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ON THE NECESSARY AND SUFFICIENT CONDITIONS TO SOLVE A HEAT EQUATION WITH GENERAL ADDITIVE GAUSSIAN NOISE*

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Abstract In this note, we consider stochastic heat equation with general additive Gaussian noise. Our aim is to derive some necessary and sufficient conditions on the Gaussian noise in order to solve the corresponding heat equation. We investigate this problem invoking two different methods, respectively, based on variance computations and on path-wise considerations in Besov spaces. We are going to see that, as anticipated, both approaches lead to the same necessary and sufficient condition on the noise. In addition, the path-wise approach brings out regularity results for the solution.

Key words Stochastic heat equation; general Gaussian noise; L^2 solution; sufficient and necessary condition; Wong-Zakai approximation; pathwise solution; Hölder continuity; Besov space

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1 Introduction

In this article, we are concerned with the following stochastic heat equation with additive noise:

$$\begin{cases} \partial_t u = \dfrac{1}{2}\Delta u + \dot{W}, \qquad t \in [0,\tau],\, x \in \mathbb{R}^d, \\ u(0,x) = 0, \end{cases} \tag{1.1}$$

where W is a general centered Gaussian field with time covariance R and spatial spectral measure μ (see Definition 2.1 for further details), and where $\dot{W} = \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$. In the recent years, there has been an active line of research aiming at a complete definition of stochastic heat equations driven by rough space-time noises. Among the numerous contributions in rough environments, let us highlight the following ones:

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(i) The first efforts in this direction concern the definition of equation (1.1) driven by a Brownian motion W in time. In this context, the articles [6, 20] give (among other results) optimal conditions on the space covariance of W so that the solution to (1.1) is function-valued.

(ii) A lot of efforts have been devoted recently to the study of multiplicative stochastic heat equations driven by fractional noises in both space and time. A particular emphasis has been made on the effects of the noise on scaling exponents and asymptotic behavior of the solution. Among the abundant literature on this topic, let us mention the references [3, 8, 13].

(iii) The acclaimed theory of regularity structures was introduced (see [11, 12]) in order to solve highly nonlinear systems in rough environments, which require renormalization techniques. Interestingly enough, this method is applied in [7] to a family of stochastic heat equations similar to (1.1).

This article can be seen as another step towards a better understanding of heat equations in rough environments. Namely, our aim is to find optimal conditions in both space and time such that equation (1.1) admits a function (versus distribution)-valued solution. Otherwise stated, we wish to give necessary and sufficient conditions so that equation (1.1) can be defined without renormalization. Specifically, we will prove the following result (see Theorem 3.10 for a more precise statement):

Theorem 1.1 Let W be a centered Gaussian field with time covariance R and spatial spectral measure μ. Suppose that R is a continuous function such that

$$|t-(s\vee s')|^{\beta}\le |R(t,t)+R(s,s')-R(s,t)-R(t,s')|\le |t-(s\wedge s')|^{\beta},\quad 0\le s,s'<t \tag{1.2}$$

for some $\beta\in(0,2]$. Then, equation (1.1) admits a random field solution $\{u(t,x);\, t\ge 0, x\in\mathbb{R}^d\}$, if and only if the following conditions is satisfied:

$$\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^{2\beta}}\mu(\mathrm{d}\xi)<\infty. \tag{1.3}$$

As the reader might see, conditions (1.2) and (1.3) are simple enough, while giving an if and only if condition of existence.

It is well-known that there are essentially two possible ways to consider equation (1.1): the stochastic method based on Wiener integrals and the path-wise method relying on Young type integration. Interestingly, we have been able to prove Theorem 1.1 resorting to both methods. Let us briefly explain our approaches:

(1) The stochastic method is based on the variance computations for Wiener integrals, involving the heat kernel and the covariance function of W. The additional ingredients in our proof with respect to previous works (see for example [2, 23]) are some subtle partial integration by parts, which are at the heart of the proof of Theorem 1.1.

(2) The path-wise method resorts to the action of the heat semigroup on Besov spaces. Once preliminary notions on harmonic analysis are given, it yields quite a simple solution to our problem and also brings some information about the regularity of the solution considered

as a Hölder continuous function for free (while the computations are more costly in the random field framework, as one can see from [23]). However, we should stress the fact that the conditions obtained in this framework are only sufficient and slightly non optimal (namely condition (1.3) in Theorem 1.1 is replaced by $\int_{\mathbb{R}^d}(1+|\xi|^{2\beta-\epsilon})^{-1}\mu(\mathrm{d}\xi)<\infty$ for some $\epsilon>0$).

Let us mention that we have been focusing on the additive Gaussian case in this article in order to investigate the limits of our methods on a simple enough case. An interesting while more demanding problem would be to handle the multiplicative case. This will be dealt with in some subsequent papers. Note that in the multiplicative case, a nonzero initial condition will be needed in order to get a non-trivial unique solution. On the other hand, for equation (1.1) with a nonzero initial condition $u(0,x)=g(x)$, an additional quantity $\int_{\mathbb{R}} p_t(x-y)g(y)\mathrm{d}y$ needs to be investigated (see for example [1] for classical results concerning the action of the heat semigroup on Besov spaces).

Here is the organization of this article. In the next section, we set up some preliminary material on the Gaussian noises we are dealing with. In Section 3, we solve equation (1.1) by the stochastic method we have mentioned above. In Section 4, we focus on path-wise techniques.

Notations In the remainder of the article, all generic constants will be denoted by K, and their value may vary from different occurrences. We denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x)=(2\pi t)^{-d/2}e^{-|x|^2/2t}$, for any $t>0$, $x\in\mathbb{R}^d$. $\mathbb{N}$ stands for the set of natural numbers: $\{0,1,2,\dots\}$. Throughout this article, Id denotes the identity function.

2 Noise Model

Let us start by introducing some basic notions on Fourier transforms of functions: the space of real-valued infinitely differentiable functions with compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ or $\mathcal{S}$. Its dual, the space of tempered distributions, is $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{S}'$. If u is a vector of tempered distributions from $\mathbb{R}^d$ to $\mathbb{R}^n$, then we write $u\in\mathcal{S}'(\mathbb{R}^d,\mathbb{R}^n)$. The Fourier transform is defined with the normalization

$$\mathcal{F}u(\xi)=\int_{\mathbb{R}^d} e^{-\imath\langle\xi,x\rangle}u(x)\mathrm{d}x,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\xi)=(2\pi)^{-d}\mathcal{F}u(-\xi)$.

Let μ be a non-negative measure on $\mathbb{R}^d$. The spatial covariance of our noise will be determined by a Hilbert space called $\mathcal{H}$, defined as the completion of $\mathcal{S}(\mathbb{R}^d)$ under the inner product:

$$\langle\varphi,\psi\rangle_{\mathcal{H}}:=\int_{\mathbb{R}^d}\mathcal{F}(\varphi)(\xi)\overline{\mathcal{F}(\psi)(\xi)}\mu(\mathrm{d}\xi). \tag{2.1}$$

As far as the time covariance of our noise is concerned, we shall consider a continuous positive definite function R on $\mathbb{R}_+^2$. For convenience, the rectangular increments of R will be denoted, for $s<t$ and $u<v$, by

$$R\begin{pmatrix} s & t\\ u & v\end{pmatrix}=R(v,t)-R(v,s)-R(u,t)+R(u,s). \tag{2.2}$$

We also denote by $\mathcal{E}(\mathbb{R}_+)$ the space of simple functions on $\mathbb{R}_+$. With those preliminary notations in hand, our noise is defined as follows:

Definition 2.1 On a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$, we consider a Gaussian noise W encoded by a centered Gaussian family $\{W(h);\, h \in \mathcal{E}(\mathbb{R}_+) \times \mathcal{S}(\mathbb{R}^d)\}$, whose covariance structure is given as follows: consider $g = \mathbf{1}_{[s,t]} \otimes \varphi$ and $h = \mathbf{1}_{[u,v]} \otimes \psi$ with $s < t$, $u < v$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Then, we have

$$\mathbb{E}(W(g)W(h)) = R\begin{pmatrix} s & t \\ u & v \end{pmatrix} \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(\xi)\overline{\mathcal{F}(\psi)(\xi)}\mu(\mathrm{d}\xi), \tag{2.3}$$

where $\mathcal{F}\varphi$ refers to the Fourier transform with respect to the space variable only.

For $g = \mathbf{1}_{[s,t]} \otimes \varphi$, where $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $s < t$, we can define the stochastic integral

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} g(r, y) W(\mathrm{d}y, \mathrm{d}r) = W(g)\,. \tag{2.4}$$

By (2.1), this definition can be easily extended to above g with $\phi \in \mathcal{H}$.

In this article, because of the singularities of the heat kernel and of our covariance function, we will define our Wiener integrals via regularization. This is the contents of the definition below.

Definition 2.2 Let g be a measurable function on $\mathbb{R}_+ \times \mathbb{R}^d$ such that $g(s, \cdot) \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space defined by (2.1). For $\varepsilon > 0$, we set $t_k = t_k^\varepsilon = k\varepsilon$, $k \in \mathbb{N}$ and also $t_\varepsilon = t - \sqrt[3]{\varepsilon}$. We define

$$\int_0^t \int_{\mathbb{R}^d} g(s, y) W(\mathrm{d}y, \mathrm{d}s) = \lim_{\varepsilon \to 0+} \sum_{t_k : 0 \le t_k < t_\varepsilon} \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} g(t_k, y) W(\mathrm{d}y, \mathrm{d}s), \tag{2.5}$$

whenever the $L^2(\Omega)$-limit of the right-hand side exists. Notice that the right-hand side of (2.5) is understood because of (2.4).

Remark 2.3 In (2.5) of Definition 2.2, we have taken t_k's from $[0, t_\varepsilon)$ in order to avoid some technical computation issues while $t_k - t$ is small. More specifically, in Lemma 3.3, the most singular term to be estimated is (3.19). It will be analyzed through a discretization procedure in Lemma 3.7. In particular, the easiest way we have found to kill the term $|\xi|^6$ in (3.35) has been to integrate in time up to $t - \varepsilon^{1/3}$.

Remark 2.4 Another natural way to introduce the Wiener integrals with respect to W, which can be found in the literature (see for example [13, 15, 17]), is to consider the linear Gaussian space $\{W(h) : h \in \mathcal{G}\}$, where $\mathcal{G}$ is the completion of $\mathcal{E} \otimes \mathcal{S}(\mathbb{R}^d)$ with respect to the inner product (2.3). However, with the general definition of Gaussian noise given in Definition 2.1, it is no longer convenient to investigate the Hilbert space $\mathcal{G}$ if one wishes to solve an additive heat equation. In Definition 2.2, we have adopted instead a discretization in time procedure and mollification in space (because we are only integrating functions φ in the Schwartz space). While the discretization in time is classical in order to define stochastic integrals with respect to a fractional Brownian motion (see [19] for details), our regularization in space is more akin to a Russo-Vallois type approximation similar to those suggested in [10, 21]. In any case, Definition 2.1 does not require a complete understanding of the space $\mathcal{G}$. This is possible because our integrand has the special heat kernel form $p_{t-\cdot}(x - \cdot)$ (see Equation (3.1) below).

3 Stochastic Heat Equation

Let W be the Gaussian field introduced in Definition 2.1. As mentioned in the introduction, we are concerned with the heat equation (1.1) with additive noise on $\mathbb{R}^d$. In this case, it is well known (see for example [5, 23]) that an explicit solution to (1.1) should be given by the so-called stochastic convolution. Namely, for all $t > 0$ and $x \in \mathbb{R}^d$, the solution u to (1.1) is expressed as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} p_{t-s}(x-y) W(\mathrm{d}y, \mathrm{d}s), \tag{3.1}$$

where p_t stands for the heat kernel mentioned in the introduction, and where (3.1) is understood as a Wiener integral compatible with Definition 2.2. More generally, we will focus on the definition of a convolution of the form:

$$\int_0^t \int_{\mathbb{R}^d} g(s,y)\, W(\mathrm{d}y, \mathrm{d}s), \tag{3.2}$$

for a deterministic kernel g. We wish to find optimal conditions on R and μ such that expression (3.1) makes sense.

3.1 A discrete integration by parts formula

In this section, our computations rely on an elementary discrete integration by parts formula. For convenience, let us first introduce the following notation:

Notation 3.1 Consider a small constant $\varepsilon > 0$, two positive numbers $s \geq 0$ and $t > 0$, and set $t_k = s + k\varepsilon$ for $k \geq 0$. Let f be a function on $\mathbb{R}_+$. We define a regularization of the integral $\int_0^t f_u \mathrm{d}u$ in the following way:

$$\int_s^t f_u \mathrm{d}^\varepsilon u = \varepsilon \sum_{s \leq t_k < t} f_{t_k}.$$

Observe that for the discretized integral defined in Notation 3.1, the following elementary change of variables formula holds true:

$$\int_0^t f_s \mathrm{d}^\varepsilon s = \int_\varepsilon^{t+\varepsilon} f_{s-\varepsilon} \mathrm{d}^\varepsilon s = \int_{-\varepsilon}^{t-\varepsilon} f_{s+\varepsilon} \mathrm{d}^\varepsilon s. \tag{3.3}$$

Let us now state our main technical tools, which is a partial integration by parts formula for the covariance function of W.

Lemma 3.2 Let t, $\tilde{t}$ be two strictly positive numbers. Consider two continuous functions R and Γ on $[0,t] \times [0,\tilde{t}]$. For $\varepsilon, \tilde{\varepsilon} > 0$, set

$$\mathcal{A}(t,\tilde{t}) := \int_0^{\tilde{t}} \int_0^t \Gamma(s,s') R \begin{pmatrix} s' & s'+\tilde{\varepsilon} \\ s & s+\varepsilon \end{pmatrix} \mathrm{d}^\varepsilon s \mathrm{d}^{\tilde{\varepsilon}} s', \qquad t, \tilde{t} \geq 0, \tag{3.4}$$

where the discretized integral is defined in Notation 3.1. Then, $\mathcal{A}(t,\tilde{t})$ can be decomposed as follows:

$$\mathcal{A}(t,\tilde{t}) = \mathcal{A}_0(t,\tilde{t}) + \mathcal{I}_0(t,\tilde{t}) + \mathcal{I}_1(t,\tilde{t}) + \mathcal{I}_2(t,\tilde{t}) + \mathcal{I}_3(t,\tilde{t}) + \mathcal{I}_4(t,\tilde{t}), \tag{3.5}$$

where $\mathcal{A}_0(t,\tilde{t})$ is the main term of an integration by parts:

$$\mathcal{A}_0(t,\tilde{t}) = \int_0^{\tilde{t}} \int_0^t \Gamma \begin{pmatrix} s'-\tilde{\varepsilon} & s' \\ s-\varepsilon & s \end{pmatrix} R(s,s') \mathrm{d}^\varepsilon s \mathrm{d}^{\tilde{\varepsilon}} s', \tag{3.6}$$

while $\mathcal{I}_0, \ldots, \mathcal{I}_4$ are boundary terms defined, respectively, by

$$\begin{aligned}\mathcal{I}_0(t,\tilde{t}) &= \left(\int_{\tilde{t}}^{\tilde{t}+\tilde{\varepsilon}}\int_t^{t+\varepsilon} - \int_0^{\tilde{\varepsilon}}\int_0^{\varepsilon}\right)\Gamma(s-\varepsilon,s'-\tilde{\varepsilon})R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &:= \mathcal{I}_{01}(t,\tilde{t}) - \mathcal{I}_{00}(t,\tilde{t}), \end{aligned} \tag{3.7}$$

$$\begin{aligned}\mathcal{I}_1(t,\tilde{t}) &= \int_{\tilde{\varepsilon}}^{\tilde{t}}\int_t^{t+\varepsilon}(\Gamma(s-\varepsilon,s'-\tilde{\varepsilon}) - \Gamma(s-\varepsilon,s'))R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &\quad - \int_0^{\tilde{\varepsilon}}\int_t^{t+\varepsilon}\Gamma(s-\varepsilon,s')R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &:= \mathcal{I}_{11}(t,\tilde{t}) - \mathcal{I}_{10}(t,\tilde{t}), \end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{I}_2(t,\tilde{t}) &= \int_{\tilde{t}}^{\tilde{t}+\tilde{\varepsilon}}\int_{\varepsilon}^{t}(\Gamma(s-\varepsilon,s'-\tilde{\varepsilon}) - \Gamma(s,s'-\tilde{\varepsilon}))R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &\quad - \int_{\tilde{t}}^{\tilde{t}+\tilde{\varepsilon}}\int_0^{\varepsilon}\Gamma(s,s'-\tilde{\varepsilon})R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &:= \mathcal{I}_{21}(t,\tilde{t}) - \mathcal{I}_{20}(t,\tilde{t}), \end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{I}_3(t,\tilde{t}) &= \int_{\tilde{\varepsilon}}^{\tilde{t}}\int_0^{\varepsilon}(\Gamma(s-\varepsilon,s') - \Gamma(s-\varepsilon,s'-\tilde{\varepsilon}))R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &\quad + \int_0^{\tilde{\varepsilon}}\int_0^{\varepsilon}\Gamma(s-\varepsilon,s')R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s', \end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{I}_4(t,\tilde{t}) &= \int_0^{\tilde{\varepsilon}}\int_{\varepsilon}^{t}(\Gamma(s,s'-\tilde{\varepsilon}) - \Gamma(s-\varepsilon,s'-\tilde{\varepsilon}))R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &\quad + \int_0^{\tilde{\varepsilon}}\int_0^{\varepsilon}\Gamma(s,s'-\tilde{\varepsilon})R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s'. \end{aligned} \tag{3.11}$$

Proof Start from the definition (3.4) of $\mathcal{A}(t,\tilde{t})$, and recall our notation (2.2) for the rectangular increments of R. Then, a series of elementary change of variables resorting to (3.3) yields

$$\begin{aligned}\mathcal{A}(t,\tilde{t}) &= \int_0^{\tilde{t}}\int_0^{t}\Gamma(s,s')\left(R(s+\varepsilon,s'+\tilde{\varepsilon}) - R(s+\varepsilon,s') - R(s,s'+\tilde{\varepsilon}) + R(s,s')\right)\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &= \int_{\tilde{\varepsilon}}^{\tilde{t}+\tilde{\varepsilon}}\int_{\varepsilon}^{t+\varepsilon}\Gamma(s-\varepsilon,s'-\tilde{\varepsilon})R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' - \int_0^{\tilde{t}}\int_{\varepsilon}^{t+\varepsilon}\Gamma(s-\varepsilon,s')R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' \\ &\quad - \int_{\tilde{\varepsilon}}^{\tilde{t}+\tilde{\varepsilon}}\int_0^{t}\Gamma(s,s'-\tilde{\varepsilon})R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s' + \int_0^{\tilde{t}}\int_0^{t}\Gamma(s,s')R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s'.\end{aligned}$$

We now rearrange those terms by separating the interval $[0,t]$ from other intervals of length ε and $\tilde{\varepsilon}$. We get

$$\mathcal{A}(t,\tilde{t}) = \mathcal{A}_0(t,\tilde{t}) + \mathcal{A}_1(t,\tilde{t}) - (\mathcal{A}_{21}(t,\tilde{t}) - \mathcal{A}_{22}(t,\tilde{t})) - (\mathcal{A}_{31}(t,\tilde{t}) - \mathcal{A}_{32}(t,\tilde{t})), \tag{3.12}$$

where $\mathcal{A}_0(t,\tilde{t})$ is defined in (3.6) and

$$\mathcal{A}_1(t,\tilde{t}) = \left(\int_{\tilde{\varepsilon}}^{\tilde{t}+\tilde{\varepsilon}}\int_{\varepsilon}^{t+\varepsilon} - \int_0^{\tilde{t}}\int_0^{t}\right)\Gamma(s-\varepsilon,s'-\tilde{\varepsilon})R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s'$$

$$\mathcal{A}_{21}(t,\tilde{t}) - \mathcal{A}_{22}(t,\tilde{t}) = \left(\int_0^{\tilde{t}}\int_t^{t+\varepsilon} - \int_0^{\tilde{t}}\int_0^{\varepsilon}\right)\Gamma(s-\varepsilon,s')R(s,s')\mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s'$$

$$\mathcal{A}_{31}(t,\tilde{t}) - \mathcal{A}_{32}(t,\tilde{t}) = \left(\int_{\tilde{t}}^{\tilde{t}+\tilde{\varepsilon}} \int_0^t - \int_0^{\tilde{\varepsilon}} \int_0^t \right) \Gamma(s, s'-\tilde{\varepsilon}) R(s,s') \mathrm{d}^{\varepsilon} s \mathrm{d}^{\tilde{\varepsilon}} s'.$$

Next, we decompose the rectangles $[\varepsilon, t+\varepsilon] \times [\tilde{\varepsilon}, \tilde{t}+\tilde{\varepsilon}]$ and $[0,t] \times [0,\tilde{t}]$ in order to get

$$\begin{aligned} \mathcal{A}_1(t,\tilde{t}) = & \Big(\int_{\tilde{t}}^{\tilde{t}+\tilde{\varepsilon}} \int_t^{t+\varepsilon} + \int_{\tilde{t}}^{\tilde{t}+\tilde{\varepsilon}} \int_\varepsilon^t + \int_{\tilde{\varepsilon}}^{\tilde{t}} \int_t^{t+\varepsilon} - \int_0^{\tilde{\varepsilon}} \int_0^{\varepsilon} \\ & - \int_0^{\tilde{\varepsilon}} \int_\varepsilon^t - \int_{\tilde{\varepsilon}}^{\tilde{t}} \int_0^{\varepsilon} \Big) \Gamma(s-\varepsilon, s'-\tilde{\varepsilon}) R(s,s') \mathrm{d}^{\varepsilon} s \mathrm{d}^{\tilde{\varepsilon}} s' \\ := & \mathcal{A}_{11}(t,\tilde{t}) + \mathcal{A}_{12}(t,\tilde{t}) + \mathcal{A}_{13}(t,\tilde{t}) - \mathcal{A}_{14}(t,\tilde{t}) - \mathcal{A}_{15}(t,\tilde{t}) - \mathcal{A}_{16}(t,\tilde{t}) . \end{aligned} \tag{3.13}$$

Now, substituting (3.13) into (3.12) and rearranging the terms, we obtain

$$\begin{aligned} \mathcal{A}(t,\tilde{t}) = & \mathcal{A}_0(t,\tilde{t}) + (\mathcal{A}_{11}(t,\tilde{t}) - \mathcal{A}_{14}(t,\tilde{t})) + (\mathcal{A}_{13}(t,\tilde{t}) - \mathcal{A}_{21}(t,\tilde{t})) \\ & + (\mathcal{A}_{12}(t,\tilde{t}) - \mathcal{A}_{31}(t,\tilde{t})) + (\mathcal{A}_{22}(t,\tilde{t}) - \mathcal{A}_{16}(t,\tilde{t})) + (\mathcal{A}_{32}(t,\tilde{t}) - \mathcal{A}_{15}(t,\tilde{t})). \end{aligned}$$

Identity (3.5) then follows from the expression of $\mathcal{I}_i$ in (3.7)–(3.11) and from observing that

$$\mathcal{I}_0 = \mathcal{A}_{11} - \mathcal{A}_{14},\ \mathcal{I}_1 = \mathcal{A}_{13} - \mathcal{A}_{21},\ \mathcal{I}_2 = \mathcal{A}_{12} - \mathcal{A}_{31},\ \mathcal{I}_3 = \mathcal{A}_{22} - \mathcal{A}_{16},\ \mathcal{I}_4 = \mathcal{A}_{32} - \mathcal{A}_{15}.$$

This completes the proof of Lemma 3.2. □

3.2 L^2 convergence for the stochastic heat equation

We will now use the integration by parts formula stated in Lemma 3.2 in order to estimate the Wiener integral defining our solution u. In the next result, we first calculate the $L^2(\Omega)$-norm of the Wong-Zakai type approximation in (2.5).

Lemma 3.3 Consider a small constant $\varepsilon > 0$, and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Denote $t_\varepsilon = t - \sqrt[3]{\varepsilon}$. Let $h : [0,t) \to \mathbb{R}^d \to \mathbb{R}_+$ be defined by $h(s,y) = p_{t-s}(x-y)$, where we recall that p_t stands for the heat kernel on $\mathbb{R}^d$ (see Notation at the end of the introduction). We define $u_\varepsilon(t,x)$ by

$$u_\varepsilon(t,x) = \sum_{0 \le t_k < t_\varepsilon} \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} h(t_k, y) W(\mathrm{d}y, \mathrm{d}s). \tag{3.14}$$

Then, the following holds true:

(i) The covariance between $u_\varepsilon(t,x)$ and $u_{\tilde{\varepsilon}}(t,x)$ can be expressed as

$$\begin{aligned} & \mathbb{E}(u_\varepsilon(t,x) u_{\tilde{\varepsilon}}(t,x)) \\ & = (\varepsilon\tilde{\varepsilon})^{-1} \left(\mathcal{A}_0(t_\varepsilon, t_{\tilde{\varepsilon}}) + \mathcal{I}_0(t_\varepsilon, t_{\tilde{\varepsilon}}) + \mathcal{I}_1(t_\varepsilon, t_{\tilde{\varepsilon}}) + \mathcal{I}_2(t_\varepsilon, t_{\tilde{\varepsilon}}) + \mathcal{I}_3(t_\varepsilon, t_{\tilde{\varepsilon}}) + \mathcal{I}_4(t_\varepsilon, t_{\tilde{\varepsilon}}) \right), \end{aligned} \tag{3.15}$$

where $\mathcal{A}_0, \mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ are defined by relations (3.6)–(3.11), with a function Γ defined on $[0,\tau]^2$ by

$$\Gamma(s,s') = \int_{\mathbb{R}^d} e^{-\frac{(2t-s-s')|\xi|^2}{2}} \mu(\mathrm{d}\xi). \tag{3.16}$$

(ii) Furthermore, the kernel Γ given by (3.16) is differentiable on $[0,t)^2$ and for $0 \le s, s' < t$, we have

$$\frac{\partial \Gamma}{\partial s}(s,s') = \frac{\partial \Gamma}{\partial s'}(s,s') = \int_{\mathbb{R}^d} \frac{|\xi|^2}{2} e^{-\frac{(2t-s-s')|\xi|^2}{2}} \mu(\mathrm{d}\xi) \tag{3.17}$$

$$\frac{\partial^2 \Gamma}{\partial s' \partial s}(s,s') = \int_{\mathbb{R}^d} \frac{|\xi|^4}{4} e^{-\frac{(2t-s-s')|\xi|^2}{2}} \mu(\mathrm{d}\xi) \tag{3.18}$$

$$\frac{\partial^3\Gamma}{\partial^2 s'\partial s}(s,s') = \frac{\partial^3\Gamma}{\partial s'\partial^2 s}(s,s') = \int_{\mathbb{R}^d} \frac{|\xi|^6}{8} e^{-\frac{(2t-s-s')|\xi|^2}{2}}\mu(\mathrm{d}\xi). \tag{3.19}$$

Proof Recall that u_ε is defined by (3.14). On each interval $[t_k, t_{k+1}]$, we use the elementary identity

$$h(t_k,y) = \frac{1}{\varepsilon}\int_{t_k}^{t_{k+1}} h(s,y)\mathrm{d}^\varepsilon s,$$

which is immediately seen from Notation 3.1. Plugging this information into (3.14), we get

$$\begin{aligned} u_\varepsilon(t,x) &= \sum_{0\le t_k<t_\varepsilon}\int_{t_k}^{t_{k+1}}\int_{\mathbb{R}^d} h(t_k,y)W(\mathrm{d}y,\mathrm{d}s)\\ &= \frac{1}{\varepsilon}\int_0^{t_\varepsilon}\mathrm{d}^\varepsilon s\int_s^{s+\varepsilon}\int_{\mathbb{R}^d} h(s,y)W(\mathrm{d}y,\mathrm{d}r). \end{aligned}$$

Therefore, taking into account the covariance function of the Gaussian field W in Definition 2.1, we obtain

$$\mathbb{E}\left[u_\varepsilon(t,x)u_{\tilde\varepsilon}(t,x)\right] = \frac{1}{\varepsilon\tilde\varepsilon}\int_0^{t_{\tilde\varepsilon}}\int_0^{t_\varepsilon}\left[\tilde\Gamma(s,s')R\begin{pmatrix} s' & s'+\tilde\varepsilon\\ s & s+\varepsilon\end{pmatrix}\right]\mathrm{d}^\varepsilon s\mathrm{d}^{\tilde\varepsilon}s',$$

where the function $\tilde\Gamma$ is defined by

$$\tilde\Gamma(s,s') = \int_{\mathbb{R}^d}\mathcal{F}(h)(\xi)\overline{\mathcal{F}(h)(\xi)}\mu(\mathrm{d}\xi). \tag{3.20}$$

Invoking Lemma 3.2, relation (3.15) is thus easily reduced to show that $\tilde\Gamma = \Gamma$, where Γ is given by (3.16). To this aim, note that

$$\mathcal{F}(p_{t-s}(x-y)) = e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\mathrm{i}\xi\cdot x}. \tag{3.21}$$

Substituting (3.21) into (3.20), we immediately have $\tilde\Gamma = \Gamma$, which finishes the proof of (3.15).

The identities (3.17)–(3.19) follows easily by noticing that

$$\mathcal{F}\left(\frac{\partial}{\partial s}p_{t-s}(x-y)\right) = \frac{\partial}{\partial s}\mathcal{F}(p_{t-s}(y-x)) = \frac{|\xi|^2}{2}e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\mathrm{i}\xi\cdot x},$$

and the fact that $e^{-\frac{(t-s)|\xi|^2}{2}}$ is an increasing function of s in order to apply monotone convergence. □

With Lemma 3.3 in hand, we will now bound the terms in (3.15) (see also (3.5) for more precise definitions) individually. Let us start by analyzing the terms $\mathcal{I}_3$ and $\mathcal{I}_4$ in (3.15).

Lemma 3.4 Let Γ be defined by (3.16) and R be a continuous function satisfying (1.2). Let $\mathcal{I}_3$ and $\mathcal{I}_4$ be given by (3.15) (see also (3.10) and (3.11)). Then, the following convergence holds true:

$$\begin{aligned} \lim_{\varepsilon,\tilde\varepsilon\to0+}\varepsilon^{-1}\tilde\varepsilon^{-1}\mathcal{I}_3(t_\varepsilon,t_{\tilde\varepsilon}) &= \int_0^t\frac{\partial\Gamma}{\partial s'}(0,s')R(0,s')\mathrm{d}s' + \Gamma(0,0)R(0,0),\\ \lim_{\varepsilon,\tilde\varepsilon\to0+}\varepsilon^{-1}\tilde\varepsilon^{-1}\mathcal{I}_4(t_\varepsilon,t_{\tilde\varepsilon}) &= \int_0^t\frac{\partial\Gamma}{\partial s}(s,0)R(s,0)\mathrm{d}s' + \Gamma(0,0)R(0,0). \end{aligned}$$

Proof The convergences follow immediately from the continuity of R and the monotonicity of Γ and its derivatives and an application of the dominated convergence theorem. Notice that when one integrates Γ defined by (3.16), there is a singularity at $s = s' = t$. However, this singularity is avoided for the terms $\mathcal{I}_3$ and $\mathcal{I}_4$, as well as for the terms (3.17)–(3.19). $\square$

In order to proceed with our estimates, we now state an elementary lemma giving an upper bound on the increments of the function R.

Lemma 3.5 Let R be a covariance function satisfying relation (1.2) with $\beta > 0$. Then for $u, v, t \in [0, \tau]$, we have $|R(t,u) - R(t,v)| \le K|u-v|^{\beta/2}$, where K is a constant depending on τ.

Proof Let X be a Gaussian process on $[0,\tau]$ with covariance function R and $X_0 = 0$. Then by relation (1.2), we have

$$|\mathbb{E}[(X_t - X_u)(X_t - X_v)]| = |R(t,t) + R(u,v) - R(u,t) - R(t,v)| \le |t - (u \wedge v)|^{\beta}. \quad (3.22)$$

The lemma then follows from the following relations

$$|R(t,u) - R(t,v)| = |\mathbb{E}[X_t(X_u - X_v)]| \le \mathbb{E}[|X_t|^2]^{1/2}\mathbb{E}[|X_u - X_v|^2]^{1/2} \le R(t,t)^{1/2}|u-v|^{\beta/2},$$

where in the last inequality we have used relation (3.22) with $u = v$. $\square$

In order to handle the term $\mathcal{A}_0(t_\varepsilon, t_{\tilde{\varepsilon}})$ in relations (3.6) and (3.15), we will artificially introduce some rectangular increments. The lemma below takes care of the convergence of the rectangular increment $\tilde{\mathcal{A}}_0^{\varepsilon,\tilde{\varepsilon}}$ derived from $\mathcal{A}_0(t_\varepsilon, t_{\tilde{\varepsilon}})$ (specifically, we replace the term $R(s,s')$ in (3.6) by its rectangular increment).

Lemma 3.6 Let Γ be defined by (3.16) and R be a continuous function satisfying relation (1.2). For $\varepsilon, \tilde{\varepsilon} > 0$, we set

$$\tilde{\mathcal{A}}_0^{\varepsilon,\tilde{\varepsilon}} = \int_0^{t_\varepsilon}\int_0^{t_\varepsilon} \Gamma\begin{pmatrix} s'-\tilde{\varepsilon} & s' \\ s-\varepsilon & s \end{pmatrix} R\begin{pmatrix} s & t \\ s' & t \end{pmatrix} \mathrm{d}^{\varepsilon}s\mathrm{d}^{\tilde{\varepsilon}}s'. \quad (3.23)$$

Suppose that the spectral measure μ satisfies relation (1.3). Then

$$\lim_{\varepsilon,\tilde{\varepsilon}\to 0+} \varepsilon^{-1}\tilde{\varepsilon}^{-1}\tilde{\mathcal{A}}_0^{\varepsilon,\tilde{\varepsilon}} = \int_0^t\int_0^t \frac{\partial^2\Gamma}{\partial s\partial s'}(s,s') R\begin{pmatrix} s & t \\ s' & t \end{pmatrix}\mathrm{d}s\mathrm{d}s' := \mathcal{J}. \quad (3.24)$$

Proof We first show that the right-hand side of (3.24) is integrable. Note first that by relation (1.2) and taking into account (3.18), we have

$$\begin{aligned}\mathcal{J} &\le \int_0^t\int_0^t \frac{\partial^2\Gamma}{\partial s\partial s'}(s,s')\left|R\begin{pmatrix} s & t \\ s' & t \end{pmatrix}\right|\mathrm{d}s\mathrm{d}s' \le \int_0^t\int_0^t \frac{\partial^2\Gamma}{\partial s\partial s'}(s,s')|t-(s\wedge s')|^{\beta}\mathrm{d}s\mathrm{d}s' \\ &\le \int_0^t\int_0^t\int_{\mathbb{R}^d} \frac{|\xi|^4}{4} e^{-\frac{(2t-s-s')|\xi|^2}{2}}\mu(\mathrm{d}\xi)|t-(s\wedge s')|^{\beta}\mathrm{d}s\mathrm{d}s'.\end{aligned}$$

We can now split our estimate by writing

$$\mathcal{J} \le \mathcal{J}_1 + \mathcal{J}_2, \quad (3.25)$$

where the terms $\mathcal{J}_1$ and $\mathcal{J}_2$ are defined by

$$\mathcal{J}_1 = \int_{|\xi|>1}\int_0^t\int_0^t \frac{|\xi|^4}{4}e^{-\frac{(2t-s-s')|\xi|^2}{2}}|t-(s\wedge s')|^{\beta}\mathrm{d}s\mathrm{d}s'\mu(\mathrm{d}\xi),$$

$$\mathcal{J}_2 = \int_{|\xi|\le 1}\int_0^t\int_0^t \frac{|\xi|^4}{4}e^{-\frac{(2t-s-s')|\xi|^2}{2}}|t-(s\wedge s')|^{\beta}\mathrm{d}s\mathrm{d}s'\mu(\mathrm{d}\xi).$$

In the following, we bound $\mathcal{J}_1$ and $\mathcal{J}_2$ separately, starting with $\mathcal{J}_1$. Indeed, because the integral defining $\mathcal{J}_1$ is symmetric in s and s', we have

$$\begin{aligned}\mathcal{J}_1 &= 2\int_{|\xi|>1}\int_0^t\int_0^s \frac{|\xi|^4}{4}e^{-\frac{(2t-s-s')|\xi|^2}{2}}|t-(s\wedge s')|^\beta \mathrm{d}s'\mathrm{d}s\mu(\mathrm{d}\xi)\\ &= 2\int_{|\xi|>1}\frac{|\xi|^4}{4}\int_0^t e^{-\frac{(t-s)|\xi|^2}{2}}\int_0^s e^{-\frac{(t-s')|\xi|^2}{2}}|t-s'|^\beta \mathrm{d}s'\mathrm{d}s\mu(\mathrm{d}\xi).\end{aligned} \tag{3.26}$$

Next, we bound the inner integral in the right-hand side above as follows:

$$\int_0^s e^{-\frac{(t-s')|\xi|^2}{2}}|t-s'|^\beta \mathrm{d}s' \le \int_0^\infty r^\beta e^{-\frac{r|\xi|^2}{2}}\mathrm{d}r = 2^{1+\beta}\Gamma(\beta+1)|\xi|^{-(2+2\beta)},$$

where the last identity is obtained by a straightforward change of variable. In the same way, it is also readily checked that

$$\int_0^t e^{-\frac{(t-s)|\xi|^2}{2}}\mathrm{d}s \le |\xi|^{-2}.$$

Plugging those two elementary estimates into (3.26), we end up with

$$\mathcal{J}_1 \le K\int_{|\xi|>1}|\xi|^4\frac{1}{|\xi|^{2\beta+4}}\mu(\mathrm{d}\xi) = K\int_{|\xi|>1}\frac{1}{|\xi|^{2\beta}}\mu(\mathrm{d}\xi) < \infty, \tag{3.27}$$

where the last inequality follows from relation (1.3).

Let us now turn to an upper bound for $\mathcal{J}_2$ defined in (3.25). When $|\xi|\le 1$, we have

$$\frac{|\xi|^4}{4}e^{-\frac{(2t-s-s')|\xi|^2}{2}} \le 1.$$

Hence, $\mathcal{J}_2$ is easily bounded as follows:

$$\mathcal{J}_2 \le \mu(\xi:|\xi|\le 1)\int_0^t\int_0^t |t-(s\wedge s')|^\beta \mathrm{d}s\mathrm{d}s' < \infty. \tag{3.28}$$

Applying the estimate of $\mathcal{J}_1$ and $\mathcal{J}_2$ obtained in (3.27) and (3.28) to relation (3.25), the integrability of the right-hand side of (3.24) is trivially satisfied.

Next, by (3.18) and the mean value theorem, we have

$$0 \le \frac{1}{\varepsilon\tilde{\varepsilon}}\Gamma\begin{pmatrix} s'-\tilde{\varepsilon} & s' \\ s-\varepsilon & s\end{pmatrix} = \frac{\partial^2\Gamma}{\partial s\partial s'}(s-\varepsilon\theta, s'-\tilde{\varepsilon}\theta') \le \frac{\partial^2\Gamma}{\partial s\partial s'}(s,s')$$

uniformly in ε and $\tilde{\varepsilon}$, where θ and θ' are some numbers between 0 and 1, and where the last inequality is because of the fact that the exponential function is monotone. Therefore, $\tilde{\mathcal{A}}_0^{\varepsilon,\tilde{\varepsilon}}$ is dominated by $\mathcal{J}$. By the dominated convergence theorem and by taking limits $\varepsilon,\tilde{\varepsilon}\to 0+$ for $\tilde{\mathcal{A}}_0^{\varepsilon,\tilde{\varepsilon}}$, we obtain the desired convergence (3.24). □

In order to get the convergence of the boundary terms $\mathcal{I}_{01}$, $\mathcal{I}_{11}$, and $\mathcal{I}_{21}$ in (3.7)–(3.9), we need the following technical lemma. In the following, we denote by Id the identity function, namely, $\mathrm{Id}(u)=u$, $u\in[0,t]$.

Lemma 3.7 Denote $t_\varepsilon = t-\sqrt[3]{\varepsilon}$ for $\varepsilon>0$, and let R and Γ be as in Lemma 3.6. For $i=1,2,3,4$, we define small constants δ_i and some instants t_i in the following way: for $i=1,3$, we take $\delta_i\in[0,\varepsilon]$ and t_i such that $t_i: 0\le t_\varepsilon - t_i\le\varepsilon$, while for $i=2,4$, we consider $\delta_i\in[0,\tilde{\varepsilon}]$ and t_i such that $t_i: 0\le t_{\tilde{\varepsilon}}-t_i\le\tilde{\varepsilon}$. On $[0,t]$, define 8 piecewise continuous functions

$$u_i = u_i(u), \quad v_i = v_i(v), \qquad \text{for } i=1,2,3,4,$$

satisfying $|u_2-u_4|\le\varepsilon$, $|v_2-v_4|\le\tilde\varepsilon$, $0\le \mathrm{Id}-u_i\le\varepsilon$, and $0\le \mathrm{Id}-v_i\le\tilde\varepsilon$ for $i=1,3$. Besides, assume that $|u_i(u)-u_i(u+\varepsilon)|\le\varepsilon$ and $|v_i(u)-v_i(u+\tilde\varepsilon)|\le\tilde\varepsilon$ for $i=1,2,3,4$. Denote $g=\frac{\partial^2\Gamma}{\partial s\partial s'}$. Set

$$\mathcal{G}_1=\int_{\delta_1}^{t_1}\int_{\delta_2}^{t_2} g(u_1,v_1)R(u_2,v_2)\mathrm{d}u\mathrm{d}v \quad\text{and}\quad \mathcal{G}_2=\int_{\delta_3}^{t_3}\int_{\delta_4}^{t_4} g(u_3,v_3)R(u_4,v_4)\mathrm{d}u\mathrm{d}v.$$

Then, the following convergence holds true:

$$\lim_{\varepsilon,\tilde\varepsilon\to 0+}|\mathcal{G}_1-\mathcal{G}_2|=0.$$

Proof Choose ε_1 and ε_2 such that $\varepsilon_1+t_3=t_1$ and $\varepsilon_2+t_4=t_2$. In the integral defining $\mathcal{G}_2$, set $u=u+\varepsilon_1$ and $v=v+\varepsilon_2$. Setting $u'_j=u_j(u-\varepsilon_1)$ and $v'_j=v_j(v-\varepsilon_1)$, we get

$$\begin{aligned}\mathcal{G}_2&=\int_{\delta_3+\varepsilon_1}^{t_3+\varepsilon_1}\int_{\delta_4+\varepsilon_2}^{t_4+\varepsilon_2} g(u'_3,v'_3)R(u'_4,v'_4)\mathrm{d}u\mathrm{d}v\\&=\int_{\delta_3+\varepsilon_1}^{t_1}\int_{\delta_4+\varepsilon_2}^{t_2} g(u'_3,v'_3)R(u'_4,v'_4)\mathrm{d}u\mathrm{d}v.\end{aligned}$$

We now introduce a slight modification of $\mathcal{G}_2$ called $\mathcal{G}'_2$:

$$\mathcal{G}'_2=\int_{\delta_1}^{t_1}\int_{\delta_2}^{t_2} g(u'_3,v'_3)R(u'_4,v'_4)\mathrm{d}u\mathrm{d}v.$$

We are going to show that

$$\mathcal{G}_1-\mathcal{G}'_2\to 0 \quad\text{and}\quad \mathcal{G}_2-\mathcal{G}'_2\to 0 \qquad \text{as}\ \ \varepsilon,\tilde\varepsilon\to 0+, \tag{3.29}$$

which then concludes the proof.

Write

$$\begin{aligned}\mathcal{G}_1-\mathcal{G}'_2 &= \int_{\delta_1}^{t_1}\int_{\delta_2}^{t_2} g(u_1,v_1)(R(u_2,v_2)-R(u'_4,v'_4))\mathrm{d}u\mathrm{d}v\\&\quad+\int_{\delta_1}^{t_1}\int_{\delta_2}^{t_2}(g(u_1,v_1)-g(u'_3,v'_3))R(u'_4,v'_4)\mathrm{d}u\mathrm{d}v\\&:=\int_{\delta_1}^{t_1}\int_{\delta_2}^{t_2}\mathcal{G}_{11}\mathrm{d}u\mathrm{d}v+\int_{\delta_1}^{t_1}\int_{\delta_2}^{t_2}\mathcal{G}_{12}\mathrm{d}u\mathrm{d}v.\end{aligned} \tag{3.30}$$

Let us briefly show how to bound the term $\mathcal{G}_{11}$. We first write

$$|R(u_2,v_2)-R(u'_4,v'_4)|\le|R(u_2,v_2)-R(u'_4,v_2)|+|R(u'_4,v_2)-R(u'_4,v'_4)|. \tag{3.31}$$

Because of the fact that $|u_2-u'_4|\le 2\varepsilon$ and $|v_2-v'_4|\le 2\tilde\varepsilon$, we can invoke Lemma 3.5 and decomposition (3.31) to get

$$|R(u_2,v_2)-R(u'_4,v'_4)|\le K(\varepsilon^{\beta/2}+\tilde\varepsilon^{\beta/2}). \tag{3.32}$$

Moreover, according to our standing assumptions, the variable u in $\mathcal{G}_{11}$ satisfies

$$u\le t_1\le t_\varepsilon\le t-\varepsilon^{1/3},$$

and similarly, $v\le t-\tilde\varepsilon^{1/3}$. Therefore,

$$K(\varepsilon^{\beta/2}+\tilde\varepsilon^{\beta/2})\le K(t-(u\wedge v))^{3\beta/2}, \tag{3.33}$$

and plugging (3.33) into (3.32), we get

$$|R(u_2,v_2)-R(u'_4,v'_4)|\le K(t-(u\wedge v))^{3\beta/2}. \tag{3.34}$$

Reporting this information into the expression of $\mathcal{G}_{11}$ in (3.30), and taking into account the fact that $g=\frac{\partial^2\Gamma}{\partial s\partial s'}$ satisfies (3.18), we thus get

$$|\mathcal{G}_{11}|\le (t-(u\wedge v))^{\frac{3\beta}{2}}\int_{\mathbb{R}^d}\frac{|\xi|^4}{4}e^{-\frac{(2t-u-v)|\xi|^2}{2}}\mu(\mathrm{d}\xi). \tag{3.35}$$

Similarly, we also have

$$|\mathcal{G}_{12}|\le (t-(u\wedge v))^{3}\int_{\mathbb{R}^d}\frac{|\xi|^6}{8}e^{-\frac{(2t-u-v)|\xi|^2}{2}}\mu(\mathrm{d}\xi). \tag{3.36}$$

In a similar way as in (3.25), it follows that both functions in the right-hand side of (3.35) and (3.35) are integrable, and thus by the dominated convergence theorem, we are able to pass the limit $\varepsilon,\tilde\varepsilon\to 0+$ inside the integrals of (3.30). This yields the convergence $\mathcal{G}_1-\mathcal{G}_2'\to 0$. Observe that a cubic power of $(t-(u\wedge v))$ is necessary in order to balance the term $|\xi|^6$ in the right-hand side of (3.36). This is the reason why we have imposed the condition $t_\varepsilon=t-\varepsilon^{1/3}$.

Summarizing our considerations so far, we have proved that $\mathcal{G}_1-\mathcal{G}_2'\to 0$ in (3.29). In order to deal with $\mathcal{G}_2-\mathcal{G}_2'$, we write

$$\begin{aligned}\mathcal{G}_2-\mathcal{G}_2'&=\left(\int_{\delta_3}^{t_1}\int_{\delta_4}^{t_2}-\int_{\delta_3-\varepsilon_1}^{t_1}\int_{\delta_4-\varepsilon_2}^{t_2}\right)g(u_3',v_3')R(u_4',v_4')\mathrm{d}u\mathrm{d}v\\&=\left(\int_{\delta_3}^{\delta_3-\varepsilon_1}\int_{\delta_4}^{t_2}+\int_{\delta_3-\varepsilon_1}^{t_1}\int_{\delta_4}^{\delta_4-\varepsilon_2}\right)g(u_3',v_3')R(u_4',v_4')\mathrm{d}u\mathrm{d}v.\end{aligned}$$

Because g and R are both continuous on $[-2\varepsilon,2\varepsilon]\times[0,t]$ and $[0,t]\times[-2\tilde\varepsilon,2\tilde\varepsilon]$, we immediately have $\mathcal{G}_2-\mathcal{G}_2'\to 0$ as $\varepsilon,\tilde\varepsilon\to 0+$. In conclusion, we have found that relation (3.29) holds true. Therefore, it is readily checked that $\lim_{\varepsilon,\tilde\varepsilon\to 0+}|\mathcal{G}_1-\mathcal{G}_2|=0$. Thus, the proof is complete. □

With Lemma 3.7 in hand, we can now estimate some of the boundary terms derived from $\mathcal{A}_0(t_\varepsilon,t_{\tilde\varepsilon})$ in Lemma 3.3.

Lemma 3.8 Let the assumptions be as in Lemma 3.6 and suppose that relations (1.2) and (1.3) hold true. Set

$$\mathcal{A}_{00}(t_\varepsilon,t_{\tilde\varepsilon})=\int_0^{t_{\tilde\varepsilon}}\int_0^{t_\varepsilon}\Gamma\begin{pmatrix}s'-\tilde\varepsilon & s'\\ s-\varepsilon & s\end{pmatrix}R(t,t)\mathrm{d}^\varepsilon s\mathrm{d}^{\tilde\varepsilon}s',$$

$$\mathcal{A}_{01}(t_\varepsilon,t_{\tilde\varepsilon})=\int_0^{t_{\tilde\varepsilon}}\int_0^{t_\varepsilon}\Gamma\begin{pmatrix}s'-\tilde\varepsilon & s'\\ s-\varepsilon & s\end{pmatrix}R(s,t)\mathrm{d}^\varepsilon s\mathrm{d}^{\tilde\varepsilon}s',$$

$$\mathcal{A}_{02}(t_\varepsilon,t_{\tilde\varepsilon})=\int_0^{t_{\tilde\varepsilon}}\int_0^{t_\varepsilon}\Gamma\begin{pmatrix}s'-\tilde\varepsilon & s'\\ s-\varepsilon & s\end{pmatrix}R(t,s')\mathrm{d}^\varepsilon s\mathrm{d}^{\tilde\varepsilon}s'.$$

Then

$$\lim_{\varepsilon,\tilde\varepsilon\to 0+}(\varepsilon\tilde\varepsilon)^{-1}(\mathcal{A}_{00}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{I}_{01}(t_\varepsilon,t_{\tilde\varepsilon}))=R(t,t)(\Gamma(0,0)-\Gamma(0,t)-\Gamma(t,0)), \tag{3.37}$$

$$\lim_{\varepsilon,\tilde\varepsilon\to 0+}(\varepsilon\tilde\varepsilon)^{-1}(-\mathcal{A}_{01}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{I}_{21}(t_\varepsilon,t_{\tilde\varepsilon}))=\int_0^t R(s,t)\frac{\partial\Gamma}{\partial s}(s,0)\mathrm{d}s, \tag{3.38}$$

$$\lim_{\varepsilon,\tilde\varepsilon\to 0+}(\varepsilon\tilde\varepsilon)^{-1}(-\mathcal{A}_{02}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{I}_{11}(t_\varepsilon,t_{\tilde\varepsilon}))=\int_0^t R(t,s')\frac{\partial\Gamma}{\partial s'}(0,s')\mathrm{d}s', \tag{3.39}$$

where $\mathcal{I}_{01}$, $\mathcal{I}_{11}$, and $\mathcal{I}_{21}$, are respectively, defined by (3.7), (3.8), and (3.9).

Remark 3.9 We highlight the fact that, because of the singularity of our equation, both terms $\mathcal{A}_{00}(t_\varepsilon, t_{\tilde{\varepsilon}})$ and $\mathcal{I}_{01}(t_\varepsilon, t_{\tilde{\varepsilon}})$ in (3.37) are divergent. However, the difference $\mathcal{A}_{00} - \mathcal{I}_{01}$ is a convergent quantity. The same holds true for the limits in (3.38) and (3.39).

Proof of Lemma 3.8 The proof is an application of Lemma 3.7. We only consider the convergence (3.37). The convergence in (3.38) and (3.39) can be shown in a similar way and will be omitted. In the following, we denote $g = \frac{\partial^2 \Gamma}{\partial s \partial s'}$.

For $s \in [0, t]$, set $\eta^\varepsilon(s) = t_k$ if $k\varepsilon \le s < (k+1)\varepsilon$, where we recall that $t_k = k\varepsilon$. Then, it is easily seen that

$$\mathcal{A}_{00}(t_\varepsilon, t_{\tilde{\varepsilon}}) = \int_0^{t_{\tilde{\varepsilon}}} \int_0^{t_\varepsilon} \Gamma \begin{pmatrix} \eta^{\tilde{\varepsilon}}(s') - \tilde{\varepsilon} & \eta^{\tilde{\varepsilon}}(s') \\ \eta^\varepsilon(s) - \varepsilon & \eta^\varepsilon(s) \end{pmatrix} R(t,t)\mathrm{d}s\mathrm{d}s'.$$

Then, resorting to the mean value theorem for the rectangular increment of Γ, we get the existence of $\theta = \theta(s, s')$ and $\theta' = \theta'(s, s')$ such that $\theta, \theta' \in [0, 1]$ and

$$\mathcal{A}_{00}(t_\varepsilon, t_{\tilde{\varepsilon}}) = \varepsilon\tilde{\varepsilon} \int_0^{t_{\tilde{\varepsilon}}} \int_0^{t_\varepsilon} g(\eta(s) - \varepsilon\theta, \eta^{\tilde{\varepsilon}}(s') - \tilde{\varepsilon}\theta') R(t,t)\mathrm{d}s\mathrm{d}s',$$

where we recall that we have set $g = \frac{\partial^2 \Gamma}{\partial s \partial s'}$. On the other hand, because of the fact that $\eta(t_\varepsilon) + \varepsilon$ is the only partition point in $[\tilde{t}, \tilde{t} + \varepsilon)$, we have

$$\mathcal{I}_{01}(t_\varepsilon, t_{\tilde{\varepsilon}}) = \varepsilon\tilde{\varepsilon}\Gamma(\eta(t_\varepsilon) \quad \text{and} \quad \eta(t_{\tilde{\varepsilon}}))R(\eta(t_\varepsilon) + \varepsilon, \eta(t_{\tilde{\varepsilon}}) + \tilde{\varepsilon}). \tag{3.40}$$

In order to ease notations, we denote $\bar{t}^\varepsilon = \eta(t_\varepsilon)$ and $\bar{t}^{\tilde{\varepsilon}} = \eta(t_{\tilde{\varepsilon}})$, so that we can recast (3.40) as

$$\mathcal{I}_{01}(t_\varepsilon, t_{\tilde{\varepsilon}}) = \varepsilon\tilde{\varepsilon}\,\Gamma(\bar{t}^\varepsilon, \bar{t}^{\tilde{\varepsilon}})\, R(\bar{t}^\varepsilon + \varepsilon, \bar{t}^{\tilde{\varepsilon}} + \tilde{\varepsilon}). \tag{3.41}$$

Introducing the rectangular increment of Γ on $[0, \bar{t}^\varepsilon] \times [0, \bar{t}^{\tilde{\varepsilon}}]$, we now write (3.41) under the form:

$$\mathcal{I}_{01} = \mathcal{A}'_{00}(t_\varepsilon, t_{\tilde{\varepsilon}}) - \varepsilon\tilde{\varepsilon}\, R(\bar{t}^\varepsilon + \varepsilon, \bar{t}^{\tilde{\varepsilon}} + \tilde{\varepsilon})(\Gamma(0,0) - \Gamma(0, \bar{t}^{\tilde{\varepsilon}}) - \Gamma(\bar{t}^\varepsilon, 0)), \tag{3.42}$$

where we set

$$\mathcal{A}'_{00}(t_\varepsilon, t_{\tilde{\varepsilon}}) = \varepsilon\tilde{\varepsilon}\, R(\bar{t}^\varepsilon + \varepsilon, \bar{t}^{\tilde{\varepsilon}} + \tilde{\varepsilon})\Gamma \begin{pmatrix} 0 & \bar{t}^{\tilde{\varepsilon}} \\ 0 & \bar{t}^\varepsilon \end{pmatrix}.$$

Furthermore, differentiating the function Γ above, $\mathcal{A}'_{00}(t_\varepsilon, t_{\tilde{\varepsilon}})$ can be expressed as

$$\begin{aligned} \mathcal{A}'_{00}(t_\varepsilon, t_{\tilde{\varepsilon}}) &= \varepsilon\tilde{\varepsilon} \int_0^{\bar{t}^{\tilde{\varepsilon}}} \int_0^{\bar{t}^\varepsilon} \frac{\partial^2 \Gamma}{\partial s \partial s'}(s, s') R(\bar{t}^\varepsilon + \varepsilon, \bar{t}^{\tilde{\varepsilon}} + \tilde{\varepsilon})\mathrm{d}s\mathrm{d}s' \\ &= \varepsilon\tilde{\varepsilon} \int_0^{\bar{t}^{\tilde{\varepsilon}}} \int_0^{\bar{t}^\varepsilon} g(s, s') R(\bar{t}^\varepsilon + \varepsilon, \bar{t}^{\tilde{\varepsilon}} + \tilde{\varepsilon})\mathrm{d}s\mathrm{d}s'. \end{aligned}$$

A direct application of Lemma 3.7 now yields that

$$\lim_{\varepsilon, \tilde{\varepsilon} \to 0+} \frac{1}{\varepsilon\tilde{\varepsilon}} \left(\mathcal{A}_{00}(t_\varepsilon, t_{\tilde{\varepsilon}}) - \mathcal{A}'_{00}(t_\varepsilon, t_{\tilde{\varepsilon}})\right) = 0. \tag{3.43}$$

In addition, some easy continuity arguments show that

$$\lim_{\varepsilon, \tilde{\varepsilon} \to 0+} R(\bar{t}^\varepsilon, \bar{t}^{\tilde{\varepsilon}})(\Gamma(0,0) - \Gamma(0, \bar{t}^{\tilde{\varepsilon}}) - \Gamma(\bar{t}^\varepsilon, 0)) = R(t,t)(\Gamma(0,0) - \Gamma(0,t) - \Gamma(t,0)).$$

Combining this relation with (3.42) and (3.43) ends the proof of (3.37). □

The following is the main result of this article:

Theorem 3.10 Let W be the Gaussian field defined in Definition 2.1 with time covariance function R and spectral measure μ. The following results on the solution to equation (1.1) hold true.

Sufficiency If μ satisfies relation (1.3) and R is such that

$$|R(t,t)-R(t,v)-R(u,t)+R(u,v)|\le K(t-u\wedge v)^{\beta} \tag{3.44}$$

for a constant $K>0$ and $\beta\in(0,2]$, then we have the followings.

(i) The solution $u(t,x)$ of (1.1) exists in the sense of (3.1) and Definition 2.2. Namely, for all $t\in\mathbb{R}_+$ and $x\in\mathbb{R}^d$, the $L^2(\Omega)$-convergence in Definition 2.2 holds for $g(s,y)=p_{t-s}(x-y)$.

(ii) The following identity for the $L^2(\Omega)$-norm of the solution holds:

$$\mathbb{E}[|u(t,x)|^2]=R\begin{pmatrix}0 & t\\ 0 & t\end{pmatrix}\Gamma(0,0)+\int_0^t R\begin{pmatrix}0 & t\\ s' & t\end{pmatrix}\frac{\partial\Gamma}{\partial s'}(0,s')\mathrm{d}s'+\int_0^t R\begin{pmatrix}s & t\\ 0 & t\end{pmatrix}\frac{\partial\Gamma}{\partial s}(s,0)\mathrm{d}s$$
$$+\int_0^t\int_0^t R\begin{pmatrix}s & t\\ s' & t\end{pmatrix}\frac{\partial^2\Gamma}{\partial s\partial s'}(s,s')\mathrm{d}s\mathrm{d}s'. \tag{3.45}$$

Necessity Suppose that the solution of (1.1) exists in the sense of (3.1), that identity (3.45) holds true, and that R is such that

$$|R(t,t)-R(t,v)-R(u,t)+R(u,v)|\ge K(t-u\vee v)^{\beta} \tag{3.46}$$

for some $\beta\in(0,2]$. Then, the spectral measure μ satisfies relation (1.3).

Proof Recall that in Lemma 3.3, we show that

$$\begin{aligned}&\mathbb{E}(u_\varepsilon(t,x)u_{\tilde\varepsilon}(t,x))\\ &=(\varepsilon\tilde\varepsilon)^{-1}\left(\mathcal{A}_0(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_0(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_1(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_2(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_3(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_4(t_\varepsilon,t_{\tilde\varepsilon})\right).\end{aligned}$$

Moreover, going back to definition (3.6) of $\mathcal{A}_0$ and (3.23) of $\tilde{\mathcal{A}}_0^{\varepsilon,\tilde\varepsilon}$, some elementary manipulations show that

$$\mathcal{A}_0(t_\varepsilon,t_{\tilde\varepsilon})=\tilde{\mathcal{A}}_0^{\varepsilon,\tilde\varepsilon}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{A}_{00}(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{A}_{00}(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{A}_{00}(t_\varepsilon,t_{\tilde\varepsilon}),$$

where $\mathcal{A}_{00}$, $\mathcal{A}_{01}$, and $\mathcal{A}_{02}$ are introduced in Lemma 3.8. Because of the decompositions (3.7), (3.8), and (3.9), we thus end up with

$$\begin{aligned}&\mathbb{E}(u_\varepsilon(t,x)u_{\tilde\varepsilon}(t,x))\\ &=(\varepsilon\tilde\varepsilon)^{-1}\Big(\tilde{\mathcal{A}}_0^{\varepsilon,\tilde\varepsilon}(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_3(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{I}_4(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{I}_{00}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{I}_{10}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{I}_{20}(t_\varepsilon,t_{\tilde\varepsilon})\\ &\quad+(\mathcal{I}_{11}(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{A}_{02}(t_\varepsilon,t_{\tilde\varepsilon}))+(\mathcal{I}_{21}(t_\varepsilon,t_{\tilde\varepsilon})+\mathcal{A}_{01}(t_\varepsilon,t_{\tilde\varepsilon}))+(\mathcal{I}_{01}(t_\varepsilon,t_{\tilde\varepsilon})-\mathcal{A}_{00}(t_\varepsilon,t_{\tilde\varepsilon}))\Big).\end{aligned}$$

Now, we can apply Lemma 3.4 to $\mathcal{I}_3$ and $\mathcal{I}_4$, Lemma 3.6 to $\tilde{\mathcal{A}}_0^{\varepsilon,\tilde\varepsilon}$, Lemma 3.8 to $(\mathcal{I}_{11}+\mathcal{A}_{02})+(\mathcal{I}_{21}+\mathcal{A}_{01})+(\mathcal{I}_{01}-\mathcal{A}_{00})$, and apply the continuity of R and Γ to $\mathcal{I}_{00}-\mathcal{I}_{10}-\mathcal{I}_{20}$, which shows the convergence of $\mathbb{E}(u_\varepsilon(t,x)u_{\tilde\varepsilon}(t,x))$ as $\varepsilon,\tilde\varepsilon\to0+$ and thus completes the proof of (i). Item (ii) can be shown immediately by rearranging the limits of these convergences. Notice that in all the aforementioned Lemma 3.4, 3.6, and 3.8, we are only using the upper bound (3.44) on the increments of R instead of relation (1.2).

To prove the necessity, we apply Fubini's theorem to the last term on the right-hand side of (3.45) and then invoke relation (3.18). This yields the inequality

$$\int_{\mathbb{R}^d} |\xi|^4 \mu(\mathrm{d}\xi) \int_0^t \int_0^t e^{-\frac{(2t-s-s')|\xi|^2}{2}} R\begin{pmatrix} s & t \\ s' & t \end{pmatrix} \mathrm{d}s\mathrm{d}s' < \infty.$$

Taking into account that R satisfies relation (3.46) and using some elementary change of variable formulas, we conclude that the spatial spectral measure μ satisfies relation (1.3). □

4 Solving the Heat Equation in a Besov Space

This section sheds a different light on the existence problem for equation (1.1). Namely, we will now consider the noise $\partial W := \frac{\partial^d W}{\partial x_1 \cdots \partial x_d}$ as a distribution in a certain Soblev space of negative order. Then, we will quantify how the heat flow regularizes W in order to give a meaning to u as a function. This analysis obviously requires some preliminary background about Littlewood-Paley theory (recalled below) and yields some slightly non optimal results. However, let us mention that the computations leading to the existence of a solution are simpler within this framework than in the previous section. Furthermore, the Besov space method also brings some regularity results for the solution u at no additional cost.

We briefly recall some elements of the Besov space theory. The readers are referred to [1, Chapter 2] for further details, and to [18] for an analysis of Besov spaces with weights. We first give a result which provides us with the dyadic partition of unity (see a more complete statement in [1, Proposition 2.10]):

Proposition 4.1 Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$. Then, there exist radial functions χ and φ, valued in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and $\mathcal{D}(\mathcal{C})$, and such that

$$\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \text{and} \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1.$$

The following support type conditions are also satisfied by φ and χ:

$$|j - j'| \ge 2 \Rightarrow \mathrm{Supp}\varphi(2^{-j}\cdot) \cap \mathrm{Supp}\varphi(2^{-j'}\cdot) = \emptyset,$$
$$j \ge 1 \Rightarrow \mathrm{Supp}\chi \cap \mathrm{Supp}\varphi(2^{-j}\cdot) = \emptyset.$$

Let us now define the dyadic blocks Δ_j, which are the basic bricks of Littlewood-Paley's analysis.

Definition 4.2 Let χ and φ be the two functions constructed in Proposition 4.1 and write $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. The nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j u = 0 \quad \text{if} \quad j \le -2, \qquad \Delta_{-1} u = \int_{\mathbb{R}^d} \tilde{h}(y) u(x-y)\mathrm{d}y,$$

and for $j \ge 0$, we have

$$\Delta_j u = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) u(x-y)\mathrm{d}y. \tag{4.1}$$

With the previous notations in hand, we shall now define a family of Besov type spaces in which our noise ∂W will sit.

Definition 4.3 Let $\kappa \in \mathbb{R}$ and $1 \leq q < \infty$. We will consider a spatial weight $\rho_\sigma(x) = \frac{1}{(1+|x|)^\sigma}$ defined for $x \in \mathbb{R}^d$ and $\sigma > d$. The non-homogeneous weighted Besov space B_q^κ consists of all tempered distributions u such that

$$\|u\|_{B_q^\kappa}^{2q} := \sum_{j\in\mathbb{Z}} 2^{2qj\kappa} \|\Delta_j u\|_{L_{\rho_\sigma}^{2q}}^{2q} < \infty. \tag{4.2}$$

Here L_ρ^{2q} denotes the space $L^{2q}(\mathbb{R}^d, \rho(x)\mathrm{d}x)$.

Remark 4.4 We have used the norms defined by (4.2) for computational sake. However, we should mention that if $f \in B_q^\kappa$ for $\kappa > \frac{d}{2q}$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$, then $f\phi \in C^\alpha$ for $\alpha = \kappa - \frac{d}{2q}$. We are thus able to embed locally our Besov spaces B_q^κ into Hölder type spaces.

Proof If $f \in B_q^\kappa$, it is shown in [18, Proposition 3.27] that $f\phi$ belongs to the non weighted Besov space $B_{2q,2q}^\kappa$. Then because of [1, Proposition 2.71], we proved that $B_{2q,2q}^\kappa$ is continuously embedded into the Hölder space C^α with $\alpha = \kappa - \frac{d}{2q}$. □

For our convenience, we are working in this section with a noise ∂W, which has to be thought of as an integrated version in time of the noise $\dot{W}$ which appears in equation (1.1). We now define this Gaussian family more rigorously.

Definition 4.5 Let W be the Gaussian family introduced in Definition 2.1. We define another centered Gaussian family $\partial W = \{\partial W_{st}(\varphi) : 0 \leq s \leq t \leq \tau, \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ by

$$\partial W_{st}(\varphi) = W(\mathbf{1}_{[s,t]} \otimes \varphi).$$

We now state the assumptions on the covariance of our noise in a slightly different way with respect to (1.2) and (1.3).

Hypothesis 1 We assume that there exists $\beta > 0$ and $\beta' \in (0, 2]$ such that the function R and the measure μ appearing in (2.3) satisfy the following conditions:

$$|R(t,t) - R(u,t) - R(t,v) + R(u,v)| \leq K|t - (u \wedge v)|^{\beta'} \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{1}{(1+|\xi|)^{2\beta}} \mu(\mathrm{d}\xi) < \infty.$$

In the following, we prove that ∂W can also be seen as a Hölder continuous function of time taking values in a weighted Besov space.

Lemma 4.6 Let W be the Gaussian field in Definition 2.1 with time covariance R and spatial spectral measure μ, and suppose that Hypothesis 1 holds true. Then $\partial W_{st} \in B_q^\kappa$ for all $\kappa < -\beta$, $q \geq 1$, and $0 \leq s \leq t$. Moreover, for all $\epsilon > 0$, $q \geq 1$, and $\kappa < -\beta$, there exists a random variable Z admitting moments of all orders such that for all $0 \leq s < t \leq \tau$, we have

$$\|\partial W_{st}\|_{B_q^\kappa} \leq Z(t-s)^{\frac{\beta'}{2} - \epsilon}. \tag{4.3}$$

Proof By Definition 4.2 of Δ_j, we first write

$$\Delta_j \partial W_{st} = W(\mathbf{1}_{[s,t]} \otimes 2^{jd} h(2^j(x - \cdot))).$$

Because $\Delta_j \partial W_{st}$ is Gaussian, by the hypercontractivity property, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\Delta_j \partial W_{st}\|_{L_{\rho_\sigma}^{2q}}^{2q}\right] &= \mathbb{E}\left[\int |\Delta_j \partial W_{st}(x)|^{2q} \rho_\sigma(x)\mathrm{d}x\right] \\ &\leq c_q \left[\int \mathbb{E}[|\Delta_j \partial W_{st}(x)|^2]^q \rho_\sigma(x)\mathrm{d}x\right]. \end{aligned} \tag{4.4}$$

Notice that the function $2^{jd}h(2^j y)$ appearing in formula (4.1) satisfies

$$\mathcal{F}[2^{jd}h(2^j\cdot)](\xi)=\varphi(2^{-j}\xi).$$

Therefore, resorting to (2.3), we can write

$$\mathbb{E}[|\Delta_j\partial W_{st}(x)|^2]=R\begin{pmatrix} s & t\\ s & t\end{pmatrix}\int_{\mathbb{R}^d}(1+|\xi|)^{2\beta}|\varphi(2^{-j}\xi)|^2\frac{\mu(\mathrm{d}\xi)}{(1+|\xi|)^{2\beta}}.$$

Hence, taking into account the fact that $(1+2^{-j}|\xi|)^{2\beta}|\varphi(2^{-j}\xi)|^2$ is uniformly bounded in ξ and Hypothesis 1 on μ, we obtain

$$\mathbb{E}[|\Delta_j\partial W_{st}(x)|^2]\le K2^{2j\beta}(R(t,t)-2R(s,t)+R(s,s)). \tag{4.5}$$

Now invoking (4.2), and then substituting (4.5) into (4.4) and taking into account the fact that $\kappa<-\beta$, we have

$$\mathbb{E}[\|\partial W_{st}\|_{B_q^\kappa}^{2q}]=\mathbb{E}\left[\sum_{j\in\mathbb{Z}}2^{2qj\kappa}\|\Delta_j\partial W_{st}\|_{L_{\rho_\sigma}^{2q}}^{2q}\right]\le K(R(t,t)-2R(s,t)+R(s,s))^q. \tag{4.6}$$

Once (4.6) is proved, our assertion (4.3) is shown because of Hypothesis 1 on R and a standard application of Garsia's lemma; see [9]. □

In order to transfer Lemma 4.6 on ∂W to properties of the heat equation, we recall some results (see [18]) about the smoothing effects of the heat flow $p_t:=e^{t\Delta}$ in Besov spaces.

Lemma 4.7 Consider $\alpha\in\mathbb{R}$ and two real numbers η,κ such that $\eta\ge\alpha$, $\kappa\ge\alpha$, and $\kappa-\alpha\le 2$. Let $q\ge 1$ be a real number. Then, there exists a constant $K<\infty$ such that uniformly over $t>0$, we have

$$\|p_tf\|_{B_q^\eta}\le\frac{K}{t^{\frac{\eta-\alpha}{2}}}\|f\|_{B_q^\alpha}\qquad\text{and}\qquad\|(\mathrm{Id}-p_t)f\|_{B_q^\alpha}\le Kt^{\frac{\kappa-\alpha}{2}}\|f\|_{B_q^\kappa}.$$

Recall that in Section 2 (see Definition 2.2), the Wiener integrals were considered as $L^2(\Omega)$ limit of Riemann sums. In this section, we introduce the same kind of regularization, whereas the limits are considered in an almost sure sense.

Definition 4.8 Consider the dyadic partition of $\mathbb{R}_+$ defined by $t_k^n=2^{-n}k$, for $k\in\mathbb{N}$ and $n\in\mathbb{N}$. The solution of equation (1.1) is defined as the almost sure limit of

$$u_t^n=\sum_{0\le t_k^n<t}p_{t-t_k^n}\partial W_{t_k^n t_{k+1}^n}, \tag{4.7}$$

whenever u_t^n converges in some Besov space B_q^η with $\eta>0$ and $q\ge 1$. In relation (4.7), $p_s\partial W_{uv}$ has to be understood as the action of the semigroup p_s on the distribution $\partial W_{uv}\in B_q^\kappa$, as introduced in Lemma 4.7.

Remark 4.9 Definition 4.8 differs from Definition 2.2 in two ways: (i) The limits in Definition 2.2 are considered in the $L^2(\Omega)$ sense, as opposed to the almost sure convergence in (4.7). (ii) In Definition 4.8, we work with dyadic partitions, while our sequence of partitions is more general in Definition 2.2. However, if we deal with dyadic partitions only, the convergences in (2.5) and (4.7) both hold in probability and the limiting object is the same. Also notice that the dyadic assumption for the sequence of partitions in Definition 4.8 is not mandatory. We could have handled the case of a generic sequence of partitions whose mesh is converging to 0, but refrained from doing so for sake of conciseness.

Remark 4.10 As in the random field setup of Section 3, the uniqueness of the solution for equation (1.1) is not a real issue because we define our solutions through an explicit formula. One could also argue by proving that if $u=\lim_{n\to\infty}u^n$ exists, then u satisfies a weak form of equation (1.1). Then, one can prove uniqueness for the weak equation. We are not pursuing this route here for sake of simplicity.

We are now ready to solve equation (1.1) within our Besov space framework. As mentioned above, this method brings out regularity results on the solution in a natural way.

Theorem 4.11 Let W be the centered Gaussian field introduced in Definition 2.1 with spatial spectral measure μ and time covariance R. We suppose that Hypothesis 1 holds true for some constants $\beta>0$ and $\beta'\in(0,2]$ such that $\beta'>\beta$. We consider an arbitrarily large time horizon $\tau>0$. Then, the following statements hold true for our stochastic heat equation:

(i) Equation (1.1) admits a random field solution $\{u(t,x);\, t\in[0,\tau], x\in\mathbb{R}^d\}$ in the sense of Definition 4.8.

(ii) In addition, for any $\epsilon,\eta>0$ such that $\eta+\epsilon<\beta'-\beta$ and for all $q\geq 1$, the solution u to (1.1) almost surely sits in the space $C^{\frac{\beta'-\beta-\eta}{2}-\epsilon}([0,\tau];B_q^\eta)$.

Proof Let $0\leq s<t\leq\tau$ and recall that the dyadic partition of $\mathbb{R}_+$ is defined by $t_k^n=k2^{-n}$ for $k,n\geq 0$. With some elementary computations, we easily get

$$\begin{aligned}
&(u_t^n-u_s^n)-(u_t^{n+1}-u_s^{n+1})\\
=&\sum_{s\leq t_{2k}^{n+1}<t}(p_{t-t_{2k}^{n+1}}-p_{t-t_{2k+1}^{n+1}})\partial W_{t_{2k+1}^{n+1}t_{2k+2}^{n+1}}\\
&+\sum_{0\leq t_{2k}^{n+1}<s}[(p_{t-t_{2k}^{n+1}}-p_{s-t_{2k}^{n+1}})-(p_{t-t_{2k+1}^{n+1}}-p_{s-t_{2k+1}^{n+1}})]\partial W_{t_{2k+1}^{n+1}t_{2k+2}^{n+1}}\\
=&\,\mathcal{U}_1+\mathcal{U}_2,
\end{aligned}\tag{4.8}$$

where we have set

$$\mathcal{U}_1=\sum_{s\leq t_{2k}^{n+1}<t}p_{t-t_{2k+1}^{n+1}}\left[(p_{2^{-(1+n)}}-\mathrm{Id})\partial W_{t_{2k+1}^{n+1}t_{2k+2}^{n+1}}\right],$$

$$\mathcal{U}_2=\sum_{0\leq t_{2k}^{n+1}<s}(p_{t-s}-\mathrm{Id})p_{s-t_{2k+1}^{n+1}}(p_{2^{-(1+n)}}-\mathrm{Id})\partial W_{t_{2k+1}^{n+1}t_{2k+2}^{n+1}}.$$

In the following, we bound $\mathcal{U}_1$ and $\mathcal{U}_2$ separately, starting with $\mathcal{U}_1$.

In order to analyze the term $\mathcal{U}_1$ in (4.8), we first tune the parameters of our Besov space. Namely, we consider three parameters κ, α, η satisfying the following conditions:

$$\kappa=-\beta-\epsilon,\quad \alpha<\kappa+\beta'-2-2\epsilon,\quad 0<\eta<2+\alpha\tag{4.9}$$

for an arbitrarily small constant $\epsilon>0$. Notice that for such values of the parameters, we have

$$\frac{\kappa-\alpha}{2}+\frac{\beta'}{2}-\epsilon-1>0\quad\text{and}\quad\frac{\alpha-\eta}{2}>-1.$$

Also observe that this choice of α and η is possible as long as $\eta<\beta'-\beta-3\epsilon$. With these values of κ, α, and η, we now apply Lemma 4.7 and estimate (4.3) to the terms of $\mathcal{U}_1$. This yields the following estimate

$$\left\|p_{t-t_{2k+1}^{n+1}}\left[(p_{2^{-(1+n)}}-\mathrm{Id})\partial W_{t_{2k+1}^{n+1}t_{2k+2}^{n+1}}\right]\right\|_{B_q^\eta}$$

$$\leq Z(t-t_{2k+1}^{n+1})^{\frac{\alpha-\eta}{2}} \left\| (p_{2^{-(1+n)}} - \mathrm{Id})\partial W_{t_{2k+1}^{n+1} t_{2k+2}^{n+1}} \right\|_{B_q^\alpha}$$

$$\leq Z(t-t_{2k+1}^{n+1})^{\frac{\alpha-\eta}{2}} (2^{-n})^{\frac{\kappa-\alpha}{2}+\frac{\beta'}{2}-\epsilon}. \tag{4.10}$$

Summing (4.10) over k and taking into account (4.8), we obtain an upper-bound for $\mathcal{U}_1$:

$$\|\mathcal{U}_1\|_{B_q^\eta} \leq \sum_{s\leq t_{2k}^{n+1}<t} \left\| p_{t-t_{2k+1}^{n+1}} \left[(p_{2^{-(1+n)}} - \mathrm{Id})\partial W_{t_{2k+1}^{n+1} t_{2k+2}^{n+1}} \right] \right\|_{B_q^\eta}$$

$$\leq Z(2^{-n})^{\frac{\kappa-\alpha}{2}+\frac{\beta'}{2}-1-\epsilon} (t-s)^{\frac{\alpha-\eta}{2}+1}. \tag{4.11}$$

The term $\mathcal{U}_2$ is bounded along the same lines as $\mathcal{U}_1$. Namely, we take η' such that $\alpha > \eta' - 2 > \eta - 2$. Applying Lemma 4.7 to $\mathcal{U}_2$ and then following the estimates of (4.10), we are left with

$$\|\mathcal{U}_2\|_{B_q^\eta} \leq \sum_{0\leq t_{2k}^{n+1}<s} Z(t-s)^{\frac{\eta'-\eta}{2}} (s-t_{2k+1}^{n+1})^{\frac{\alpha-\eta'}{2}} (2^{-n})^{\frac{\kappa-\alpha}{2}+\frac{\beta'}{2}-\epsilon}$$

$$\leq Z(t-s)^{\frac{\eta'-\eta}{2}} (2^{-n})^{\frac{\kappa-\alpha}{2}+\frac{\beta'}{2}-1-\epsilon} s^{\frac{\alpha-\eta'}{2}+1}. \tag{4.12}$$

We now plug inequalities (4.11) and (4.12) into (4.8), which yields

$$\|(u_t^n - u_s^n) - (u_t^{n+1} - u_s^{n+1})\|_{B_q^\eta} \leq Z(t-s)^{\frac{\eta'-\eta}{2}} (2^{-n})^{\frac{\kappa-\alpha}{2}+\frac{\beta'}{2}-1-\epsilon}. \tag{4.13}$$

Note that a similar estimate for $u_t^0 - u_s^0$ also holds:

$$\|u_t^0 - u_s^0\|_{B_q^\eta} \leq Z(t-s)^{\frac{\eta'-\eta}{2}}, \tag{4.14}$$

which can be shown in a similar way as for (4.13).

It now follows easily from (4.13) and (4.14) and the inequality $\frac{\kappa-\alpha}{2} + \frac{\beta'}{2} - \epsilon - 1 > 0$ that u_t^n converges to u_t in B_q^η and that

$$\sup_{s,t\in[0,\tau]} \frac{\|u_t - u_s\|_{B_q^\eta}}{|t-s|^{\frac{\eta'-\eta}{2}}} \leq Z. \tag{4.15}$$

Let us now compute the order of magnitude of $\eta' - \eta$. Recall that we had to impose

$$\eta < \eta' < \alpha + 2.$$

In addition, according to (4.9) we have $\alpha < \beta' - \beta - 2 - 3\epsilon$. Hence, it is readily checked that η' is at most $(\beta' - \beta - 3\epsilon)-$. In conclusion, the exponent $\eta' - \eta$ in (4.15) can take any value in $(0, \beta' - \beta)$, and we get $u \in C^{\frac{\beta'-\beta-\eta}{2}-3\epsilon}([0,\tau]; B_q^\eta)$. Thus, the proof is now complete. □

Remark 4.12 Combining Theorem 4.11, Remark 4.4 and because of the fact that we can consider an arbitrarily large number q in Theorem 4.11, we get the fact that the random field solution u to equation (1.1) is a $[\frac{1}{2}(\beta' - \beta - \eta) - \epsilon, \eta]$-Hölder function on $[0,\tau] \times [-M, M]^d$ for ϵ arbitrarily small, M arbitrarily large, and any $\eta \in (0, \beta' - \beta)$. Namely, we obtain the existence of a random variable $L = L(\omega)$, such that for all $s, t \in [0, \tau]$ and all $x, y \in [-M, M]^d$, we have

$$\frac{|u_t(x) - u_t(y) - u_s(x) + u_s(y)|}{|t-s|^{\frac{1}{2}(\beta'-\beta-\eta)-\epsilon} |x-y|^\eta} \leq L(\omega).$$

As an easy consequence (invoking Besov spaces embeddings), we also get (for a different random variable L)

$$\frac{|u_t(x) - u_s(y)|}{|t-s|^{\frac{1}{2}(\beta'-\beta-\varepsilon)} + |x-y|^{\beta'-\beta-\varepsilon}} \leq L(\omega),$$

and those Hölder continuity exponents are compatible with the optimal exponents obtained in [6, 23].

5 Examples of Application

Let us now give some examples of covariance functions satisfying Hypothesis (1.2) and (1.3). We will also compare Theorems 3.10 and 4.11 with the numerous results available for equation (1.1) and for the parabolic Anderson model. Recall that the parabolic Anderson model is given by

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + u\dot{W}, \qquad t\in[0,\tau],\, x\in\mathbb{R}^d \\ u(0,x) = u_0(x), \end{cases} \tag{5.1}$$

where u_0 is a given smooth and non-degenerate initial condition. In (5.1), the product between u and $\dot{W}$ is understood in the Wick sense.

Example 5.1 Assume that the noise is white in time, namely, $R(s,t)=s\wedge t$. In this case $\beta=1$ and condition (1.3) recovers Dalang's condition (see [6]):

$$\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\mu(\mathrm{d}\xi)<\infty.$$

In particular, if the spatial covariance is given by a Riesz kernel $\Lambda(x)=|x|^{-\eta}$ for some $\eta>0$:

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y))=\delta(t-s)\Lambda(x-y)\,,$$

then $\mu(\mathrm{d}\xi)=c_{\eta,d}|\xi|^{-(d-\eta)}\,\mathrm{d}\xi$. Condition (1.3) is thus equivalent to $\eta<2$, as in [6].

Still in the Brownian case in time, suppose that $d=1$ and that the spatial noise is fractional with Hurst parameter $H\in(0,1)$, which is equivalent to $\mu(\mathrm{d}\xi)=C_H|\xi|^{1-2H}$. Condition (1.3) is thus satisfied for all $H\in(0,1)$. This is in sharp contrast with the multiplicative case of equation (5.1). Indeed, in the multiplicative case with $d=1$, one has to assume $H>1/4$ in order to get the existence and uniqueness of a function valued solution (see for example [14]).

Consider now a Brownian motion in time whose spatial covariance is given by the Bessel kernel

$$\Lambda(x)=\int_0^\infty w^{\frac{\eta-d}{2}}e^{-w}e^{-\frac{|x|^2}{4w}}\mathrm{d}w\,.$$

Then, we have $\mu(\mathrm{d}\xi)=c_{\eta,d}(1+|\xi|^2)^{-\frac{\eta}{2}}\mathrm{d}\xi$ and (1.3) is satisfied if and only if $\eta>d-2$. This result is implicitly contained in [6].

Example 5.2 Assume that the noise is fractional in time with Hurst parameter $H_0\in(0,1)$, namely, $R(s,t)=\frac{1}{2}\left(|s|^{2H_0}+|t|^{2H_0}-|s-t|^{2H_0}\right)$. In this situation, take $0<s<s'<t$. Then, the rectangular covariance in time, defined by (2.2), can be written as

$$R\begin{pmatrix} s & t\\ s' & t\end{pmatrix}=\mathbb{E}[(B_t-B_s)(B_t-B_{s'})]=\frac{1}{2}\left[(t-s')^{2H_0}+(t-s)^{2H_0}-(s'-s)^{2H_0})\right]\,,$$

where B stands for a 1-dimensional fractional Brownian motion. It is then obvious to see that

$$\frac{1}{2}(t-s')^{2H_0}\le R\begin{pmatrix} s & t\\ s' & t\end{pmatrix}\le (t-s)^{2H_0}\,.$$

Therefore, the coefficient β in (1.2) and (1.3) is $\beta = 2H_0$.

The particular case $H_0 > 1/2$ was considered in [2] for equation (1.1), where some necessary and sufficient conditions on the covariance function in space were obtained for the existence of the solution.

Suppose now that the noise is also fractional in space with Hurst parameters $(H_1, \cdots, H_d)$, which means that the spatial fractional covariance is given on $\mathbb{R}^{2d}$ by

$$\prod_{i=1}^{d} R_{H_i}(x_i, y_i), \quad \text{where} \quad R_{H_i}(u,v) = \frac{1}{2}(|u|^{2H_i} + |v|^{2H_i} - |u-v|^{2H_i}).$$

Then, we have $\mu(\mathrm{d}\xi) = C_H \prod_{i=1}^{d} |\xi_i|^{1-2H_i}\mathrm{d}\xi$, where C_H is a constant depending on the parameters H_i. In this situation, condition (1.3) becomes

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{4H_0}} \prod_{i=1}^{d} |\xi_i|^{1-2H_i}\mathrm{d}\xi < \infty,$$

and an easy calculation shows that this is equivalent to

$$2H_0 + \sum_{i=1}^{d} H_i > d\,. \tag{5.2}$$

This condition has to be compared to what is obtained for multiplicative equations like (5.1). In this context, [13, Example 2.6] asserts the existence of an L^2 solution u under the condition $H_i > 1/2$ for all $i = 0, 1, \cdots, d$ and the additional lower bound $\sum_{i=1}^{d} H_i > d-1$, which is a stronger assumption than (5.2). In a recent article by Chen [4], this condition was improved to rough cases with some of the $H_1, \cdots, H_d$ less than 1/2. For example, when $d = 1$, it is shown that (5.1) admits a unique solution as soon as $H_0 > 1/2$ and $H_0 + H_1 > 3/4$.

Let us particularize condition (5.2) to a white noise in space, that is $H_1 = \cdots = H_d = 1/2$. In this case, equation (5.2) becomes $H_0 > d/4$. We can compare this result to two situations studied in [16]:

- When $d = 1$, our condition reads as $H_0 > 1/4$, while [16] was assuming $H_0 > 1/2$ in the multiplicative case (5.1).
- When $d = 2$, equation (5.2) becomes $H_0 > 1/2$, while only the existence for small time for (5.1) was established in [16] under the condition $H_0 > 1/2$ and $H_1 = H_2 = 1/2$.

Let us also mention a recent result in [7], which considered a Stratonovich-type nonlinear heat equation with fractional noise in time and space and with $d = 1$ in space. It was shown in [7] that in this case the solution exists when $2H_0 + H_1 > 2$, while a renormalization of the system is required to solve the equation when $2 \geq 2H_0 + H_1 > 5/3$.

Example 5.3 Assume that the noise is independent of the time parameter t. In this case, $R(s,t) = st$ which means that $\beta = 2$. Condition (1.3) becomes

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^4}\mu(\mathrm{d}\xi) < \infty\,. \tag{5.3}$$

Condition (5.3) can be compared again to the multiplicative case (5.1). Namely, we can quote [13, Theorem 3.9], where Dalang's condition $\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(\mathrm{d}\xi) < \infty$ had to be assumed in order to solve equation (5.1).

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