = c_H t^{1/2-H}\big[I_{T^-}^{H-1/2}\big(u^{H-1/2}\varphi_u\big)\big]_t, \tag{41}$$

where I_{T-}^{α} stands for the fractional integration of order α on $[0,T]$ (see [29] for further details). Furthermore, by [1], the inner product in $\mathcal{H}$ can be written as

$$\langle \varphi, \psi\rangle_{\mathcal{H}} = c_H \int_0^T \int_0^T \varphi_u |u-v|^{2H-2}\psi_v\,du\,dv.$$

The isometry K_H^* allows us to introduce the version of the Reproducing Kernel Hilbert space $\mathcal{H}_H$ associated with the process B. Namely, Let $\mathcal{K}_H$ be given by

$$\begin{aligned}
&\mathcal{K}_H : L^2\big([0,T];\mathbb{R}^d\big) \to \mathcal{H}_H := \mathcal{K}_H\big(L^2\big([0,T];\mathbb{R}^d\big)\big),\\
&(\mathcal{K}_H h)(t) = \int_0^t K_H(t,s)h(s)\,ds.
\end{aligned}$$

The space $\mathcal{H}$ is continuously and densely embedded in Ω. Indeed, it is not difficult to see that the operator $\mathcal{R}_H : \mathcal{H} \to \mathcal{H}_H$ defined by

$$\mathcal{R}_H \phi = \int_0^{\cdot} K_H(\cdot,s)\big(K_H^*\phi\big)(s)\,ds$$

embeds $\mathcal{H}$ continuously and densely into Ω, because, as was pointed out in [31], $\mathcal{R}_H(\phi)$ is H-Hölder continuous. Thus, we see that $(\Omega, \mathcal{H}, P)$ is an abstract Wiener space.

Now we introduce the derivative in the Malliavin calculus sense of a random variable. We say that a random variable F is a smooth functional if it has the form

$$F = f\big(B(h_1), \dots, B(h_n)\big),$$

where $h_1, \dots, h_n \in \mathcal{H}$ and f and all its partial derivatives have polynomial growth. In the remainder of this paper, $\mathcal{S}$ denotes the family of smooth functionals. The derivative of this smooth functional is the $\mathcal{H}$-valued random variable given by

$$\mathcal{D}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(B(h_1), \dots, B(h_n)\big)h_i.$$

For $p > 1$, the operator $\mathcal{D}$ is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ (see [28]). The closure of this operator is also denoted by $\mathcal{D}$ and its domain by $\mathbb{D}^{1,p}$, which is the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p = E\big(|F|^p\big) + E\big(\|\mathcal{D}F\|_{\mathcal{H}}^p\big).$$

The operator $\mathcal{D}$ has the local property (i.e., $\mathcal{D}F = 0$ on $A \subset \Omega$ if $\mathbf{1}_A F = 0$). This allows us to extend the domain of the operator $\mathcal{D}$ as follows. We say that $F \in \mathbb{D}_{\text{loc}}^{1,p}$

if there is a sequence $\{(\Omega_n, F_n), n \geq 1\} \subset \mathcal{F} \times \mathbb{D}^{1,p}$ such that $\Omega_n \uparrow \Omega$ w.p.1 and $F = F_n$ on Ω_n. In this case, we define $\mathcal{D}F = \mathcal{D}F_n$ on Ω_n.

It is known that, in the abstract Wiener space $(\Omega, \mathcal{H}, P)$, we can consider the differentiability of random variable F in the directions of $\mathcal{H}$. That is, we say that F is $\mathcal{H}$-differentiable if for almost all $\omega \in \Omega$ and $h \in \mathcal{H}$, the map $\varepsilon \mapsto F(\omega + \varepsilon \mathcal{R}_H h)$ is differentiable. The following result due to Kusuoka [20] (see also [28], Proposition 4.1.3) will be fundamental in the study of the existence of smooth densities of the solution of equation (40).

Proposition 4.1 *Let F be an $\mathcal{H}$-differentiable random variable. Then F belongs to the space $\mathbb{D}^{1,p}_{\mathrm{loc}}$, for any $p > 1$.*

We will apply this result to the solution of equation (40) as follows. Note that for $\varphi \in \mathcal{H}$, we have the inequality

$$\left|(\mathcal{R}_H\varphi)^i(t) - (\mathcal{R}_H\varphi)^i(s)\right| = \left(E\left[\left|B_t^i - B_s^i\right|^2\right]\right)^{1/2} \|\varphi\|_{\mathcal{H}} \leq \|\varphi\|_{\mathcal{H}} |t-s|^H.$$

Consequently, Proposition 3.10 (see also Lemma 4.2 below) implies that the random variable y_t defined in equation (40) is also $\mathcal{H}$-differentiable, which, together with Proposition 4.1, yields that y_t^i belongs to $\mathbb{D}^{1,p}_{\mathrm{loc}}$ for every $t \in [0,T]$, $p > 1$ and $i \in \{1, \ldots, n\}$. Moreover, the relation between the $\mathcal{H}$-derivative and $\mathcal{D}$ is given by (see also Lemma 4.3),

$$\left\langle \mathcal{D}y_t^i, h\right\rangle_{\mathcal{H}} = \frac{d}{d\varepsilon} y_t^i(\omega + \varepsilon \mathcal{R}_H h)|_{\varepsilon=0}, \quad h \in \mathcal{H}. \tag{42}$$

More generally, if $\omega \mapsto X(\omega)$ is infinitely Fréchet differentiable in the directions of $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R})$, then

$$\begin{aligned}
&\left\langle \mathcal{D}^n X, h_1 \otimes \cdots \otimes h_n\right\rangle_{\mathcal{H}^n} \\
&\quad = D_{\mathcal{R}_H h_1, \ldots, \mathcal{R}_H h_n} X = \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_n} X(\omega + \varepsilon_1 \mathcal{R}_{h_1} + \cdots + \varepsilon_n \mathcal{R}_{h_n})|_{\varepsilon_1 = \cdots = \varepsilon_n = 0}.
\end{aligned}$$

4.2 Existence of the Density of the Solution

In this section we establish that, for each $t \in [0,T]$, the random variable y_t introduced in equation (40) has a density.

Let us start with two important technical tools. The first one relates the derivative of the vector-valued quantity y_t with the derivative of y as a function.

Lemma 4.2 *Let y be the solution of (40) and $t \in [0,T]$. Then almost surely, $h \mapsto y_t(B+h)$ is Fréchet differentiable from $\mathcal{C}^{\lambda}_{0,0,T}(\mathbb{R}^d)$ into $\mathbb{R}^n$. Furthermore*

$$Dy_t(B)(h) = \left[Dy(B)(h)\right]_t.$$

Proof The proof is an immediate consequence of

$$\begin{aligned}
&\left|y_t(x+h) - y_t(x) - \left(Dy(x)(h)\right)(t)\right| \\
&\quad = \left|y_t(x+h) - y_t(x) - \left(Dy(x)(h)\right)(t)\right.
\end{aligned}$$

$$
\begin{aligned}
&- y_0(x+h) - y_0(x) - \big(Dy(x)(h)\big)(0)\big| \\
&\le \big\|y(x+h) - y(x) - Dy(x)(h)\big\|_\lambda t^\lambda,
\end{aligned}
$$

with $x, h \in \mathcal{C}^\lambda_{0,0,T}(\mathbb{R}^d)$. □

Lemma 4.3 *Let y be the solution of (40). Then y^i_t belongs to $\mathbb{D}^{1,2}_{\mathrm{loc}}$ for every $t \in [0, T]$ and $i \in \{1, \ldots, n\}$. Moreover, for $h \in \mathcal{H}$, we have*

$$
\langle \mathcal{D}y^i_t, h\rangle_{\mathcal{H}} = \big[Dy^i(B)(\mathcal{R}_H h)\big]_t. \tag{43}
$$

Proof By Proposition 4.1 and Lemma 4.2, we have already shown that y^i_t is in $\mathbb{D}^{1,2}_{\mathrm{loc}}$ for every $t \in [0, T]$ and $i \in \{1, \ldots, n\}$.

Furthermore, from (42) and Lemma 4.2, we have

$$
\langle \mathcal{D}y^i_t, h\rangle_{\mathcal{H}} = D_{\mathcal{R}_H h} y^i_t = Dy^i_t(B)(\mathcal{R}_H h) = \big(Dy^i(B)(\mathcal{R}_H h)\big)(t).
$$

Thus, the proof is complete. □

We now use the ideas of Nualart and Saussereau [31] to state one of the main results of this section:

Theorem 4.4 *Let us assume that Hypotheses 1–3 hold, recall that ξ is the (functional) initial condition of equation (40), and assume that the space spanned by $\{(f(\xi)^{1j}, \ldots, f(\xi)^{nj});\ 1 \le j \le d\}$ is $\mathbb{R}^n$. Then for $t \in (0, T]$, the random variable y_t given by (40) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.*

Proof By Lemma 4.3, we see that y^i_t belongs to $\mathbb{D}^{1,2}_{\mathrm{loc}}$. Therefore we only need to see that the Malliavin covariance matrix

$$
Q^{ij}_t := \langle \mathcal{D}y^i_t, \mathcal{D}y^j_t\rangle_{\mathcal{H}} \tag{44}
$$

is invertible almost surely.

For $v \in \mathbb{R}^n$, following [31] (proof of Theorem 8), we have

$$
v^T Q_t v = \sum_{m=1}^{\infty} \big|\langle Dy(B)(\mathcal{R}_H h_m)(t), v\rangle_{\mathbb{R}^n}\big|^2,
$$

where $\{h_m, m \ge 1\}$ is a complete orthonormal system of $\mathcal{H}$.

Now assume that the Malliavin matrix Q_t is not almost surely invertible. Then, on the set of strictly positive probability where Q_t is not invertible, there exists $v_0 \in \mathbb{R}^n$, $v_0 \ne 0$ such that $v_0^T Q_t v_0 = 0$. Moreover, recalling our notation (28), it is clear from equation (31) that $D_2F(k, Z)$ is a linear homomorphism. Hence, we obtain that

$$
\begin{aligned}
0 &= \big\langle D_1 F\big(0, y\big(B - \tilde{\xi}\big)\big)(\mathcal{R}_H h_m)(t), v_0\big\rangle_{\mathbb{R}^n} \\
&= -\left\langle \int_0^t \mathcal{U}^{(\mathbf{T})}\big(y(B)\big)_s \, d\mathcal{R}_H h_m(s), v_0\right\rangle_{\mathbb{R}^n}
\end{aligned}
$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{d} v_0^i \int_0^t \big(\mathcal{U}^{(\mathbf{T})}\big(y(B)\big)\big)_s^{ij}\, d\mathcal{R}_H h_m^j(s)$$

$$= -\sum_{i=1}^{n}\big\langle v_0^i \big(\mathcal{U}^{(\mathbf{T})}\big(y(B)\big)\big)^i \mathbf{1}_{[0,t]}, h_m\big\rangle_{\mathcal{H}}, \quad \text{for all } m \geq 0,$$

where the last equality follows from [31]. For $t > 0$, taking into account the definition of $\mathcal{U}^{(\mathbf{T})}$ given in Lemma 3.1, we obtain that $\sum_{i=1}^{n} v_0^i f^{ij}(\xi) = 0$, which contradicts the fact that $\mathbb{R}^n$ coincides with the space spanned by

$$\big\{\big(f(\xi)^{1j}, \ldots, f(\xi)^{nj}\big);\ 1 \leq j \leq d\big\}.$$

So we see that the Malliavin matrix Q_t is invertible for any $t \in (0, T]$, as we wished to prove. □

4.3 Smoothness of the Density of the Solution

In order to avoid lengthy lists of hypotheses on our coefficients, we focus in this section on the example of the weighted delay treated in Sect. 3.3. As usual in the stochastic analysis context, we study the smoothness of the density of the random variable under consideration by bounding the L^{-p} moments of its Malliavin matrix. Toward this aim, it will be useful to produce an equation solved by the Malliavin derivative of the solution y_t of equation (40). This is contained in the following lemma:

Lemma 4.5 *Under the conditions of Proposition* 3.12, *let y be the solution to equation* (40). *Assume furthermore that B is a fBm with Hurst parameter $H > H_0$, where H_0 is defined in Remark* 3.15. *Then $y_t \in \mathbb{D}^{1,p}$ for any $p \geq 1$, and $\Phi_t(r) := \mathcal{D}_r y_t$ is the unique solution to the following equation*:

$$\Phi_t(r) = \big[\mathcal{U}^{(\mathbf{T})}(y)\big]_t + V_t(r),$$

$$\text{where } V_t^{ij}(r) = \sum_{m=1}^{n}\sum_{l=1}^{d}\int_r^t \Big(\big(\big[\nabla\mathcal{U}^{(\mathbf{T})}(y)\big]^m\big)^{il}\big(\Phi^{mj}(s)\big)\Big)_s\, dB_s^l, \tag{45}$$

with the additional constraint $\Phi_t(r) = 0$ for all $0 \leq t < r \leq T$.

Proof The fact that $\mathcal{D}y$ follows equation (45) is a direct consequence of relation (43) and Proposition 3.12. The fact that $y_t \in \mathbb{D}^{1,p}$ when $H > H_0$ stems now from Proposition 3.14. □

Now we are able to state the second main result of this section, for which we need an additional notation: for two symmetric non-negative matrices $M, N \in \mathbb{R}^{n\times n}$, we write $M \geq N$ when the matrix $M - N$ is symmetric non-negative.

Theorem 4.6 *Let f, σ, ν and B as in Lemma 4.5. Assume that σ has bounded derivatives of any order and that*

$$\sigma(\eta)\sigma(\eta)^* \geq \varepsilon Id_{\mathbb{R}^n}, \quad \textit{for all } \eta \in \mathbb{R}^n. \tag{46}$$

Then, for $t \in (0, T]$, y_t has a $\mathcal{C}^\infty$-density.

Proof The proof follows closely the lines of [21, Theorem 3.5], which is classical in the Malliavin calculus setting, and we shall thus proceed without giving too many details. Nevertheless, we shall divide our proof in two steps.

Step 1: *Reduction of the problem.* Let Q_t be the Malliavin matrix of y_t, defined by (44). The standard conditions to verify in order to get a $\mathcal{C}^\infty$ density are: (i) $y_t \in \mathbb{D}^\infty$, and (ii) $[\det(Q_t)]^{-1} \in L^p$ for all $p \geq 1$. Condition (i) is obtained by iterating the derivatives of y, similarly to what is done in [31], so that we will focus on point (ii).

In order to check that $[\det(Q_t)]^{-1} \in L^p$ for all $p \geq 1$, let us recall from [28, Lemma 2.3.1] that it is enough to show that for all $p \geq 1$ there exists $\varepsilon_0 := \varepsilon_0(p)$ such that for all $\varepsilon \leq \varepsilon_0$ we have

$$\sup_{|\alpha|=1} P\big(\alpha^* Q_t \alpha \leq \varepsilon\big) \leq \varepsilon^p,$$

where α stands for a generic vector of $\mathbb{R}^n$.

To this end, recalling our notation (32), notice that

$$\alpha^* Q_t \alpha = \sum_{j=1}^{d} \sum_{p,q=1}^{n} \alpha_p \left(\int_0^t \int_0^t \Phi_t^{pj}(u) |u-v|^{2H-2} \Phi_t^{qj}(v)\, du\, dv \right) \alpha_q.$$

Moreover, invoking the positivity of the kernel $|u-v|^{2H-2}$, it is readily checked that

$$\sum_{p,q=1}^{n} \alpha_p \left(\int_{t-\rho}^t \int_{t-\rho}^t \Phi_t^{pj}(u) |u-v|^{2H-2} \Phi_t^{qj}(v)\, du\, dv \right) \alpha_q \geq 0,$$

for any $0 < \rho < t$ and $j \leq d$. Set then $A_\rho = [t-\rho, t] \times [0, t] \cup [0, t] \times [t-\rho, t]$ and

$$\langle \varphi, \psi \rangle_{\mathcal{H}_\rho} = c_H \int_{A_\rho} \varphi_u |u-v|^{2H-2} \varphi_v\, du\, dv.$$

It should be noticed at this point that $\langle \varphi, \psi \rangle_{\mathcal{H}_\rho}$ cannot be considered as an inner product for its lack of positivity. However, the previous considerations show that

$$\alpha^* Q_t \alpha \geq \alpha^* Q_{t,\rho} \alpha, \quad \text{where } Q_{t,\rho}^{pq} = \sum_{j=1}^{d} \big\langle \Phi_t^{pj}, \Phi_t^{qj} \big\rangle_{\mathcal{H}_\rho}.$$

The remainder of the proof will now consist in bounding from below the quantity $\alpha^* Q_{t,\rho} \alpha$. Specifically, our problem will be reduced to show that

$$\sup_{|\alpha|=1} P\big(\alpha^* Q_{t,\rho} \alpha \leq \eta\big) \leq \eta^p, \tag{47}$$

for any $p \geq 1$ and $\eta \leq \eta_0(p)$.

Step 2: *Bounds for* $Q_{t,\rho}$. Going back to our notation (45), let us decompose $Q^{pq}_{t,\rho}$ as $Q^{pq}_{t,\rho} = J^{pq}_{\rho,1} + 2J^{pq}_{\rho,2} + J^{pq}_{\rho,3}$, with

$$J^{pq}_{\rho,1} = \langle [\mathcal{U}^{(\mathbf{T})}(y)]^{pj}_t, [\mathcal{U}^{(\mathbf{T})}(y)]^{qj}_t \rangle_{\mathcal{H}_\rho}, \qquad J^{pq}_{\rho,2} = \langle [\mathcal{U}^{(\mathbf{T})}(y)]^{pj}_t, V^{qj}_t \rangle_{\mathcal{H}_\rho},$$
$$J^{pq}_{\rho,3} = \langle V^{qj}_t, V^{qj}_t \rangle_{\mathcal{H}_\rho},$$

where we have used the summation convention over the repeated index j. In order to bound $Q_{t,\rho}$ from below, the basic idea is now to rely on the fact that the term $[\mathcal{U}^{(\mathbf{T})}(y)]_t$ is bounded deterministically from below under the non-degeneracy condition (46), while V is a highly fluctuating quantity, since it is given by a stochastic integral with respect to B.

Let us thus bound $J_{\rho,1}$ from below: observe that

$$J_{\rho,1} = \sigma\sigma^*\left(\int_{-h}^0 y_{t+\theta}\,\nu(d\theta)\right)\int_{A_\rho} |u-v|^{2H-2}\,du\,dv.$$

Besides, we have assumed the elliptic condition (46), and it is easily shown that $\int_{A_\rho} |u-v|^{2H-2}\,du\,dv \ge c\rho^H$ for a certain positive constant c. Hence, in the matrix sense, the following deterministic bound holds true:

$$J_{\rho,1} \ge c\varepsilon\rho^H Id_{\mathbb{R}^n}.$$

As far as $J_{\rho,3}$ is concerned, recall that $V^{pj}_t(r)$ is given by

$$V^{pj}_t(r) = \sum_{m=1}^n \sum_{l=1}^d \int_r^t \left[\left(\left[\nabla\mathcal{U}^{(\mathbf{T})}(y)\right]^m\right)^{pl}\left(\Phi^{mj}(s)\right)\right]_s dB^l_s.$$

In particular, Theorem 2.5 yields, for $1/2 < \lambda < \gamma < H$,

$$\begin{aligned}\left|V^{pj}_t(r)\right| &\le c\sum_{m=1}^n \sum_{l=1}^d \left|\left(\left[\nabla\mathcal{U}^{(\mathbf{T})}(y)\right]^m\right)^{pl}\left(\Phi^{mj}(s)\right)\right|_{\lambda,[r,t]}\left|B^l\right|_{\gamma,[0,t]}|t-r|^\gamma \\ &\le cX^{(t)}\rho^\gamma,\end{aligned} \tag{48}$$

where we have set $X^{(t)} = |([\nabla\mathcal{U}^{(\mathbf{T})}(y)]^m)^{pl}(\Phi^{mj}(s))|_{\lambda,[r,t]}\,|B^l|_{\gamma,[0,t]}$. Owing to Lemma 4.5, and since we have assumed $H > H_0$, $X^{(t)}$ is a L^p random variable for any $p \ge 1$. Therefore, plugging the estimate (48) into the definition of $J_{\rho,3}$, we obtain for any $\alpha \in \mathbb{R}^n$ satisfying $|\alpha| = 1$ and any $\eta > 0$:

$$\begin{aligned} &E\left[\left|\sum_{p,q=1}^n \alpha_p J^{pq}_{\rho,3}\alpha_q\right|^p\right] \le c_p\rho^{(2\gamma+H)p} \\ &\quad\Longrightarrow\quad P\left(\left|\sum_{p,q=1}^n \alpha_p J^{pq}_{\rho,3}\alpha_q\right| \ge \eta\right) \le \frac{c_p\rho^{(2\gamma+H)p}}{\eta^p}.\end{aligned} \tag{49}$$

Notice that the same kind of bound is available for $J_{\rho,2}$, except that $\frac{c_p \rho^{(\gamma+H)p}}{\eta^p}$ is obtained in the right hand side of equation (49).

Step 3: *Conclusion.* We are now ready to prove our claim (49). Indeed, for $\eta > 0$ take $\varepsilon\rho^H = 2\eta$, i.e. $\rho = (2\eta/\varepsilon)^{1/H}$. Appealing to the decomposition of $Q_{t,\rho} = J_{\rho,1} + 2J_{\rho,2} + J_{\rho,3}$ we obtain, for $|\alpha| = 1$ and $\hat{p} \geq 1$,

$$P\big(\alpha^* Q_{t,\rho}\alpha \leq \eta\big) \leq P\big(\alpha^*\big(2J_{\rho,2} + J_{\rho,3}\big)\alpha \geq \eta\big) \leq \frac{c_{\hat{p}}\rho^{(\gamma+H)\hat{p}}}{\eta^{\hat{p}}} = c_{\hat{p},\varepsilon}\eta^{(\gamma+H-1)\hat{p}}.$$

Since we have chosen γ such that $\gamma + H > 1$, it is now sufficient to pick $\hat{p}$ satisfying $(\gamma + H - 1)\hat{p} > p$ in order to prove (47). □

Remark 4.7 As mentioned before, the restriction $H > H_0$ for the smoothness of the density of the random variable y_t is due to the continuous dependence of our coefficient f on the past of the solution. Indeed, in case of a discrete delayed coefficient of the form $\sigma(y_t, y_{t-r_1}, \ldots, y_{t-r_q})$, with $q \geq 1$ and $r_1 < \cdots < r_q \leq h$, it can be seen that equation (40) can be reduced to an ordinary differential equation driven by B. This allows us to apply the criteria given in [18], which are valid up to $H = 1/2$.

In order to get convinced of this fact, consider the simplest discrete delay case, which is an equation of the form

$$y_t = \xi_0 + \int_0^t \sigma(y_u, y_{u-r})\, dB_u, \quad 0 \leq t \leq T, \tag{50}$$

with $r > 0$. The initial condition of this process is given by $\xi \in \mathcal{C}_1^\gamma$ on $[-r, 0]$, and we also assume that σ and B are real valued. Without loss of generality, one can assume that $T = mr$ for $m \in \mathbb{N}^*$. In this case, set $y(k) = \{y_{s+kr};\ s \in [0, r)\}$, and adopt the same notation for B. Then one can recast (50) as

$$y_t(k) = y_r(k-1) + \int_0^t \sigma\big(y_u(k), y_u(k-1)\big)\, dB_u(k), \quad t \in [0, r],\ k \leq m-1. \tag{51}$$

Setting now $\mathbf{y} = (y(1), \ldots, y(m))^t$, $\mathbf{B} = (B(1), \ldots, B(k))^t$ and defining $\hat{\sigma} : \mathbb{R}^m \to \mathbb{R}^{m,m}$ by

$$\hat{\sigma}\big(\eta(1), \ldots, \eta(m)\big) = \mathrm{Diag}\big(\sigma\big(\eta(1)\big), \ldots, \sigma\big(\eta(m)\big)\big),$$

we can express (51) in a matrix form as

$$\mathbf{y}_t = \mathbf{y}_0 + \int_0^t \hat{\sigma}\big(\mathbf{y}_u(1), \ldots, \mathbf{y}_u(m)\big)\, d\mathbf{B}_u,, \quad t \in [0, r]. \tag{52}$$

This is now an ordinary equation driven by a m-dimensional fBm $\mathbf{B}$. Whenever $|\sigma(\eta)| \geq \varepsilon > 0$ and $H > 1/2$, one can apply the non-degeneracy criterion of [18] in order to see that y_t possesses a smooth density for any $t \in (0, T]$. The case of a vector-valued original equation (50) can also be handled through cumbersome matrix notations. As far as the case of a coefficient $\sigma(y_t, y_{t-r_1}, \ldots, y_{t-r_q})$ is concerned, it can also be reduced to an equation of the form (52) by introducing all the quantities

$$y_t(k_1, k_2, \ldots, k_r) = y_{t+\sum_{j=1}^r k_j(r_j - r_{j-1})},$$

where we have used the convention $r_0 = 0$.

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